On the Hyers-Ulam-Rassias stability of a nonlinear integral equation

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Abstract. In this paper, By the fixed point method, we investigate the generalized Hyers-Ulam-Rassias stability of a non-linear integral equation in a Banach space X, which determines the solution of a singular initial value problem. In the case where X is the Euclidean space, this problem was solved by B. Fajmon and Z. Šmarda in a paper published in [Journal of Applied Mathematics, Volume III (2010), number II, p. 53-59].

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1 Introduction

In 1940, Ulam (see [46] and [47]) asked the following question:

Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$.

Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $h : G_1 \to G_2$ such that $d(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

If the answer is yes, then we say that the equation of group homomorphisms is stable in the sense of Ulam.

In 1941, Hyers [19] solved the case of approximately additive mappings, when G_1 and G_2 are Banach spaces.

More precisely, Hyers [19] proved that for all Banach spaces E_1 and E_2, if a function $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon, \quad \forall x, y,$$

for some given $\epsilon > 0$, then there exists a unique additive function $h : E_1 \to E_2$ such that $\|f(x) - h(x)\| \leq \epsilon$, for all $x \in E_1$.

Hyers proved that h is given by the limit

$$h(x) = \lim_{n\to\infty} 2^{-n} f(2^n x),$$

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which exists for all $x \in E_1$.

Another important result was published in 1950 by T. Aoki (see [5]) concerning equations involving unbounded Cauchy differences.

In 1978, Th. M. Rassias [39] investigated approximately additive mappings involving unbounded Cauchy differences and established the following important stability result:

Theorem 1.1. *Let E_1 and E_2 be two Banach spaces and let $f : E_1 \to E_2$ be a mapping satisfying the follwing properties:*
(i) The map $t \mapsto f(tx)$ is continuous in t for each fixed x in E_1.
(ii) There exists a positive number θ and $0 \leq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \quad \forall x, y \in E_1.$$

Then there exists an unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p, \quad \forall x, y \in E_1.$$

For $p = 0$, we recapture Hyers' Theorem.

The contributions of Ulam, Hyers and Rassias are recognized nowadays as the basis of a theory of stability called *stability in the sense of Ulam-Hyers-Rassias*. For more informations on this theory, the reader is invited to consult the books [33], [22], [26], [11], [43] and the references.

Stability in the sense of Ulam-Hyers-Rassias are investigated and studied not only for the classical functional equations, but for various kinds of equations like differential, integral or algebraic equations. Now, the stability of equations in the sense of Ulam-Hyers-Rassias makes use of various methods of different kinds.

The fixed point method is a powerful tool to obtain stability results. In 1991, J. A. Baker (see [6]) has inaugurated this method and studied the Hyers-Ulam stability for a nonlinear functional equation by using the Banach fixed point theorem. In several papers, V. Radu [38] (see also [8] and [9]) applied the fixed point alternative theorem (due to J. B. Diaz and B. Margolis [12]) in order to investigate the Hyers-Ulam-Rassias stability. D. Miheţ [34] used the Luxemburg-Jung fixed point theorem in generalized metric spaces to study the Hyers-Ulam stability for two functional equations in a single variable. L. Găvruţa [17] obtained a general result concerning the Hyers-Ulam stability of a functional equation in a single variable by using a fixed point theorem of Matkowski.

In 2007, S.-M. Jung [30] used the alternative fixed point theorem to establish the stability of a Volterra integral equation. The results of [30] were generalized in [10].

In 2010, M. Gachpazan and O. Baghani [14] (see also [15]) studied the stability of certain Volterra integral equations on finite intervals by using the successive approximation method.

By using the fixed point alternative theorem, the stability of a class of nonlinear Volterra integral equations Hyers-Ulam-Rassias was studied by M. Akkouchi in [2].

In [4], the stability of the mild solutions of a general abstract Cauchy problem was investigated by using the Banach fixed point theorem. A stability result in the sense of Ulam-Hyers was established in [3] for a general class of nonlinear functional equations by using a fixed point theorem of L.J. Ćirić.

Other methods exist to deal with the stability of equations in the sense of Hyers-Ulam-Rassias. For instance, P. Găvruţa and L. Găvruţa (see [18]) have recently provided a new method called the *weighted space method* to deal with the generalized Hyers-Ulam-Rassias stability.

The aim of this paper is to study the generalized HyersUlam-Rassias stability of a non-linear integral equation (see Equation (2.3)) which describes the solutions of a class of singular initial value problem introduced by B. Fajmon and Z. Šmarda in their paper [13].

To study the generalized HyersUlam-Rassias stability of Equation (2.3), we adopt the fixed point method using the classical Banach contraction principle. The main result of this paper (see Theorem 3.1) is established in the third section. In the second section, we make some preliminaries where we set some definitions, notations and state the problem with the associated assumptions.

Our work will provide a natural continuation to the work initiated in [13] by B. Fajmon and Z. Šmarda.

2 Preliminaries and statement of the problem

2.1 Statement of the problem

Let $(X, \|.\|)$ be a real or complex Banach space endowed with a norm $\|.\|$. Let $T > 0$ be a given positive number.

We consider the following singular initial value problem which extends the problem solved in [13] by B. Fajmon and Z. Šmarda in the case where $X = \mathbb{R}^n$.

$$y'(t) = F(t, y(t), \int_{0^+}^{t} K(t, s, y(t), y(s)))\, ds,\ y(0^+) = 0, \quad y(t) \in X,\ \forall t \in (0, T]. \tag{2.1}$$

We make three sets of assumptions similar to those used in [13].

(I) $F : \Omega \to X$ is continuous on the set Ω given by :

$\Omega := \{(t, u_1, u_2) \in J \times X \times X : \|u_1\| \leq \phi(t), \|u_2\| \leq \psi(t)\}$, where $J := (0, T]$, $\phi, \psi : J \to (0, +\infty)$ are continuous with $\phi(0^+) = 0$ and there exist two non-negative numbers M_1 and M_2 such that

$$\|F(t, u_1, u_2) - F(t, v_1, v_2)\| \leq M_1\|u_1 - v_1\| + M_2\|u_2 - v_2\|, \forall (t, u_1, u_2), (t, v_1, v_2) \in \Omega.$$

(II) $K : \Omega_1 \to X$ is a continuous function defined on the set Ω_1 given by :

$\Omega_1 := \{(t, s, w_1, w_2) \in J \times J \times X \times X : \|w_1\| \leq \phi(t), \|w_2\| \leq \phi(t)\}$ which is satisfying the following conditions:

(a) there exist two non-negative numbers N_1 and N_2 such that

$$\|K(t, s, w_1, w_2) - K(t, s, z_1, z_2)\| \leq N_1\|w_1 - z_1\| + N_2\|w_2 - z_2\|,$$

for all $(t,s,w_1,w_2),(t,s,z_1,z_2) \in \Omega_1$, and

(b) $\int_{0+}^{t} \|K(t,s,h(t),h(s)\| \, ds \le \psi(t)$, for all $t \in J$ and $h \in E_\phi$, where E_ϕ is given by

$$E_\phi := \{h : [0,T] \to X : h \text{ is continuous and } \|h(t)\| \le \phi(t), \, \forall t \in [0,T]\}. \tag{2.2}$$

(III) There exist two continuous functions $g_1, g_2 : J \to (0,+\infty)$ and two non-negative numbers α, β with $\alpha + \beta \le 1$ such that the following conditions hold true:

(a) $\|F(t,u_1,u_2)\| \le g_1(t)\|u_1\| + g_2(t)\|u_2\|$, for all $(t,u_1,u_2) \in \Omega$,

(b) $\int_{0+}^{t} g_1(s)\phi(s)\, ds \le \alpha\phi(t)$ and $\int_{0+}^{t} g_2(s)\psi(s)\, ds \le \beta\phi(t)$, for all $t \in (0,T]$.

The initial value problem (2.1) is equivalent to find the solutions (in the set E_ϕ) of the following integral equation:

$$y(t) = \int_{0^+}^{t} F\left(s, y(s), \int_{0^+}^{s} K(s,w,y(s),y(w))\, dw\right) ds, \quad \forall t \in J. \tag{2.3}$$

We notice that the solutions of (2.1) are given by the solutions of the integral equation (2.3). As in [13], the integral equation (2.3) can be solved by using the iteration method and the Banach fixed point theorem.

In this paper, we intend to establish the generalized stability of the integral equation (2.3) in the sense of Ulam-Hyers-Rassias. This concept will be precised in the next subsection.

2.2 Concepts of Ulam-Hyers-Rassias stability

We keep in mind the assumption (I), (II) and (III). We set $I = [0,T]$. The set of all continuous functions from I to X will be denoted by $\mathcal{E} := \mathcal{C}(I,X)$. We recall that

$$E_\phi := \{h : I \to X : h \text{ is continuous and } \|h(t)\| \le \phi(t), \, \forall t \in I\}.$$

For any $h \in E_\phi$, we set

$$\Lambda(h)(t) := \int_{0^+}^{t} F\left(s, h(s), \int_{0^+}^{s} K(s,w,h(s),h(w))\, dw\right) ds, \quad \forall t \in I. \tag{2.4}$$

With the assumptions (I), (II) and (III) made above, it is easy to see that the map $h \mapsto \Lambda(h)$ is a self-mapping of the set E_ϕ. Indeed, for any $h \in E_\phi$, the function $\Lambda(h)$ is continuous on I. Moreover, we have

$$\begin{aligned}
\|\Lambda(h)(t)\| &\le \int_{0^+}^{t} \left\| F\left(s, h(s), \int_{0^+}^{s} K(s,w,h(s),h(w))\, dw\right)\right\| ds \\
&\le \int_{0^+}^{t} \left[g_1(s)\|h(s)\| + g_2(s) \int_{0^+}^{s} \|K(s,w,h(s),h(w))\|\, dw\right] ds \\
&\le \int_{0^+}^{t} g_1(s)\phi(s)\, ds + \int_{0^+}^{t} g_2(s)\psi(s)\, ds \\
&\le \alpha\phi(t) + \beta\phi(t) = (\alpha+\beta)\phi(t) \le \phi(t), \qquad \forall t \in I.
\end{aligned} \tag{2.5}$$

(2.5) proves that Λ transforms the set E_ϕ into itself.

Let $\epsilon > 0$ and let $G \in \mathcal{C}(I, (0, +\infty))$ be given. We consider the following equation

$$g(t) = \Lambda(g)(t), \quad t \in I, \tag{2.6}$$

where the unknown function g is in the set E_ϕ. Beside this integral equation, we consider the following inequalities:

$$\|f(t) - \Lambda(f)(t)\| \leq \epsilon, \quad t \in I, \tag{2.7}$$

$$\|f(t) - \Lambda(f)(t)\| \leq G(t), \quad t \in I, \tag{2.8}$$

where the unknown function f is in the set E_ϕ.

As in [45], we introduce the following definitions.

Definition 2.1. The integral equation (2.6) is Ulam-Hyers stable if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution $f \in E_\phi$ of (2.7) there exists a solution $g \in E_\phi$ of (2.6) such that

$$\|f(t) - g(t)\| \leq c\epsilon, \quad \forall t \in I.$$

Definition 2.2. The integral equation (2.6) is generalized Ulam-Hyers stable if there exists $\theta \in \mathcal{C}([0, +\infty), [0, +\infty))$, $\theta(0) = 0$, such that for each $\epsilon > 0$ and for each solution $f \in E_\phi$ of (2.7) there exists a solution $g \in E_\phi$ of (2.6) such that

$$\|f(t) - g(t)\| \leq \theta(\epsilon), \quad \forall t \in I.$$

Definition 2.3. The integral equation (2.6) is generalized Ulam-Hyers-Rassias stable, with respect to $G \in \mathcal{C}([0, +\infty), (0, +\infty))$, if there exists $c_G > 0$ such that for each solution $f \in E_\phi$ of (2.8) there exists a solution $g \in E_\phi$ of (2.6) such that

$$\|f(t) - g(t)\| \leq c_G\, G(t), \quad \forall t \in I.$$

In the sequel, we are interested by the stability of the equation (2.6) in the sense of Definition 2.3.

3 Main result

The main result of this paper reads as follows.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a (real or complex) Banach space. Let $T > 0$ be a given positive number. Let F and K satisfying the conditions (I), (II) and (III). Let E_ϕ defined by (2.2). Let $G : [0, T] \to (0, \infty)$ be a continuous function.*

Then there exists a constant $c_G > 0$ such that for every $f \in E_\phi$ satisfying the following inequality:

$$\left\| f(t) - \int_{0^+}^{t} F\left(s, f(s), \int_{0^+}^{s} K(s, w, f(s), f(w))\, dw\right) ds \right\| \leq G(t), \quad \forall t \in [0, T], \tag{3.1}$$

there exists a (unique) function $g \in E_\phi$ such that

$$g(t) = \int_{0^+}^{t} F\left(s, g(s), \int_{0^+}^{s} K(s, w, g(s), g(w))\, dw\right) ds, \quad \forall t \in [0, T], \tag{3.2}$$

and

$$\|f(t) - g(t)\| \le c_G\, G(t), \quad \forall t \in [0, T]. \tag{3.3}$$

Proof. We recall that E_ϕ is the set of all continuous functions $h : [0, T] \to X$ such that $\|h(t)\| \le \phi(t)$, for all $t \in [0, T]$.

Let $S > 0$ be such that

$$S[M_1 + M_2N_1T] + S^2M_2N_2 < 1. \tag{3.4}$$

We choose a continuous function $\theta : [0, T] \to (0, \infty)$ such that

$$\int_0^t \theta(s)ds \le S\,\theta(t), \quad \forall t \in [0, T]. \tag{3.5}$$

Such functions exist. For example, we can take $\theta(t) := \exp(\frac{t}{\lambda})$ and set $S := \frac{1}{\lambda}$, which tends to zero when λ tends to $+\infty$, then (3.4) is realized for large values of λ.

To simplify notations, we set $q_S := S[M_1 + M_2N_1T] + S^2M_2N_2$. By (3.4), we know that $q_S \in [0, 1)$.

Let f be satisfying the inequality (3.1). Let α_G and β_G be two positive numbers such that

$$\alpha_G\theta(t) \le G(t) \le \beta_G\theta(t), \quad \forall t \in [0, T]. \tag{3.6}$$

For all $h, g \in E_\phi$, we set

$$d_\theta(h, g) := \inf\{C \in [0, \infty) : \|h(t) - g(t)\| \le C\theta(t), \forall t \in [0, T]\},$$

It is easy to see that (E_ϕ, d_θ) is a metric space and that (E_ϕ, d_ϕ) is complete.

Now, consider the operator $\Lambda : E_\phi \to E_\phi$ defined by

$$(\Lambda h)(t) := \int_{0^+}^{t} F\left(s, h(s), \int_{0^+}^{s} K(s, w, h(s), h(w))\, dw\right) ds, \quad \forall t \in [0, T].$$

We shall prove that Λ is strictly contractive on the metric space (E_ϕ, d_θ). Indeed, let $h, g \in \mathcal{E}$ and let $C(h, g) \in [0, \infty)$ be an arbitrary constant such that

$$\|h(t) - g(t)\| \le C(h, g)\theta(t), \quad \forall t \in [0, T].$$

Then, by using the assumptions (I), (II), (III), (3.5) and (3.6), we have the following inequalities:

$$
\begin{aligned}
&\|(\Lambda h)(t) - (\Lambda g)(t)\| \\
&\leq \int_{0^+}^{t} \|F\left(s, h(s), \int_{0^+}^{s} K(s,w,h(s),h(w))\,dw\right) \\
&\qquad\qquad - F\left(s, g(s), \int_{0^+}^{s} K(s,w,g(s),g(w))\,dw\right)\|\,ds \\
&\leq \int_{0^+}^{t} \left[M_1\|h(s)-g(s)\| + M_2 \int_{0^+}^{s} \|K(s,w,h(s),h(w)) - K(s,w,g(s),g(w))\|dw\right] ds \\
&\leq M_1 \int_{0^+}^{t} C(f,g)\theta(s)\,ds + M_2 \int_{0^+}^{t} \left[\int_{0^+}^{s} \left(N_1\|h(s)-g(s)\| + N_2\|h(w)-g(w)\|\right) dw\right] ds \\
&\leq M_1 C(f,g) \int_{0^+}^{t} \theta(s)\,ds \\
&+ M_2N_1 \int_{0^+}^{t} s\|h(s)-g(s)\|ds + M_2N_2 \int_{0^+}^{t} \left[\int_{0^+}^{s} \|h(w)-g(w)\|dw\right] ds \\
&\leq M_1C(f,g)S\theta(t) + M_2N_1C(f,g)\int_{0^+}^{t} s\theta(s)\,ds + M_2N_2C(f,g)\int_{0^+}^{t}\left[\int_{0^+}^{s}\theta(w)dw\right] ds \\
&\leq C(f,g)\left(M_1S\theta(t) + M_2N_1\int_{0^+}^{t} T\theta(s)\,ds + M_2N_2S\int_{0^+}^{t}\theta(s)\,ds\right) \\
&\leq C(f,g)\left(M_1S\theta(t) + M_2N_1TS\theta(t) + M_2N_2S^2\theta(t)\right) \\
&= C(f,g)\left(M_1S + M_2N_1TS + M_2N_2S^2\right)\theta(t) \\
&= q_SC(f,g)\theta(t), \quad \text{for all} \quad t \in [0,T].
\end{aligned}
$$

Therefore, we have $d_\theta(\Lambda(h),\Lambda(g)) \leq q_SC(h,g)$, from which we deduce that

$$d_\theta(\Lambda(h),\Lambda(g)) \leq q_S d_\theta(h,g).$$

Since $q_S < 1$, it follows that Λ is strictly contractive on the complete metric space (E_ϕ, d_θ). By the Banach fixed point principle, there exits a unique function (say) g in E_ϕ such that $g = \Lambda(g)$.

By the triangle inequality, we have

$$d_\theta(f,g) \leq d_\theta(f,\Lambda(f)) + d_\theta(\Lambda(f),\Lambda(g))) \leq \beta_G + q_S d_\theta(f,g),$$

which implies that

$$d_\theta(f,g) \leq \frac{\beta_G}{1-q_S},$$

from which, we deduce the following inequality

$$\|f(t)-g(t)\| \leq \frac{\beta_G}{(1-q_S)}\theta(t) \leq \frac{\beta_G}{(1-q_S)}\frac{G(t)}{\alpha_G} \leq c_G\,G(t), \quad \forall t \in [0,T], \tag{3.7}$$

where

$$c_G := \frac{\beta_G}{(1 - S[M_1 + M_2N_1T + SM_2N_2])\alpha_G}$$

which is the desired inequality (3.4). This ends the proof. □

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