Existence of positive solutions for a new class of nonlocal $p(x)$-Kirchhoff elliptic systems via sub-super solutions concept

Bilel Mairi, Rafik Guefaifia, Salah Boulaaras and Tahar Bouali

Abstract. Motivated by the idea which has been introduced in [3] combined with the properties of Kirchhoff type operators, we prove the existence of positive solutions for a class of nonlocal $p(x)$-Kirchhoff elliptic systems by using the sub and super solutions concept, which is a new research idea for the presented problems.

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1 Introduction

The study of differential equations and variational problems with nonstandard $p(x)-$growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [1], [5], [7], [23] and [27]). Many existence results have been obtained on this kind of problems, see for example [14], [18], [25] and [26]. In [8], [10], [13], [15]-[17], X.L. Fan et al. studied the regularity of solutions for differential equations with nonstandard $p(x)$-growth conditions.

In this article, we are interested in the $p(x)$-Kirchhoff systems of the form

$$(1.1)\qquad \begin{cases} -M\left(I_0\left(u\right)\right)\triangle_{p(x)}u=\lambda^{p(x)}\left[\lambda_1 a\left(x\right)f\left(v\right)+\mu_1 c\left(x\right)h\left(u\right)\right] \text{ in } \Omega \\ -M\left(I_0\left(v\right)\right)\triangle_{p(x)}v=\lambda^{p(x)}\left[\lambda_2 b\left(x\right)g\left(u\right)+\mu_2 d\left(x\right)\tau\left(v\right)\right] \text{ in } \Omega, \\ u=v=0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded smooth domain with C^2 boundary $\partial\Omega, 1<p\left(x\right)\in C^1\left(\overline{\Omega}\right)$ is a functions with $1<p^-:=\inf_\Omega p\left(x\right)\leq p^+:=\sup_\Omega p\left(x\right)<\infty, \triangle_{p(x)}u=\operatorname{div}\left(\left|\nabla u\right|^{p(x)-2}\nabla u\right)$ is called $p(x)-$Laplacian, $\lambda,\lambda_1,\lambda_2,\mu_1$,and μ_2 are positive parameters, $I_0\left(u\right)=\int\limits_\Omega \frac{1}{p\left(x\right)}\left|\nabla u\right|^{p(x)}dx$ and $M(t)$ is a continuous function.

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Problem (1.1) is a generalization of a model introduced by Kirchhoff [21]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0 \tag{1.2}$$

where $\rho,\ P_0,\ h, E, L$ are constants, which extends the classical D'Alembert's wave equation, by consideringthe effects of the changes in the length of the strings during the vibrations. In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [2, 4, 8, 9, 20, 21, 22, 24], in which the authors have used variational method and topological method to get the existence of solutions.

In this paper,motivated by the ideas introduced in ([18]) and the properties of Kirchhoff type operators in [18], we study the existence of positive solutions for system(1.2) by using the sub- and super solutions techniques. To our best knowledge, this is a new research topic for nonlocal problems. The remainder of this paper is organized as follows. In Section 2, we present some preleminary results on the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$ and the method of sub- and super solutions. In Section 3 is devoted to state and prove the main result.

2 Preliminary results

In order to discuss problem (1.1), we need some theories on $W_0^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space. Firstly we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [14]). Let us define

$$L^{p(x)}(\Omega)=\left\{u : u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx<\infty\right\}.$$

We introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u(x)|_{p(x)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le 1\right\},$$

and

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega);|\nabla u|\in L^{p(x)}(\Omega)\right\},$$

with the norm

$$\|u\|=|u|_{p(x)}+|\nabla u|_{p(x)},\forall u\in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1. *(See [13]). The spaces $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexiveBanach spaces.*

Throughout the paper, we will assume that:

$(H1)$ $M : [0,+\infty) \to [m_0, m_\infty]$ is a continuous and increasing function with $m_0 > 0$;

$(H2)$ $p \in C^1\left(\overline{\Omega}\right)$ and $1 < p^- \leq p^+$;

$(H3)$ $f, g, h, \tau : [0,+\infty[\to \mathbb{R}$ are C^1, monotone functions such that

$$\lim_{u\to+\infty} f(u) = +\infty, \lim_{u\to+\infty} g(u) = +\infty, \lim_{u\to+\infty} h(u) = +\infty, \lim_{u\to+\infty} \tau(u) = +\infty;$$

$(H4)$ $\lim_{u\to+\infty} \frac{f\left(L(g(u))^{\frac{1}{p^- -1}}\right)}{u^{p^- -1}} = 0$,for all $L > 0$;

$(H5)$ $\lim_{u\to+\infty} \frac{h(u)}{u^{p^- -1}} = 0$, and $\lim_{u\to+\infty} \frac{\tau(u)}{u^{p^- -1}} = 0$.

$(H6)$ $a, b, c, d : \overline{\Omega} \to (0, \ +\infty)$ are continuous functions, such that

$$\begin{aligned} a_1 &= \min_{x\in\overline{\Omega}} a(x), b_1 = \min_{x\in\overline{\Omega}} b(x),\ c_1 = \min_{x\in\overline{\Omega}} c(x), d_1 = \min_{x\in\overline{\Omega}} d(x) \\ a_2 &= \max_{x\in\overline{\Omega}} a(x), b_2 = \max_{x\in\overline{\Omega}} b(x),\ c_2 = \max_{x\in\overline{\Omega}} c(x), d_2 = \max_{x\in\overline{\Omega}} d(x) \end{aligned}$$

Definition 2.1. If $u, v \in W_0^{1,p(x)}(\Omega)$, we say that

$$-M(I_0(u)) \triangle_{p(x)} u \leq -M(I_0(v)) \triangle_{p(x)} v$$

if for all $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, we have

$$M(I_0(u)) \int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla\varphi dx \leq M(I_0(v)) \int_\Omega |\nabla v|^{p(x)-2} \nabla v.\nabla\varphi dx, \tag{2.1}$$

where

$$I_0(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx.$$

Definition 2.2. 1. If $u, v \in W_0^{1,p(x)}(\Omega)$, (u, v) is called a weak solution of (1.1) if it satisfies

$$\begin{cases} M(I_0(u)) \int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla\varphi dx = \int_\Omega \lambda^{p(x)} [\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)] \varphi dx, \\ M(I_0(v)) \int_\Omega |\nabla v|^{p(x)-2} \nabla v.\nabla\varphi dx = \int_\Omega \lambda^{p(x)} [\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)] \varphi dx, \end{cases}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$, with $\varphi \geq 0$.

2. We say that (u, v) is called a sub solution (respectively a super solution) of (1.1) if

$$M(I_0(u)) \int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla\varphi dx \leq (\text{respectively} \geq) \int_\Omega \lambda^{p(x)} [\lambda_1 a(x) f(v) + \mu_1 c(x) h(u)] \varphi dx,$$

$$M(I_0(v)) \int_\Omega |\nabla v|^{p(x)-2} \nabla v.\nabla\varphi dx \leq (\text{respectively} \geq) \int_\Omega \lambda^{p(x)} [\lambda_2 b(x) g(u) + \mu_2 d(x) \tau(v)] \varphi dx.$$

Lemma 2.2. *(See [20] Comparison principle) Let* $u, v \in W^{1,p(x)}(\Omega)$ *and* $(H1)$ *holds.If*

$$-M(I_0(u))\triangle_{p(x)}u \leq -M(I_0(v))\triangle_{p(x)}v$$

and $(u-v)^+ \in W_0^{1,p(x)}(\Omega)$ *then* $u \leq v$ *in* Ω.

Lemma 2.3. *(See [20]).Let* $(H1)$ *hold.*$\eta > 0$ *and let* u *be the unique solution of the problem*

$$-\int_\Omega M(I_0(u))\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \mu \text{ in } \Omega,$$

Set $h = \frac{m_0 p^-}{2|\Omega|^{\frac{1}{N}} C_0}$*.Then, when*

$$\mu \geq h, |u|_\infty \leq C^* \mu^{\frac{1}{p^- - 1}},$$

and when

$$\mu < h, |u|_\infty \leq C_* \mu^{\frac{1}{p^+ - 1}},$$

where C^* *and* C_**are positive constants depending* $p^+, p^-, N, |\Omega|, C_0$ *and* m_0.

Here and hereafter, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to denote the distance of Ω.Denote $d(x) = d(x, \partial\Omega)$ and

$$\partial\Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}.$$

Since $\partial\Omega$ is C^2 regularly, there exists a constant $\delta \in (0,1)$ such that $d(x) \in C^2\left(\overline{\partial\Omega_{3\delta}}\right)$ and $|\nabla d(x)| = 1$.
Denote

$$v_1(x) = \begin{cases} \gamma d(x), & d(x) < \delta \\ \gamma\delta + \int_\delta^{d(x)} \gamma\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (\lambda_1 a_1 + \mu_1 c_1)^{\frac{2}{p^- - 1}} dt, & \delta \leq d(x) < 2\delta, \\ \gamma\delta + \int_\delta^{2\delta} \gamma\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (\lambda_1 b_1 + \mu_1 d_1)^{\frac{2}{p^- - 1}} dt, & 2\delta \leq d(x). \end{cases}$$

$$v_2(x) = \begin{cases} \gamma d(x), & d(x) < \delta \\ \gamma\delta + \int_\delta^{d(x)} \gamma\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (\lambda_2 a_2 + \mu_2 c_2)^{\frac{2}{p^- - 1}} dt, & \delta \leq d(x) < 2\delta, \\ \gamma\delta + \int_\delta^{2\delta} \gamma\left(\frac{2\delta - t}{\delta}\right)^{\frac{2}{p^- - 1}} (\lambda_2 b_2 + \mu_2 d_2)^{\frac{2}{p^- - 1}} dt, & 2\delta \leq d(x). \end{cases}$$

Obviously,$0 \leq v_1(x), v_2(x) \in C^1\left(\overline{\Omega}\right)$.Considering

$$
(2.2) \qquad \begin{cases} -M\left(\int\limits_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\right) \triangle_{p(x)} \omega(x) = \eta \text{ in } \Omega \\ \omega = \text{ on } \partial\Omega, \end{cases}
$$

we have the following result

Lemma 2.4. *(See [12]). If the positive parameter η is large enough and ω is the unique solution of* (2.2)*, then we have*

(*i*) *For any $\theta \in (0,1)$ there exists a positive constant C_1 such that*

$$
C_1 \eta^{\frac{1}{p^+ - 1 + \theta}} \le \max_{x \in \overline{\Omega}} \omega(x);
$$

(*ii*) *There exists a positive constant C_2 such that*

$$
\max_{x \in \overline{\Omega}} \omega(x) \le C_2 \eta^{\frac{1}{p^- - 1}}.
$$

3 The main result

In the following, when there is no misunderstanding, we always use C_i to denote positive constants.

Theorem 3.1. *Assume that the conditions $(H1) - (H6)$ are satisfied. Then problem (1.1) has a positive solution when λ is large enough.*

Proof. We shall establish Theorem 3.1 by constructing a positive subsolution (ϕ_1, ϕ_2) and supersolution (z_1, z_2) of (1.1). such that $\phi_1 \le z_1$ and $\phi_2 \le z_2$.that is, (ϕ_1, ϕ_2) and (z_1, z_2) satisfies

$$
\begin{cases} M(I_0(\phi_1)) \int\limits_{\Omega} |\nabla \phi_1|^{p(x)-2} \nabla \phi_1 . \nabla q dx \le \int\limits_{\Omega} \lambda^{p(x)} [\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)] q dx, \\ M(I_0(\phi_2)) \int\limits_{\Omega} |\nabla \Phi_2|^{p(x)-2} \nabla \phi_2 . \nabla q dx \le \int\limits_{\Omega} \lambda^{p(x)} [\lambda_2 b(x) g(\phi_1) + \mu_2 d(x) \tau(\phi_2)] q dx, \end{cases}
$$

and

$$
\begin{cases} M(I_0(z_1)) \int\limits_{\Omega} |\nabla z_1|^{p(x)-2} \nabla z_1 . \nabla q dx \ge \int\limits_{\Omega} \lambda^{p(x)} [\lambda_1 a(x) f(z_2) + \mu_1 c(x) h(z_1)] q dx, \\ M(I_0(z_2)) \int\limits_{\Omega} |\nabla z_2|^{p(x)-2} \nabla z_2 . \nabla q dx \ge \int\limits_{\Omega} \lambda^{p(x)} [\lambda_2 b(x) g(z_1) + \mu_2 d(x) \tau(z_2)] q dx, \end{cases}
$$

for all $q \in W_0^{1,p(x)}(\Omega)$ with $q \ge 0$.According to the sub-super solution method for $p(x)$-Kirchhoff type equations (see [20]), then (1.1) has a positive solution.

Step 1. We will construct a subsolution of (1.1). Let $\sigma \in (0, \delta)$ is small enough.

Denote

$$\phi_1(x)=\begin{cases} e^{kd(x)}-1,\ d(x)<\sigma \\ e^{k\sigma}-1+\int\limits_{\sigma}^{d(x)} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}(\lambda_1 a_1+\mu_1 c_1)^{\frac{2}{p^{-}-1}}\,dt, \sigma\le d(x)<2\delta, \\ e^{k\sigma}-1+\int\limits_{\sigma}^{2\delta} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}(\lambda_1 a_1+\mu_1 c_1)^{\frac{2}{p^{-}-1}}\,dt, 2\delta\le d(x). \end{cases}$$

$$\phi_2(x)=\begin{cases} e^{kd(x)}-1, & d(x)<\sigma \\ e^{k\sigma}-1+\int\limits_{\sigma}^{d(x)} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}(\lambda_2 b_1+\mu_2 d_1)^{\frac{2}{p^{-}-1}}\,dt, \sigma\le d(x)<2\delta, \\ e^{k\sigma}-1+\int\limits_{\sigma}^{2\delta} ke^{k\sigma}\left(\frac{2\delta-t}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}(\lambda_2 b_1+\mu_2 d_1)^{\frac{2}{p^{-}-1}}\,dt, 2\delta\le d(x). \end{cases}$$

It is easy to see that $\phi_1,\phi_2\in C^1\left(\overline{\Omega}\right)$,Denote

$$\begin{aligned} \alpha &= \min\left\{\frac{\inf p(x)-1}{4(\sup|\nabla p(x)|+1)},1\right\}, \\ \zeta &= \min\left\{\lambda_1 f(0)+\mu_1 h(0),\lambda_2 g(0)+\mu_2\tau(0),-1\right\}. \end{aligned}$$

By some simple computations we can obtain

$$-\triangle_{p(x)}\phi_1=\begin{cases} -k\left(e^{kd(x)}\right)^{p(x)-1}\left[(p(x)-1)+\left(d(x)+\frac{\ln k}{k}\right)\nabla p\nabla d+\frac{\triangle d}{k}\right],\ d(x)<\sigma \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^{-}-1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}\nabla p\nabla d+\triangle d\right]\right\} \\ \times\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^{-}-1}-1}(\lambda_1 a_1+\mu_1 c_1),\ \sigma\le d(x)<2\delta, \\ 0,\ 2\delta\le d(x). \end{cases}$$

$$-\triangle_{p(x)}\phi_2=\begin{cases} -k\left(e^{kd(x)}\right)^{p(x)-1}\left[(p(x)-1)+\left(d(x)+\frac{\ln k}{k}\right)\nabla p\nabla d+\frac{\triangle d}{k}\right],\ d(x)<\sigma \\ \left\{\frac{1}{2\delta-\sigma}\frac{2(p(x)-1)}{p^{-}-1}-\left(\frac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^{-}-1}}\nabla p\nabla d+\triangle d\right]\right\} \\ \times\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\frac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^{-}-1}-1}(\lambda_2 b_1+\mu_2 d_1),\sigma\le d(x)<2\delta, \\ 0,\ 2\delta\le d(x). \end{cases}$$

from $(H4)$ there exists a positive constant $L>1$ such that

$$f(L-1)\ge 1, g(L-1)\ge 1, h(L-1)\ge 1, \tau(L-1)\ge 1.$$

Let $\sigma=\frac{1}{k}\ln L$,then

$$\sigma k=\ln L \tag{3.1}$$

If k is sufficiently large, from (3.1), we have

$$-\triangle_{p(x)}\phi_1 \le -k^{p(x)}\alpha, \quad d(x) < \sigma \tag{3.2}$$

Let $\frac{\lambda\zeta}{m_\infty} = k\alpha$,then

$$-k^{p(x)}\alpha \ge -\lambda^{p(x)}\frac{\zeta}{m_\infty}.$$

From (3.2), we have

$$\begin{cases} -M\left(I_0\left(\phi_1\right)\right)\triangle_{p(x)}\phi \le M\left(I_0\left(\phi_1\right)\right)\lambda^{p(x)}\frac{\zeta}{m_\infty} \\ \le \lambda^{p(x)}\zeta \\ \le \lambda^{p(x)}\left(\lambda_1 a_1 f(0) + \mu_1 c_1 h(0)\right) \\ \le \lambda^{p(x)}\left(\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)\right), \qquad d(x) < \sigma. \end{cases}$$

Since $d(x) \in C^2\left(\overline{\partial\Omega_{3\delta}}\right)$, there exists a positive constant C_3 such that

$$\begin{aligned} -M\left(I_0\left(\phi_1\right)\right)\triangle_{p(x)}\phi_1 &\le m_\infty\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\tfrac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2(p(x)-1)}{p^- - 1}-1}\left(\lambda_1+\mu_1\right)\times \\ &\times\left|\left\{\tfrac{1}{2\delta-\sigma}\tfrac{2(p(x)-1)}{p^- - 1} - \left(\tfrac{2\delta-d}{2\delta-\sigma}\right)\left[\left(\ln ke^{k\sigma}\right)\left(\tfrac{2\delta-d}{2\delta-\sigma}\right)^{\frac{2}{p^- - 1}}\nabla p\nabla d + \triangle d\right]\right\}\right| \\ &\le C_3 m_\infty\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\lambda_1 a_1 + \mu_1 c_1\right)\ln k, \ \sigma \le d(x) < 2\delta, \end{aligned}$$

If k is sufficiently large, let $\frac{\lambda\zeta}{m_\infty} = k\alpha$, then we have

$$\begin{aligned} C_3 m_\infty\left(Ke^{k\sigma}\right)^{p(x)-1}\left(\lambda_1 a_1 + \mu_1 c_1\right)\ln k &= C_3 m_\infty (KL)^{p(x)-1}\left(\lambda_1 a_1 + \mu_1 c_1\right)\ln k \\ &\le \lambda^{p(x)}\left(\lambda_1 a_1 + \mu_1 c_1\right) \end{aligned}$$

then

$$-M\left(I_0\left(\phi_1\right)\right)\triangle_{p(x)}\phi_1 \le \lambda^{p(x)}\left(\lambda_1 a_1 + \mu_1 c_1\right), \quad \sigma \le d(x) < 2\delta, \tag{3.3}$$

Since $\phi_1(x), \phi_2(x)$ and f, h are monotone, when λ is large enough we have

$$-M\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\right)\triangle_{p(x)}\phi_1 \le \lambda^{p(x)}\left(\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)\right), \ \sigma \le d(x) < 2\delta,$$

(3.4)

$$-M\left(I_0\left(\phi_1\right)\right)\triangle_{p(x)}\phi_1 = 0 \le \lambda^{p(x)}\left(\lambda_1 a_1 + \mu_1 c_1\right) \le \lambda^{p(x)}\left(\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)\right), \ 2\delta \le d(x).$$

Combining (3.3) and (3.4), we can conclude that

$$-M\left(I_0\left(\phi_1\right)\right)\triangle_{p(x)}\phi_1 \le \lambda^{p(x)}\left(\lambda_1 a(x) f(\phi_2) + \mu_1 c(x) h(\phi_1)\right), \text{ a.e.on } \Omega \tag{3.5}$$

Similarly

$$-M\left(I_0\left(\phi_2\right)\right)\triangle_{p(x)}\phi_2 \le \lambda^{p(x)}\left(\lambda_2 b(x) g(\phi_1) + \mu_2 d(x) \tau(\phi_2)\right), \text{ a.e.on } \Omega \tag{3.6}$$

From (3.5 and (3.6), we can see that (ϕ_1, ϕ_2) is a subsolution of problem (1.1).

Step 2. We will construct a supersolution of problem (1.1).

We consider

$$\begin{cases} -M\left(I_0\left(z_1\right)\right) \triangle_{p(x)} z_1 = \frac{\lambda^{p^+}}{m_0}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu \text{ in } \Omega \\ -M\left(I_0\left(z_2\right)\right) \triangle_{p(x)} z_2 = \frac{\lambda^{p^+}}{m_0}\left(\lambda_2 b_2 + \mu_2 d_2\right) g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)\right) \text{ in } \Omega \\ z_1 = z_2 = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\beta = \beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right) = \max_{x \in \overline{\Omega}} z_1(x)$.We shall prove that (z_1, z_2) is a supersolution of problem (1.1).

For $q \in W_0^{1,p(x)}(\Omega)$ with $q \geq 0$, it is easy to see that

$$\text{(3.7)} \quad \begin{cases} M\left(I_0\left(z_2\right)\right) \int\limits_\Omega \left|\nabla z_2\right|^{p(x)-2} \nabla z_2 . \nabla q dx \\ = \frac{1}{m_0} M\left(I_0\left(z_2\right)\right) \int\limits_\Omega \lambda^{p^+}\left(\lambda_2 b_2 + \mu_2 d_2\right) g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)\right) q dx \\ \geq \int\limits_\Omega \lambda^{p^+} \lambda_2 b(x) g\left(z_1\right) q dx + \int\limits_\Omega \lambda^{p^+} \mu_2 d(x) g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 + \mu_1\right)\mu\right)\right) q dx \end{cases}$$

By $(H6)$, for μ large enough, using Lemma 2.6, we have

$$\text{(3.8)} \quad \begin{gathered} g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)\right) \geq \\ \tau\left(C_2\left[\lambda^{p^+}\left(\lambda_2 b_2 + \mu_2 d_2\right) g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)\right)\right]^{\frac{1}{p^- - 1}}\right) \geq \tau\left(z_2\right) \end{gathered}$$

Hence

(3.9)

$$M\left(I_0\left(z_2\right)\right) \int\limits_\Omega \left|\nabla z_2\right|^{p(x)-2} \nabla z_2 . \nabla q dx \geq \int\limits_\Omega \lambda^{p^+} \lambda_2 b(x) g\left(z_1\right) q dx + \int\limits_\Omega \lambda^{p^+} \mu_2 d(x) \tau\left(z_2\right) q dx$$

Also

$$\begin{aligned} M\left(I_0\left(z_1\right)\right) \int\limits_\Omega \left|\nabla z_1\right|^{p(x)-2} \nabla z_1 . \nabla q dx &= \frac{1}{m_0} M\left(I_0\left(z_1\right)\right) \int\limits_\Omega \lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu q dx \\ &\geq \int\limits_\Omega \lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu q dx \end{aligned}$$

By $(H4), (H5)$ and Lemma 2.6, when μ is sufficiently large, we have

$$\begin{aligned} \left(\lambda_1 a_2 + \mu_1 c_2\right)\mu &\geq \frac{1}{\lambda^{p^+}}\left[\frac{1}{C_2}\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)\right]^{p^- - 1} \\ &\geq \mu_1 h\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)\right) \\ &\quad + \lambda_1 f\left(C_2\left[\lambda^{p^+}\left(\lambda_2 b_2 + \mu_2 d_2\right) g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)\right)\right]^{\frac{1}{p^- - 1}}\right). \end{aligned}$$

Then
(3.10)
$$M\left(I_0\left(z_1\right)\right)\int_\Omega \left|\nabla z_1\right|^{p(x)-2}\nabla z_1.\nabla q dx \geq \int_\Omega \lambda^{p+}\lambda_1 a\left(x\right) f\left(z_2\right) q dx+\int_\Omega \lambda^{p+}\mu_1 c\left(x\right) h\left(z_1\right) q dx.$$

According to (3.9) and (3.10), we can conclude that (z_1, z_2) is a supersolution of problem (1.1).It only remains to prove that $\phi_1 \leq z_1$ and $\phi_2 \leq z_2$.

In the definition of $v_1(x)$, let

$$\gamma = \frac{2}{\delta}\left(\max_{\overline{\Omega}}\phi_1\left(x\right) + \max_{\overline{\Omega}}\left|\nabla\phi_1\right|\left(x\right)\right).$$

We claim that

$$\phi_1\left(x\right) \leq v_1\left(x\right), \forall x \in \Omega. \tag{3.11}$$

From the definition of v_1, it is easy to see that

$$\phi_1\left(x\right) \leq 2\max_{\overline{\Omega}}\phi_1\left(x\right) \leq v_1\left(x\right), \text{when } d\left(x\right) = \delta$$

and

$$\phi_1\left(x\right) \leq 2\max_{\overline{\Omega}}\phi_1\left(x\right) \leq v_1\left(x\right), \text{when } d\left(x\right) \geq \delta.$$

$$\phi_1\left(x\right) \leq v_1\left(x\right), \text{when } d\left(x\right) < \delta.$$

Since $v_1 - \phi_1 \in C^1\left(\overline{\partial\Omega_\delta}\right)$,there exists a point $x_0 \in \overline{\partial\Omega_\delta}$ such that

$$v_1\left(x_0\right) - \phi_1\left(x_0\right) = \min_{x_0\in\overline{\partial\Omega_\delta}}\left(v_1\left(x_0\right) - \phi_1\left(x_0\right)\right).$$

If $v_1(x_0) - \phi_1(x_0) < 0$,it is easy to see that $0 < d(x) < \delta$ and then

$$\nabla v_1\left(x_0\right) - \nabla\phi_1\left(x_0\right) = 0.$$

From the definition of v_1, we have

$$\left|\nabla v_1\left(x_0\right)\right| = \gamma = \frac{2}{\delta}\left(\max_{\overline{\Omega}}\phi_1\left(x_0\right) + \max_{\overline{\Omega}}\left|\nabla\phi_1\right|\left(x_0\right)\right) > \left|\nabla\phi_1\right|\left(x_0\right).$$

It is a contradiction to

$$\nabla v_1\left(x_0\right) - \nabla\phi_1\left(x_0\right) = 0.$$

Thus (3.11) is valid.

Obviously, there exists a positive constant C_3 such that

$$\gamma \leq C_3\lambda.$$

Since $d(x) \in C^2\left(\overline{\partial\Omega_{3\delta}}\right)$, according to the proof of Lemma 2.6, there exists a positive constant C_4 such that

$$M\left(I_0\left(v_1\right)\right) - \triangle_{p(x)}v_1\left(x\right) \leq C_*\gamma^{p(x)-1+\theta} \leq C_4\lambda^{p(x)-1+\theta}.\ a.e \text{ in } \Omega, \text{where } \theta \in \left(0,1\right).$$

When $\eta \geq \lambda^{p^+}$ is large enough, we have

$$-\triangle_{p(x)} v_1(x) \leq \eta.$$

According to the comparison principle, we have

$$v_1(x) \leq \omega(x), \forall x \in \Omega. \tag{3.12}$$

From (3.11) and (3.12), when $\eta \geq \lambda^{p^+}$ and $\lambda \geq 1$ is sufficiently large, we have

$$\phi_1(x) \leq v_1(x) \leq \omega(x), \forall x \in \Omega. \tag{3.13}$$

According to the comparison principle, when μ is large enough, we have

$$v_1(x) \leq \omega(x) \leq z_1(x), \forall x \in \Omega.$$

Combining the definition of $v_1(x)$ and (3.13), it is easy to see that

$$\phi_1(x) \leq v_1(x) \leq \omega(x) \leq z_1(x), \forall x \in \Omega.$$

When $\mu \geq 1$ and λ is large enough, from Lemma 2.6, we can see that $\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)$ is large enough, then

$$\frac{\lambda^{p^+}}{m_0}\left(\lambda_2 b_2 + \mu_2 d_2\right) g\left(\beta\left(\lambda^{p^+}\left(\lambda_1 a_2 + \mu_1 c_2\right)\mu\right)\right)$$

is large enough. Similarly, we have $\phi_2 \leq z_2$. This completes the proof. □

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Authors' addresses:

Bilel Mairi
Department of Mathematics and Computer Science,
Larbi Tebessi University, 12002 Tebessa, Algeria.
E-mail: bilelmr83@gmail.com

Rafik Guefaifia
Department of Mathematics and Computer Science,
Larbi Tebessi University, 12002 Tebessa, Algeria.
E-mail: rafikik982@gmail.com

Salah Boulaaras
Department Of Mathematics, College Of Sciences and Arts,
Al-Rass, Qassim University , Kingdom Of Saudi Arabia.

Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran 1, 31000, Ahmed Benbella. Algeria.
E-mail: saleh_boulaares@yahoo.fr

Tahar Bouali
Department of Mathematics and Computer Science,
Larbi Tebessi University, 12002 Tebessa, Algeria.
E-mail: botahar@gmail.com