The coset space of the unified field theory

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Abstract. The coset space of the unified field theory is postulated, based on the automorphism group of the spinor space, which has been determined to be a direct sum as a result of the unitarity of the CKM matrix. Reduction sequences are considered, particularly in connection with vector bosons of the strong interactions and the six-dimensional theory yielding the Weinberg-Salam model upon integration over S^2. The embedding of the manifold in a twelve-dimensional unified theory is established, and a solution to the field equations with a given form of the Ricci tensor is found. A coset space space $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$ satisfying the holonomy condition for the Ricci tensor derived from the spinor equations is either nonsupersymmetric or admits a maximal $N = 1$ supersymmetry. The greater weighting of a compactification of the reduced ten-dimensional theory over $G_2/SU(3)$ is verified, confirming the relevance of the elementary particle interactions.

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1 Introduction

The unification of the elementary particle interactions in higher dimensions requires a compact space which will yield the standard model upon reduction to four dimensions. The compactification of space-time over a coset manifold produces the gauge group $SU(3)\times SU(2)\times U(1)$ with known quark and lepton numbers if the internal symmetry space $L^{klm} = \frac{SU(3)\times SU(2)\times U(1)}{SU(2)\times U(1)'}$ occurs as a limit of solutions to the 11-dimensional supergravity equations, $M^{klm} = \frac{SU(3)\times SU(2)\times U(1)}{SU(2)\times U(1)'\times U(1)''}$ [10][69], when the embedding parameters for the $U(1)'$ and $U(1)''$ factors are chosen to coincide at certain values [70]. Since the internal symmetry manifold is eight-dimensional, it appears then that a twelve-dimensional theory would be necessary for a consistent description of the particle spectrum.

Supergravity theories formulated in space-times with a signature (n-1,1) and a dimension larger than eleven necessarily include spins larger than two [56] and have generated inconsistencies upon the introduction of interactions [32][38]. There exists

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a class of superalgebras in twelve-dimensional space-times with signatures (10,2) that yield spins less than or equal to two [61]. The gauging of this superalgebra would yield a twelve-dimensional supergravity theory. Since there are two time coordinates in this space-time, compactification to a four-dimensional Lorentzian space time would introduce a non-compact internal manifold. Although the (11,1) signature had not been allowed because of the upper bound on the particle spins, a rotation of the coordinate then would be necessary for the consideration of a compact space. The possibility of the compact space $\frac{G_2 \times SU(2) \times U(1)}{SU(3) \times U(1)' \times U(1)''}$ will be considered. The isometry group will be shown to be a continuously connected component of automorphism group of the spinor space of the standard model, and the compact space contains $G_2/SU(3)$, which occurs in a phenomenologically viable solution to the leading-order heterotic string effective field equations.

A generic property of a phenomenologically realistic four-dimensional effective field theory is $N = 1$ supersymmetry [25], [62], since it is consistent with the approximate stability of energy levels and the relative magnitude of the elementary particle masses and the Planck mass. The hierarchy problem can be solved similarly, since divergences resulting from radiative corrections cancel. The solution to the eleven-dimensional supergravity equations yielding the known particle multiplets has been found to occur at those values of the embedding parameters such that the coset manifold admits supersymmetries [10] [69]. It will be shown that the coset space $\frac{G_2 \times SU(2) \times U(1)}{SU(3) \times U(1)' \times U(1)''}$ generally would not have solutions to the holonomy condition on spinors as an Einstein manifold. This result does not necessarily affect the physical relevance of these types of spaces, since a ground state without supersymmetry may be compatible with the solution of theoretical problems that are resolved through the supersymmetric invariance of the action. Nevertheless, it is suggestive of the existence of a choice of topology of the space that is distinguished by the presence of supersymmetry.

Furthermore, by basing the compactification of the heterotic string theory on a coset manifold, the gauge group symmetries are evident, whereas symmetries in the Calabi-Yau compactifications must be derived from commutation with the holonomy group operator [8]. The first approach is closer to the existence of symmetry groups in higher-dimensional field theories, whereas the other method is based on an intrinsically string-theoretic description of the matter sector, and the field-theory limit can be more easily found for the compactification over the coset manifold. Because there exist Calabi-Yau manifolds with Euler number of magnitude 6, it is possible to list models with three generations [4], which is a necessary property of a standard model based on the contribution of neutrinos to the width of the Z^0 resonance [9]. Three generations of leptons and quarks also can be produced by the coset compactification based on a dimensional reduction with symmetric gauge fields [33]. From the physical equivalence of configurations related by a gauge transformation, it may be deduced that reduction of theory with gauge group G resulting from compactification of the higher-dimensional manifold over the coset space S/R yields an action on the base manifold with a residual gauge group H that is the centralizer of the image of R in G. The example of $G = E_8$ and $S/R = G_2/SU(3)$ has been considered for ten-dimensional super-Yang-Mills theory, and the resulting anomaly-free E_6-invariant four-dimensional action has soft terms which break $N = 1$ supersymmetry [52].

The symmetry breaking patterns, gauge bosons and fermion multiplets have been derived for E_6, and it has been shown that there is an extra triplet of quarks in the three-generation model, extra gauge bosons and a large number of Higgs fields, while approximate conservation of $B-L$ [17] and equality of the Weinberg angle with the experimental value after renormalization is predicted [42]. Since there is no direct evidence for additional gauge bosons or quarks, the known elementary particle multiplets can be derived from an $E_8 \times E_8$ heterotic string theory only if a solution is chosen such that complementary $E_6 \times E_8$ fields do not propagate. However, if these gauge fields are set equal to zero, the condition for anomaly cancelation in the heterotic string theory [8] then cannot be satisfied. Instead, the identification of the gauge connection with the spin connection implies that the the gravitational fields may be redefined, thereby removing the dependence on the gauge fields. This particular choice of field variables reduces one set of internal gauge symmetries to E_6, leaving the group resulting from compactification over the coset space. Nevertheless, the initial $E_8 \times E_8$ invariance, which would be necessary for cancelation of diffeomorphism anomalies, and the subsequent $E_6 \times E_8$ symmetry, is sufficient to preserve properties such $B-L$ conservation and the value of the Weinberg angle in perturbation theory. Consequently, a physically viable four-dimensional theory may be constructed from a ground state of the ten-dimensional heterotic string theory with the internal symmetry manifold described by a coset space $G_2/SU(3)$ and the compactification of the solution to the twelve-dimensional effective field equations over $\frac{G_2 \times SU(2) \times U(1)}{SU(3) \times U(1)' \times U(1)''}$.

2 The spinor space of a unified theory of the elementary particle interactions

The gauge groups of the standard model have been derived as subgroups of a Clifford group for ten-dimensional space-time with a Lorentz metric. Specifically, the Clifford algebra $R_{1,9}$, where $R_{p,q}$ is generated by the elements γ_α such that $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\eta_{\alpha\beta} I_{p+q}$, $\eta_{\alpha\beta} = diag(1, ..., 1, -1, ..., -1)$, has a complexification which is the Dirac algebra in ten-dimensional Minkowski space-time. The corresponding Pauli algebra is the left adjoint algebra of the spinor space $\mathbb{R} \otimes \mathbb{C} \otimes \mathbf{H} \otimes \mathbf{O}$ [26]. The subspace of two-vectors of the $R_{0,6}$ subalgebra, which is the adjoint algebra of the octonionic module, generates an $SU(4)$ group that is embedded in the ten-dimensional Lorentz group of the ten-dimensional theory [27]. This Lorentz group is a subgroup of a larger twelve-dimensional Lorentz group and a diffeomorphism group that represent local and general coordinate symmetries respectively, indicating the consistent use of string theory to describe the strong interactions and gravity. The standard gauge group of the strong interactions then can be derived either by breaking the $SU(4)$ symmetry or restricting the invariance group to that of an intersection with G_2. The latter method is useful for the description of quarks and shall provide a resolution of a potential problem arising from the coset space formalism for the unified field theory. By contrast, the $SU(2) \times U(1)$ groups would be regarded as internal symmetries commuting with the action of the left adjoint algebra of the complex and quaternionic modules in the spinor space [28].

It was shown that the fermion multiplets comprising leptons and quarks in the standard model in each generation could be described by the spinor space $\mathbb{R} \otimes \mathbb{C} \otimes \mathbf{H} \otimes \mathbf{O}$ [28]. The components of the spinor space would be modules for groups that are spontaneously broken to $SU(3) \times SU(2) \times U(1)$. Furthermore, the modules of any semisimple Lie group were decomposed into the components $\mathbb{C}$, $\mathbf{H}$ and $\mathbf{O}$. This led to the postulate of $\oplus_{i=1}^{k} \mathbb{C}^{n_{\mathbb{C}i}} \otimes \mathbf{H}^{n_{\mathbf{H}i}} \otimes \mathbf{O}^{n_{\mathbf{O}i}}$ as the most general spinor space of a theory of the elementary particle interactions. For the standard model, $k = 3$ and $n_{\mathbb{C}i} = n_{\mathbf{H}i} = n_{\mathbf{O}i} = 1$. It was suggested, moreover, that the modules would be chosen such that there were no interaction terms in the Lagrangian representing coupling between the components of the direct sum [21].

While the flavour quantum number is conserved in the electromagnetic and strong nuclear processes, the W-boson is charged and flavour can be changed in a weak interaction. For example, the Wud coupling is an interaction of this kind, and it is restricted to a single generation. With Cabibbo mixing, the Wus couplings also can occur with a relative probability of $tan^2\theta_c$. However, the mixing between the d, s and b quarks is represented by the Cabibbo-Kobayashi-Maskawa matrix [7][48]

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{2.1}$$

Therefore, if this matrix is unitary, a rotation in flavour space would restore a diagonal form of the interaction terms in a sum over generations. Experimental evidence based on the half-life and branching ratio of ^{14}O has led to revised estimates of the first row of matrix elements, $|V_{ud}| = 0.9738 \pm 0.0005$, $|V_{us}| = 0.2272 \pm 0.0030$, $|V_{ub}| = 0.0035 \pm 0.0015$, such that $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.9999 \pm 0.00017$, which confirms the unitarity of the CKM matrix [6]. Two elements of the second row have been determined to be $|V_{cd}| = 0.224 \pm 0.012$ and $|V_{cb}| = 0.0413 \pm 0.0015$ [11]. The predicted value of $|V_{cs}|$ by unitarity is 0.9739, which is consistent with the value derived from neutrino production of charm and semileptonic D decays, $|V_{cs}|(expt.) = 0.97 \pm 0.16$ [1].

The classification of the elementary particles through modules isomorphic to the division algebras can be related to the multiplets of the standard model because all such spinor spaces may be decomposed into a product of modules of these semisimple Lie groups [21]. Furthermore, the inclusion of both the automorphism groups of the division algebras and $SU(3) \times SU(2) \times U(1)$ in the larger exceptional groups [64] is known to be consistent with the phenomenology of the particle interactions, and an isomorphism between the representations of these groups follows from the equal dimensions of the entire spinor spaces for each generation and the projections from the Clifford algebra to the Lie algebras of standard model gauge groups [28].

3 The reduction sequences of the higher-dimensional Unified Theory

Defining the coset manifolds

$$\begin{aligned} M^{k\ell m} &= \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1)' \times U(1)''} \\ L^{k\ell m} &= \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1)'} \end{aligned} \tag{3.1}$$

it follows from a consideration of various reductions of the coset manifold that a description of the vector bosons of the strong interactions depends very much on the choice of the sequence. For example, it has been shown that a non-perturbative formulation of vector boson bound states can be based partially on the group $SU(2)\times SU(2)$ [24][51]. If the projection $S^7 \to S^3 \times S^3$ is used to derive this group, then the second, third and fourth sequences given in this section do not yield this reduction. However, a sequence containing a more general Aloff-Wallach space, such as $SU(3) \to SU(3)/\Delta_{m,n}(S^1) \to SU(2) \times SU(2)$, will be shown to be compatible with a reduction of the space L^{klm} in §5. Nevertheless, because L^{klm} arises from a limit of a set of solutions to the equations of motion of eleven-dimensional supergravity, the coset space $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$ shall be considered in connection with the heterotic or superstring models and a twelve dimensional theory, which has been proposed for a fundamental unification of string theories in higher dimensions. It will be demonstrated in §6 that a reduction sequence beginning from the coset space $\frac{G_2\times SU(2)\times SU(1)}{SU(3)\times U(1)'\times U(1)''}$ and containing the projection $S^7 \to S^3\times S^3$ exists, thereby providing an embedding of the geometrical formulation of the bosonic part of the theory of strong interactions into the coset model and, since the fermions also must transform under a representation of a subgroup of the ten-dimensional Lorentz group [44], it also yields a consistent description of the interactions of strongly interacting bosons and fermions.

The following sequences can be considered:

(3.2)
$$\begin{aligned} &(i)\quad SU(3) \to SU(3)/\Delta_{m,n}(S^1)\times S^3\times S^3 \\ &(ii)\quad L^{klm} \to SU(3)/\Delta_{m,n}(S^1) \to S^3\times S^3 \\ &(iii)\quad M^{klm} \to S^5\times S^3/S^1\times S^1 \\ &(iv)\quad L^{klm} \to S^5\times S^3/S^1 \to S^5\times S^3/S^1\times S^1 \\ &(v)\quad \frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''} \to S^7 \to S^3\times S^3 \\ &(vi)\quad \frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''} \to S^6. \end{aligned}$$

The sphere S^7, which is an S^3 bundle over S^4, could arise as

(3.3)
$$S^3\times S^3 \leftarrow S^7 = P(S^4, S^3; SU(2)) \cup P\left(\frac{G_2}{SU(3)}, SU(2); SU(2)\right).$$

Since the $SU(2)\times SU(2)$ subgroup of G_2 is not entirely contained in $SU(3)$, the first sequence does not arise in the reduction of a twelve-dimensional the compact manifold $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$ arising as the eight-dimensional component of a solution to the equations of motion to a ten-dimensional string theory.

The field theoretic projection from S^7 to $S^3\times S^3$ yields a theory that includes the gauge bosons of the strong interactions, but the coset manifold $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$ may be used to descibe the fermion multiplets. Furthermore, $G_2\times SU(2)\times U(1)\ wr\ S_3$ is the group of automorphisms, $\{\sigma|\langle\sigma(a),\sigma(b)\rangle = \langle a,b\rangle = \bar{a}b\}$, of $\mathbb{C}\otimes\mathbf{H}\otimes\mathbf{O}$, the spinor space of each generation of fermions in the standard model.

It is known that there are two remaining dimensions after the compactification of the superstring of heterotic string effective field theory. String theory has been developed to provide a theoretical basis for quantum gravity and the strong interactions, where experiments confirm the presence of string-like structure in hadrons. Therefore, an elucidation of the electromagnetic and weak interactions would be given by the reduction over the two-dimensional compact manifold.

Let E/B represent the fibre F in a sequence $F \to E \to B$, which may be a generalized fibration with a countable number of singularities. After reduction over $S^3 \times S^3$, the other two coordinates parameterize
$\frac{S^6}{S^4} \times \frac{S^7}{S^3\times S^3} \times \frac{U(1)}{U(1)'\times U(1)''} \simeq \frac{S^6}{S^4} \times \frac{S^4}{S^3} \times \frac{U(1)}{U(1)'\times U(1)''} \simeq \frac{S^6}{S^3\times S^1}$.
While $S^1 \times S^3 \subset S^5$, $S^1 \times S^{2n}$ has no great k-sphere fibrations for any n or $k > 1$. It follows that there is no bundle with $S^1 \times S^6$ as the total space and S^3, $S^3 \times S^1$ or $S^3 \times S^1 \times S^1$ as a fibre. From the homotopy groups of the spheres [65],

$$\pi_i(S^4) \approx \pi_{i-1}(S^3) + \pi_i(S^7) \tag{3.4}$$

and

$$\pi_7(S^4) \approx \pi_6(S^3) + \mathbf{Z}. \tag{3.5}$$

Since $S^7/S^4 \simeq S^3$, which is an S^1 bundle over S^2, there is a homotopy equivalence between $\frac{S^7}{S^4\times S^1}$ and $\frac{S^6}{S^3\times S^1}$ in an identity component of an ∞-1 covering of the homotopy groups. Nevertheless, the degree of the covering map prevents an identification of the fibre with S^2.

This reduction shows that it is not feasible to use the fifth sequence to determine the remaining gauge groups for a model of the electromagnetic and weak interactions. Instead it is useful only for the description of the geometrical structures and symmetry groups arising in a non-perturbative formalism for the vector bosons of the strong interactions.

Furthermore, it has been found that $G_2/SU(3)$ is a six-dimensional component of the solution of the equations of motion of heterotic string theory [35]. With this solution, $\frac{SU(2)\times U(1)}{U(1)'\times U(1)''} \simeq S^2$ would be the two-dimensional compact space yielding the bosonic sector of the Weinberg-Salam model after reduction of a six-dimensional Yang-Mills theory [54]. The model can be extended to include a fermionic sector [44][54], and the sixth sequence is compatible with the reduction to this theory.

$$(3.6)\quad I = \frac{1}{4}\int d^4x \Big\{ -(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^{\mu b}A^{\nu c})(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + \epsilon^{abc}A^{\mu b}A^{\nu c})$$
$$- (\partial^\mu B^\nu - \partial^\nu B^\mu)(\partial_\mu B_\nu - \partial_\nu B_\mu)$$
$$+ 4(\partial_\mu \phi - \frac{1}{2}i A_\mu{}^a \sigma^a \phi - \frac{1}{2} i B_\mu \phi\, tan\ \theta)^\dagger$$
$$(\partial^\mu \phi - \frac{1}{2} i A^{\mu a}\sigma^a \phi - \frac{1}{2} i B^\mu \phi tan\ \theta)$$
$$- 2\left(\frac{1}{sin^2\theta_W} - 2\phi^\dagger\phi + (\phi^\dagger\phi)^2\right)\Big\}.$$

Based on the reduction over S^2 and the embedding of the normalizer of $U(1)$ in G_2, a value of θ_W, $\frac{\pi}{6}$, which nearly equals the experimental result $sin^2\theta_W = 0.23120 \pm 0.00015$ [68], is derived. Furthermore, $\frac{M_H}{M_{W^\pm}} = \frac{1}{cos\ \theta_W} = \frac{2}{\sqrt{3}}$ and $\frac{M_{Z^0}}{M_{W^\pm}} = \frac{2}{\sqrt{3}}$. The experimental values of the masses of the intermediate vector bosons, $M_{Z^0} = 91.1876 \pm 0.0021\ GeV$ and $M_W = 80.403 \pm 0.029\ GeV$ [68], are consistent with the ratio derived from reduction of the six-dimensional theory.

The predicted mass of the Higgs particle also would be equal to this value without the inclusion of radiative corrections, instead of the conventionally higher value of $100 - 200\ GeV$. A search for the Higgs boson at energies of $90 - 100\ GeV$ has provided evidence for this range [43]. The ALEPH collaboration has found from the analysis of several events that $M_H = 91.8 \pm 2\ GeV$ [2][3]. Because the Z boson and the Higgs particles are neutral, there would not be a significant difference in the signatures in scattering experiments. It follows that observations of the Z bosons also may be an indication of the presence of the Higgs particle in these collisions.

It is known also that radiative corrections with Higgs particles are proportional to $log\frac{M_H}{M_Z}$ [47]. It follows that, when $M_H = M_Z$, there would be no radiative corrections from loops containing the Higgs bosons. There are also no three-W or WZZ interactions in the Weinberg-Salam Lagrangian, and the Z and W propagators would not receive corrections from W boson loops. Consequently, the ratio $\frac{M_Z}{M_W}$ should not be significantly altered by the perturbative expansion of scattering matrix elements, and the close agreement with the experimental value confirms the theoretical predictions for M_H and M_Z.

The rotational invariance in the extra dimensions of a six-dimensional Yang-Mills theory which can be reduced to the bosonic part of the Weinberg-Salam model would eliminate fermions expanded in generalized spherical harmonics with half-integer indices [54]. The anticommutator of the supercharges in a super-Yang-Mills theory in six dimensions contains the generator of the rotations, and the action could not be consistently truncated over S^2. If supersymmetry is broken and the six-dimensional spinor fields are not rotationally symmetric, a reduced theory with fermions exists, but it would not have an $SU(2) \times U(1)$ symmetry.

Since fermions cannot be added to the six-dimensional Yang-Mills action to attain the Weinberg-Salam model, it is necessary for the spinor to transform under the representation of the isometry groups of other manifolds in the reduction sequence. This separation of the bosonic and fermionic fields is evident in the connection between the vector bosons and the projection of the vector fields on S^7 to $SU(2) \times SU(2)$ together with the quarks transforming under the fundamental representation of $SU(3)$. The splitting between the bosons and fermions is reflected again in the choice of the bosonic fields for reduction of the six-dimensional Yang-Mills theory over S^2 and the absence of the fermions.

4 Classification of bundles with seven-dimensional fibres

The seven-dimensional manifolds $M^{klm} = \frac{SU(3)\times SU(2)\times U(1)}{SU(2)\times U(1)'\times U(1)''}$ provide a scheme for Kaluza-Klein unification in eleven dimensions. These manifolds differ from the M^{pqr} spaces initially used for the unification of the gauge groups in higher dimensions in the characterization of the topology by six rational numbers k', ℓ', m', k'', ℓ'', m'' corresponding to the embeddings of $U(1)'$ and $U(1)''$ in $SU(3) \times SU(2) \times U(1)$ orthogonal to $U(1)$ with a generalized definition of orthogonality [69], instead of three integers p, q, r used when the orthogonality is fixed by the embedding of $U(1)$ in the tangent space of M^{klm}. The diffeomorphism classes of seven-dimensional Einstein manifolds with $SU(3) \times SU(2) \times U(1)$ symmetry, including the M^{pqr} spaces, have been classified [49].

It has been found that consistency with the lepton and quark multiplets in the standard model occurs in a limit of the M^{klm} spaces which leads to the addition of an extra dimension. Although a twelve-dimensional unified theory is not compatible with the critical dimension of superstring theory, consistency could be obtained after the reduction to ten coordinates, which would follow from the projection of $SU(3)$. The embedding of group submanifolds defining theories with a specified number of vector bosons in seven-dimensional manifolds would imply a reduction of the higher-dimensional theory defined by the L^{klm} spaces after projection of $SU(3)$ to a six-dimensional group submanifold of S^7. The seven-sphere with a squashed metric is isomorphic to $SU(4)/SU(3) = SU(4)/[SU(4) \cap G_2]$, and it is therefore complementary to the subgroup of $SU(4)$ that overlaps with G_2, the automorphism group of the octonions. This automorphism group shall arise in the choice of the coset space of the unified field theory, and it is evidently the invariance group of an inner product between two elements of the octonionic module, implying its relevance for the symmetries of the fermionic terms. The existence of two different subspaces of $SU(4)$ determining the relevant symmetries of bosons and fermions in the strong interactions follows.

The occurrence of n-spheres in the reduction of the higher-dimensional theory would be related to theorems on parallelizability and the diffeomorphism between n-dimensional simply connected manifolds with nonnegative lower bounds on the sectional curvature [39][60]. The seven-sphere, which is the total space of an S^3 Hopf fibration over S^4, can be given 28 non-diffeomorphic structures. Fifteen of the exotic spheres are S^3 bundles over S^4 with Euler class $e = \pm 1$, or Milnor spheres, which are known to have non-negative curvature [41]. The existence of these bundles suggests a new type of gauge theory based on these fibrations. Since the fibre is diffeomorphic to S^3, a variant of the standard $SU(2)$ gauge theory perhaps can be constructed. More generally, there are manifolds $M_{m,n}$ that are the total spaces of fibre bundles over S^4

with S^3-fibre and structure group SO(4), labelled by the elements $\pi_3(SO(4)) \simeq \mathbf{Z} \oplus \mathbf{Z}$ [16]. These manifolds have been classified up to homotopy, homeomorphism and diffeomorphism equivalence.

The physical constraint of fibre-coordinate independence of the connection form leads to the condition of the globally integrable parallelism of the fibre. While the seven-sphere admits a parallelism, the fibre coordinate cannot be eliminated from the transformation rule of the connection form. Nevertheless, the bundle can be projected to a subbundle such as S^3, and the action of the structure group on the projected vector fields may be determined. The identification of the coefficients of the vector fields with the components of the gauge potential then can be used to obtain a theory with a number of vector bosons which is effectively different from the dimension of the fibre. It has been shown that when the linearized internal symmetry group is $SU(2) \times SU(2)$, the effective number of gauge bosons is consistent with a theoretical explanation of the slope of the pomeron trajectory [24].

Since it is not possible for S^7 to arise in a projection of $SU(3)$, the reduction from $SU(3)/\Delta_{m,n}(S^1)$ to a six-dimensional group submanifold could be relevant in the reduction from twelve to ten dimensions. The classification of the Aloff-Wallach manifolds through the Kreck-Stolz invariants then would define entirely a reduction sequence. The connection with the L^{klm} spaces and the sequence $L^{k\ell m} \to SU(3)/\Delta_{m,n}(S^1) \to S^3 \times S^3$ is considered in §5.

The projection of $S^7 \to S^3 \times S^3$ would require another coset space such as $\frac{G_2 \times SU(2) \times U(1)}{SU(3) \times U(1)' \times U(1)''}$. The choice of embedding parameters of the $U(1) \times U(1)' \times U(1)''$ subgroup in $G_2 \times SU(2) \times U(1)$ required for the reduction sequence including this projection is then outlined in §6.

5 The reduction of $SU(3)$ to a six-dimensional group submanifold

Following the reduction from the twelve-dimensional unified theory to the 10-dimensional heterotic string theory, a six-dimensional group submanifold of $SU(3)$ shall be considered. If the group is $SU(2) \times SU(2)$, the embedding would not arise from the union of two sets of $SU(2)$ triplets

$$(5.1) \quad \left\{\frac{1}{2}I_1, \frac{1}{2}I_2, \frac{1}{2}I_3\right\} = \left\{\frac{1}{2}\lambda_1, \frac{1}{2}\lambda_2, \frac{1}{2}\lambda_3\right\}$$

$$\left\{\frac{1}{2}U_1, \frac{1}{2}U_2, \frac{1}{2}U_3\right\} = \left\{\frac{1}{2}\lambda_6, \frac{1}{2}\lambda_7, -\frac{1}{4}\lambda_3 + \frac{\sqrt{3}}{4}\lambda_8\right\}$$

$$\left\{\frac{1}{2}V_1, \frac{1}{2}V_2, \frac{1}{2}V_3\right\} = \left\{\frac{1}{2}\lambda_4, \frac{1}{2}\lambda_5, \frac{1}{4}\lambda_3 + \frac{\sqrt{3}}{4}\lambda_8\right\}$$

where $\lambda_i,\ i = 1, ..., 8$ are the Gell-Mann matrices, since $\lambda_4,\ \lambda_5,\ \lambda_6$ and λ_7 have nontrivial commutation relations with $\lambda_1,\ \lambda_2$ and λ_3 even though λ_8 does commute with $\lambda_i\ i = 1, 2, 3$, but it can be found from the diffeomorphism between $SU(3)/SU(2)$ and the $U(1)$ bundle over $\mathbb{C}\mathbf{P}^2$, where the base space has a three-dimensional representation dependent on $W_1 = z_5 + iz_4$, $W_2 = x_7 + iz_6$ and $R = 1 + |W_1|^2 + |W_2|^2$, because S^3 is contained in $\mathbb{C}\mathbf{P}^2$ through the constraint $|W_1|^2 + |W_2|^2 = 1$. Therefore, since L^{klm} has isometry group $SU(3)\times SU(2)\times U(1)$, a decrease of two dimensions would be consistent with the projection $SU(3)\to SU(2)\times SU(2)$ for all values of the embedding parameters.

The intermediate manifold in the sequence $SU(3)\to SU(3)/\Delta_{m,n}(S^1)\to SU(2)\times SU(2)$ would be described by an Aloff-Wallach space, where $\Delta_{m,n}: e^{i\theta}\to diag(e^{in\theta}, e^{im\theta}, e^{-i(m+n)\theta})$ [43]. The following classification of these manifolds has been given. Two Aloff-Wallach spaces are homeomorphic if

$$(5.2)\qquad \begin{aligned} &(i)\ \ r = n^2+nm+m^2 = \tilde{n}^2+\tilde{n}\tilde{m}+\tilde{m}^2.\\ &(ii)\ \ nm(n+m)\equiv \pm\tilde{n}\tilde{m}(\tilde{n}+\tilde{m})\quad mod\ 24r.\end{aligned}$$

Diffeomorphism equivalence follows if

$$(5.3)\qquad \begin{aligned} &(i)\ \ |H^4(M;\mathbb{Z})| = |H^4(N;\mathbb{Z})|\\ &(ii)\ \ s_i(M) = s_i(N)\in\mathbb{Q}/\mathbb{Z}\end{aligned}$$

where

$$(5.4)\qquad \begin{aligned} s_1(M) &= -\frac{1}{2^5\cdot 7}sign(W)+\frac{1}{2^7\cdot 7}p_1^2\\ s_2(M) &= -\frac{1}{2^4\cdot 3}z^2p_1+\frac{1}{2^3\cdot 3}z^4\\ s_3(M) &= -\frac{1}{2^2\cdot 3}z^2p_1+\frac{2}{3}z^4\end{aligned}$$

for spin manifolds M with coboundary W, such that p_1 is the Pontryagin class and z is a generator of $H^2(W)$ [49].

Homotopy equivalence [50] is valid if

$$(5.5)\qquad \begin{aligned} &(i)\ \ r^2 = n^2+nm+m^2 = \tilde{n}^2+\tilde{n}\tilde{m}+\tilde{m}^2\\ &(ii)\ \ nm(n+m)\equiv \tilde{n}\tilde{m}(\tilde{n}+\tilde{m})\qquad mod\ r\end{aligned}$$

Aloff-Wallach spaces have weak G_2 holonomy [36] and one Killing spinor. Since $S^3\times S^3$ has six Killing spinors, there exists a submanifold of $SU(3)/\Delta_{m,n}(S^1)$ as many as five additional supersymmetries. For a theory with $N=2$ supersymmetry in higher dimensions, one supersymmetry invariance is restored upon the restriction to $S^3\times S^3$.

To determine if the L^{klm} spaces are compatible with the sequence $SU(3) \to SU(3)/\Delta_{m,n}(S^1) \to SU(2) \times SU(2)$, a condition for the topological equivalence of M^{klm} and $M^{\tilde{k}\tilde{l}\tilde{m}}$

$$\left(\frac{k'm'' - k''m'}{\ell'm'' - \ell''m'}\right)^2 = \left(\frac{\tilde{k}'\tilde{m}'' - \tilde{k}''\tilde{m}'}{\tilde{\ell}'\tilde{m}'' - \tilde{\ell}''\tilde{m}'}\right)^2 \tag{5.6}$$

is trivially satisfied for L^{klm} spaces, with $k' = k''$, $\ell' = \ell''$ and $m' = m''$, it is necessary to refine the set of integers k, ℓ, m to belong to the same homeomorphism class.

For the $\frac{SU(3)\times SU(2)}{SU(2)\times U(1)}$ spaces, characterized by two integers defining the first Chern class of the S^1 bundle over $\mathbb{C}\mathbf{P}^2 \times \mathbb{C}\mathbf{P}^1$, two different spaces are homeomorphically equivalent if relations between $k,\ \ell$ and $\tilde{k},\ \tilde{\ell}$ are satisfied:

$$\bar{s}_i(M_{k,\ell}) = \bar{s}_i(M_{\tilde{k},\ell}) \tag{5.7}$$

and

$$\begin{aligned} \frac{1}{2^5\cdot 3\cdot \ell^2} t_i(\ell,k,m) &\equiv \frac{1}{2^5\cdot 3\cdot \ell^2} t_i(\ell,\tilde{k},m) \qquad\qquad mod\ \mathbb{Z} \\ t_i(\ell,k,m) &= a_i(\ell^2+3)(\ell^2-1)k + b_i(-3mk(\ell^2+1)+2(\ell^2+3))\cdot m \\ &\quad + c_i(5mk-4)m^3 \end{aligned} \tag{5.8}$$

where the integers m and n are defined here to satisfy $mk + n\ell = 1$, and $z = mx - ny$, with x and y being the generators of $H^2(\mathbb{C}\mathbf{P}^2)$ and $H^2(\mathbb{C}\mathbf{P}^1)$ [54]. The coboundary can be chosen such that $sign(W) = 0$, and

$$\begin{aligned} s_1(M_{k,\ell}) &= \frac{1}{2^7\cdot 7}\frac{3k}{\ell^2}(\ell^2+3)(\ell^2-1) \\ \bar{s}_1(M_{k,\ell}) &= 28 s_1(M_{k,\ell}) = \frac{3^2}{2^5\cdot 3\cdot \ell^2}(\ell^2+3)(\ell^2-1)k \\ \bar{s}_2(M_{k,\ell}) &= s_2(M_{k,\ell}) = -\frac{1}{2^4\cdot 3}z^2 p_1 + \frac{1}{2^3\cdot 3}z^4 \\ &= -\frac{1}{2^4\cdot 3}\frac{m}{\ell^2}(3mk(\ell^2+1) - 2(\ell^2+3)) + \frac{1}{2^3\cdot 3}\frac{m^3}{\ell^2}(5mk-4) \\ &\quad + \frac{1}{2^5\cdot 3}\left[2\frac{m}{\ell^2}(-3mk(\ell^2+1)+2(\ell^2+3)) + 2^2\frac{m^3}{\ell^2}(5mk-4)\right] \\ \bar{s}_3(M_{k,\ell}) &= s_3(M_{k,\ell}) = -\frac{1}{2^2\cdot 3}z^2 p_1 + \frac{2}{3}z^4 \\ &= \frac{1}{2^5\cdot 3\cdot \ell^2}\left[2^3(-3mk(\ell^2+1)+2(\ell^2+3))m + 2^6(5mk-4)m^3\right]. \end{aligned} \tag{5.9}$$

The coefficients for even k for $\{a_i\} = \{3^2, 0, 0\}$, $\{b_i = 0, 2, 2^2\}$ and $\{c_i = 0, 2^3, 2^6\}$.

There are two separate cases, when $3|\ell$ and $3 \nmid \ell$. It can be shown that the conditions with $i = 1$ imply that diffeomorphism equivalence between $M_{k,\ell}$ and $M_{\tilde{k},\ell}$ requires the congruence relations $k \equiv \tilde{k}\ (mod\ \ell^2)$ if $3 \nmid \ell$ and $k \equiv \tilde{k}\ \left(mod\ \frac{\ell^2}{3}\right)$ if $3|\ell$ [49].

First suppose that $3 \nmid \ell$. Then the congruence relation between $t_i(\ell, k, m)$ and $t_i(\ell, \tilde{k}, m)$ is

(5.10)
$$\frac{1}{2^5 \cdot 3 \cdot \ell^2}\Big[a_i(\ell^2+3)(\ell^2-1)k + b_i(-3mk(\ell^2+1) + 2(\ell^2+3))m + c_i(5mk-4)m^3\Big]$$
$$\equiv \frac{1}{2^5 \cdot 3 \cdot \ell^2}[a_i(\ell^2+3)(\ell^2-1)(k+\rho\ell^2) + b_i(-3mk(k+\rho\ell^2)(\ell^2+1) + 2(\ell^2+3))m + c_i(5m(k+\rho\ell^2)-4)m^3] \quad (mod\ \mathbb{Z})$$

and

(5.11)
$$\rho[a_i(\ell^2+3)(\ell^2-1) - 3b_i(\ell^2+1)m^2 + 5c_i m^4] \equiv 0 \quad (mod\ 2^5 \cdot 3).$$

If k and $\tilde{k}$ are even, ℓ is odd and ρ is even and $\tilde{k} \equiv k\ (mod\ 2\ell^2)$. Precisely, when $2^{\mu_1} \parallel (\ell^2+3)(\ell^2-1)$,

(5.12)
$$\rho \equiv 0\ (mod\ 2^{5-\mu_1}).$$

When $i = 2$,

(5.13)
$$\rho[-3b_2(\ell^2+1)m^2 + 5c_2 m^4] \equiv 0 \quad (mod\ 2^5 \cdot 3)$$
$$\rho[-3 \cdot 2 \cdot (\ell^2+1)m^2 + 5 \cdot 2^2 m^4] \equiv 0 \quad (mod\ 2^5 \cdot 3).$$

If $3|m$, let $m = 3m'$. Then

(5.14)
$$\rho[-2 \cdot (\ell^2+1) \cdot 3^2 m'^2 + 5 \cdot 2^2 \cdot 3^4 \cdot m'^4] \equiv 0 \quad (mod\ 2^5).$$

Dividing by $2 \cdot 3^2$ gives

$$\rho m'^2[m'^2 \cdot 5 \cdot 2 \cdot 3^2 - (\ell^2+1)] \equiv 0 \quad (mod\ 2^4). \tag{5.15}$$

If $2 \not| m'$ and $2 \parallel [5 \cdot 2 \cdot 3^2 m'^2 - (\ell^2+1)]$, $\rho \equiv 0\ (mod\ 2^3)$. When $2 \parallel m'$, $2 \parallel (\ell^2+1)$, then $\rho \equiv 0\ (mod\ 2)$. If $2|m$, $2^2|(\ell^2+1)$, there is no further constraint on ρ.

When $3 \not| m$, $3|\rho$. If $2 \parallel m$, $3 \not| m$, $2 \parallel (\ell^2+1)$, $2^4|[5 \cdot 2^2 \cdot m^4 - 3 \cdot 2 \cdot (\ell^2+1)m^2]$ and $\rho \equiv 0\ (mod\ 6)$. If $2 \parallel m$, $3 \not| m$, $2^2|(\ell^2+1)$, or $2^2|m$, $3 \not| m$, $2^5|[5 \cdot 2^2 \cdot m^4 - 3 \cdot 2 \cdot (\ell^2+1)m^2]$ and $\rho \equiv 0\ (mod\ 3)$. If $2 \not| m$ and $3 \not| m$, $\rho \equiv 0\ (mod\ 2^3 \cdot 3)$.

When $i = 3$, the condition (5.11) is

$$\begin{aligned} \rho[-3b_3(\ell^2+1)m^2 + 5c_3 m^4] &\equiv 0 && (mod\ 2^5 \cdot 3) \\ \rho[-3 \cdot 2^3(\ell^2+1)m^2 + 5 \cdot 2^6 \cdot m^4] &\equiv 0 && (mod\ 2^5 \cdot 3) \\ \rho[-3 \cdot (\ell^2+1)m^2 + 5 \cdot 2^3 \cdot m^4] &\equiv 0 && (mod\ 2^2 \cdot 3). \end{aligned} \tag{5.16}$$

If $2 \not| m$, $2|[5 \cdot 2^3 \cdot m^4 - 3(\ell^2+1)]$ and $\rho \equiv 0\ (mod\ 6)$. If $2|m$, $\rho \equiv 0\ (mod\ 3)$.

If ℓ is even and $3 \not| \ell$, $(\ell^2+3)(\ell^2-1)$ is odd, and the equation for $i = 1$ is

$$\begin{aligned} \rho[3^2(\ell^2+3)(\ell^2-1)] &\equiv 0 && (mod\ 2^5 \cdot 3) \\ \rho \cdot 3(\ell^2+3)(\ell^2-1) &\equiv 0 && (mod\ 2^5) \\ \rho &\equiv 0 && (mod\ 2^5). \end{aligned} \tag{5.17}$$

The equation for $i = 2$ is

$$\begin{aligned} \rho[-3b_2(\ell^2+1)m^2 + 5c_2 m^4] &\equiv 0 && (mod\ 2^5 \cdot 3) \\ \rho[-3 \cdot (\ell^2+1)m^2 + 5 \cdot 2 \cdot m^4] &\equiv 0 && (mod\ 2^4 \cdot 3). \end{aligned} \tag{5.18}$$

If $3 \nmid m$, $\rho \equiv 0\ (mod\ 2^4 \cdot 3)$, and when $2 \parallel m$, $\rho \equiv 0\ (mod\ 2^2 \cdot 3)$, whereas, if $2^2|m$, $\rho \equiv 0\ (mod\ 3)$. When $3|m$, $\rho \equiv 0\ (mod\ 2^4)$, and if $2 \parallel m$, $\rho \equiv 0\ (mod\ 2^2)$, while there is no constraint on ρ following from the divisibility condition $2^2|m$. The equation for $i = 3$ is

$$\begin{aligned} \rho\,[-3 \cdot 2^3 \cdot (\ell^2+1)m^2 + 5 \cdot 2^6 \cdot m^4] &\equiv 0 && (mod\ 2^5 \cdot 3) \\ \rho\,[-3 \cdot (\ell^2+1)m^2 + 5 \cdot 2^3 \cdot m^4] &\equiv 0 && (mod\ 2^2 \cdot 3). \end{aligned} \tag{5.19}$$

If $3 \nmid m$, $\rho \equiv 0\ (mod\ 2^2 \cdot 3)$, and, when $2|m$, $\rho \equiv 0\ (mod\ 3)$. When $3|m$, $\rho \equiv 0\ (mod\ 2^2)$, and, if $2|m$, ρ is not constrained.

Since $mk + n\ell = 1$, the congruence condition $mk \equiv 1$ $(mod\, 3)$ follows from the divisibility condition $3|\ell$. Therefore, $3 \nmid m$. Consequently, when $3|\ell$, $\tilde{k} = k + \rho' \left(\frac{\ell}{3}\right)^2$ and

$$\rho'[a_i(\ell^2+3)(\ell^2-1) - 3b_1(\ell^2 = 1)m^2 + 5c_i m^4] \equiv 0 \quad (mod\, 2^5 \cdot 3^3). \tag{5.20}$$

If ℓ is odd and $i = 1$,

$$\begin{aligned} \rho'[3^2(\ell^2+3)(\ell^2-1)] &\equiv 0 \qquad (mod\, 2^5 \cdot 3^3) \\ \rho'(\ell^2+3)(\ell^2-1) &\equiv 0 \qquad (mod\, 2^5 \cdot 3). \end{aligned} \tag{5.21}$$

When $2^{\mu_1}|(\ell^2+3)(\ell^2-1)$,

$$\rho' \equiv 0 \quad (mod\, 2^{5-\mu_1} \cdot 3). \tag{5.22}$$

If $i = 2$,

$$\rho'[-3 \cdot 2 \cdot (\ell^2+1)m^2 + 5 \cdot 2^2 m^4] \equiv 0 \quad (mod\, 2^5 \cdot 3^3). \tag{5.23}$$

When $2 \parallel m$, $2 \parallel (\ell^2+1)$, $2^4|[5 \cdot 2^2 \cdot m^4 - 3 \cdot 2 \cdot (\ell^2+1)m^2]$ and $\rho' \equiv 0$ $(mod\, 2 \cdot 3^3)$. If $2 \parallel m$, $2^2|(\ell^2+1)$, or $2^2|m$, $\rho' \equiv 0$ $(mod\, 3^3)$, and when $2 \nmid m$, $\rho' \equiv 0$ $(mod\, 2^3 \cdot 3^3)$.

For $i = 3$,

$$\begin{aligned} \rho'[-3 \cdot 2^3 \cdot (\ell^2+1)m^2 + 5 \cdot 2^6 \cdot m^4] &\equiv 0 \qquad (mod\, 2^5 \cdot 3^3) \\ \rho'[-3 \cdot (\ell^2+1)m^2 + 5 \cdot 2^3 \cdot m^4] &\equiv 0 \qquad (mod\, 2^2 \cdot 3^3). \end{aligned} \tag{5.24}$$

If $2|m$, $\rho' \equiv 0$ $(mod\, 3^3)$, whereas, if $2 \not| m$, $\rho' \equiv 0$ $(mod\, 2 \cdot 3^3)$.

If ℓ is even and $3|\ell$, the first congruence relation implies that

$$\rho' \equiv 0 \quad (mod\, 2^5 \cdot 3). \tag{5.25}$$

The equation for $i = 2$ is

$$\rho'[-3\cdot 2\cdot(\ell^2+1)m^2+5\cdot 2^2 m^4]\equiv 0 \qquad (mod\ 2^5\cdot 3^3). \tag{5.26}$$

Since $3\nmid\ m$, $\rho'\equiv 0\ \ (mod\ 2^4\cdot 3^3)$. If $2\parallel m$ also, $\rho'\equiv 0 (mod\ 2^2\cdot 3)$, while $2^2|m$ would imply that $\rho'\equiv 0\ (mod\ 3)$. The equation for $i=3$ is

$$\rho'[-3\cdot(\ell^2+1)m^2+5\cdot 2^3 m^4]\equiv 0 \qquad (mod\ 2^2\cdot 3^3). \tag{5.27}$$

Since $3\not| m$, $\rho'\equiv 0\ \ (mod\ 2^2\cdot 3^3)$. If $2|m$, $\rho'\equiv 0\ (mod\ 3^3)$.

Using the minimal conditions for $i=1,2,3$, the following congruence relations are found for spin manifolds with k and $\tilde{k}$ even.

$\underline{\ell\ is\ odd,\ 3\not|\ell}$

$$\begin{aligned}\rho\equiv 0 \qquad & (mod\ 2^{5-\mu_1})\ \ if\ 2^{\mu_1}\parallel(\ell^2+3)(\ell^2-1)\\ & (mod\ 2) \qquad if\ 2\not| m,\ 3|m\\ & (mod\ 6) \qquad if\ 2\not| m,\ 3\not| m\\ & (mod\ 3) \qquad if\ 2|m,\ 3\not| m\end{aligned}$$

$$\begin{aligned}\rho &\equiv 0 \qquad (mod\ 6)\\ \tilde{k} &\equiv k \qquad (mod\ 6\ell^2)\end{aligned}$$

$\underline{\ell\ is\ odd,\ 3|\ell}$

$$\begin{aligned}\rho'\equiv 0 \qquad & (mod\ 2^{5-\mu_1}\cdot 3)\quad if\ 2^{\mu_1}\parallel(\ell^2+3)(\ell^2-1)\\ & (mod\ 3^3) \qquad if\ 2|m\\ & (mod\ 2\cdot 3^3) \qquad if\ 2\not| m\end{aligned}$$

$$\begin{aligned}\rho' &\equiv 0 \quad (mod\ 6)\\ \tilde{k} &\equiv k \quad \left(mod\ \frac{2\ell^2}{3}\right)\end{aligned}$$

$\underline{\ell \ is \ even, \ 3 \not| \ \ell}$

$$\begin{array}{lll} \rho \equiv 0 & (mod\ 2^5) & \\ & (mod\ 2^2 \cdot 3) & if\ 2 \not| \ m,\ 3 \not| \ m \\ & (mod\ 3) & if\ 2|m,\ 3 \not| \ m \\ & (mod\ 2^2) & if\ 2 \not| \ m,\ 3 \not| \ m \end{array}$$

$$\begin{array}{ll} \rho \equiv 0 & (mod\ 2^2 \cdot 3) \\ \tilde{k} \equiv k & (mod\ 12\ell^2) \end{array}$$

$\underline{\ell \ is \ even, \ 3|\ell}$

$$\begin{array}{lll} \rho' \equiv 0 & (mod\ 2^5 \cdot 3) & \\ & (mod\ 2^2 \cdot 3^3) & if\ 2 \not| \ m \\ & (mod\ 3^3) & if\ 2|m \end{array}$$

$$\begin{array}{ll} \rho' \equiv 0 & (mod\ 2^2 \cdot 3) \\ \tilde{k} \equiv k & \left(mod\ \dfrac{4\ell^2}{3}\right) \end{array}$$

For the M^{klm} spaces, the $U(1)$ factor can be replaced by $U(1)'$ or $U(1)''$ in describing the quotient space, after an identification of the remaining parameters, and the homeomorphism equivalence conditions become

$$
\begin{aligned}
&\tilde{k}' \equiv k' \quad mod\ a_{\ell'}\ell'^2 \\
&\qquad where\ a_{\ell'} = \begin{cases} \frac{2}{3} \quad if\ \ell'\ is\ odd,\ 3 \not|\ \ell';\ 6 \quad if\ \ell'\ is\ odd,\ 3\ |\ell' \\ 12 \quad if\ \ell'\ is\ even,\ 3 \not|\ \ell';\ \frac{4}{3} \quad if\ \ell'\ is\ even,\ 3|\ell' \end{cases} \\
&\tilde{\ell}' = \pm\ell' \\
&\tilde{k}'' = k'' \\
&\tilde{\ell}'' = \ell'' \\
&or \\
&\tilde{k}' = k' \\
&\tilde{\ell}' = \ell' \\
&\tilde{k}'' \equiv k'' \quad mod\ a_{\ell''}\ell''^2 \\
&\qquad where\ a_{\ell''} = \begin{cases} \frac{2}{3} \quad if\ \ell''\ is\ odd,\ 3 \not|\ \ell'';\ 6 \quad if\ \ell''\ is\ odd,\ 3|\ell'' \\ 12 \quad if\ \ell''\ is\ even,\ 3 \not|\ \ell'';\ \frac{4}{3} \quad if\ \ell''\ is\ even,\ 3|\ell'' \end{cases} \\
&\tilde{\ell}'' = \pm\ell''
\end{aligned}
\tag{5.28}
$$

In the first instance $\tilde{\ell}'^2 + \tilde{\ell}'\tilde{k}' + \tilde{k}'^2 = \ell'^2 \pm \ell'\tilde{k}' + \tilde{k}''$. If the positive sign is chosen, $\tilde{k}' = k'$. With the negative sign and $\tilde{k}' = k' + \nu a_{\ell'}\ell'^2$,

$$
\tilde{k}'^2 = (k' + \nu a_{\ell'}\ell'^2)^2 = k'^2 + 2\nu a_{\ell'} l'^2 k' + \nu^2 a_{\ell'}^2 \ell'^4 \tag{5.29}
$$

and

$$
\ell'^2 - \ell'\tilde{k}' + \tilde{k}'^2 = \ell'^2 + (2\nu a_{\ell'}\ell' - 1)k'\ell' + k'^2 + (\nu^2 a_{\ell'}^2 \ell'^4 - \nu a_{\ell'}\ell'^3). \tag{5.30}
$$

Equality with $\ell'^2 + k'\ell' + k'^2$ implies that

$$
\begin{gathered}
2\nu a_{\ell'}\ell' - 1 = 1 \\
\nu^2 a_{\ell'}^2 \ell'^4 - \nu a_{\ell'}\ell'^3 = 0
\end{gathered}
\tag{5.31}
$$

or $\nu a_{\ell'}\ell' = 1$. However, ν must be integer, which is not possible since $a_{\ell'}\ell' > 1$ in each of the four cases. A similar result holds for k'', ℓ'', $\tilde{k}''$, and $\tilde{\ell}''$. Also

$$
\tilde{\ell}'\tilde{k}'(\tilde{\ell}' + \tilde{k}') = \pm\ell'k'(\ell' + k') + [\ell'^2 \pm 2\ell'k' \pm \nu a_{\ell'}\ell'^3]\nu a_{\ell'}\ell'^2 \tag{5.32}
$$

which is congruent to $\ell'k'(\ell' + k')$ only if $\tilde{\ell}' = \ell'$ and $\nu = 0$.

It follows that k', ℓ' or k'', ℓ'' may be identified with $\{\bar{m}, \bar{n}\}$, the indices parameterizing the Aloff-Wallach spaces. There is a projection of L_{klm} to $L_{klm}/U(1)'''$ to a codimension-two group submanifold which can be mapped, without changing the relative topological type, to the sequence $SU(3) \to SU(3)/\Delta_{\bar{m},\bar{n}}(S^1) \to SU(2) \times SU(2)$, because the homeomorphism conditions for the former are included in those for the latter.

6 The field theoretic projection of S^7 to $S^3 \times S^3$ and the coset space $\frac{G_2 \times SU(2) \times U(1)}{SU(3) \times U(1)' \times U(1)''}$

While the projection of S^7 to $S^3 \times S^3$ cannot be derived from the L^{klm} spaces, it may occur in the reduction of a bundle with total space diffeomorphic to $\frac{G_2 \times SU(2) \times U(1)}{SU(3) \times U(1)' \times U(1)''}$ and twisting of the $SU(2)$ action. There exists an S^3 bundle over $\frac{G_2}{SU(3)} = S^6$ such that its restriction to S^4 is S^7.

For the embedding of $U(1)'$ and $U(1)''$ in $G_2 \times SU(2) \times U(1)$, let $\bar{\lambda}_8$, $\bar{\sigma}_3$ and $\bar{Y}_3$ be the images of λ_8, σ_3 and Y_3 in the set of sixteen-dimensional matrices representing G_2, $SU(2)$ and $U(1)$ and

$$Z' = k'\left(\frac{i}{2}\sqrt{3}\bar{\lambda}_8\right) + \ell'\left(\frac{i}{2}\bar{\sigma}_3\right) + m'(i\bar{Y}_3) \tag{6.1}$$

$$Z'' = k''\left(\frac{i}{2}\sqrt{3}\bar{\lambda}_8\right) + \ell''\left(\frac{i}{2}\bar{\sigma}_3\right) + m''(i\bar{Y}_3). \tag{6.2}$$

The generators have been chosen to commute with the $su(2)$ subalgebra of the Lie algebra of G_2. It is not necessary for the first generator to commute with the Gell-Mann matrices as σ_3 has nontrivial commutation relations with the other Pauli matrices, For the third $U(1)$ factor, there are two possible embeddings. Either it is selected such that there exists a projection to the space M^{klm}

$$Z^{(1)} = k\left(\frac{i}{2}\sqrt{3}\bar{\lambda}_8\right) + \ell\left(\frac{i}{2}\bar{\sigma}_3\right) + m(i\bar{Y}_3) \tag{6.3}$$

through the contractions $G_2 \to SU(3)$ and $SU(3) \to SU(2)$ or it belongs to the tangent space of the coset space

$$Z^{(2)} = k\left(\frac{i}{2}\sqrt{3}\bar{\rho}_{14}\right) + \ell\left(\frac{i}{2}\bar{\sigma}_3\right) + m(i\bar{Y}_3) \tag{6.4}$$

and the manifold then may be denoted as N^{klm} and represented as the product $\frac{G_2}{SU(3)\times U(1)''} \times \frac{SU(2)}{U(1)'} \times U(1)$. Since the $U(1)$ factor belongs to the tangent space of $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$, the standard embedding can be derived by setting

$$\begin{aligned} k &= 0 \qquad \ell = 0 \qquad m = 1 \\ k' &= 0 \qquad \ell' = 1 \qquad m' = 0 \\ k'' &= 1 \qquad \ell'' = 0 \qquad m'' = 0. \end{aligned} \tag{6.5}$$

It is the space N^{001}, $\frac{S^6}{U(1)''} \times \frac{S^3}{U(1)'} \times U(1)$, which can be used to construct S^7 through the restriction to S^4 after a twisting of an S^3 fibre over S^6. The reduction of N^{001} by one dimension is consistent with the field theoretic projection of S^7 to $S^3 \times S^3$, $S^3 \subset S^6$.

7 The particle multiplets of the Coset Space Theory

To extend the analysis of the M^{klm} spaces to $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$, it is necessary to determine the representation of $G_2 \times SU(2) \times U(1)$ which overlaps with the tangent space group $SO(8)$ in a minimum of one representation of the subgroup $SU(3) \times U(1)'\times U(1)''$. For example, the generators of the stability group $SU(3)\times U(1)'\times U(1)''$ are determined by the commutation relations [69]

$$[Q_H, Q_\alpha] = C^\beta{}_{H\alpha} Q_\beta \qquad\qquad C^\beta{}_{H\alpha} = -\frac{1}{2} Q^\beta{}_{H\alpha}. \tag{7.1}$$

The image of the Gell-Mann matrices will have nontrivial commutators with the complementary set of generators of G_2, and, if $\bar{m}$ is the index for this group

$$[Q_{\bar{m}}, Q_A] = C_{\bar{m}AB} Q_B. \tag{7.2}$$

where A, B are indices representing six matrices. The representation of these generators is

$$Q^{\alpha\beta}_{\bar{m}} = \begin{pmatrix} f'_{\bar{m}AB} & 0 \\ 0 & 0 \end{pmatrix} \tag{7.3}$$

with $(f'_{\bar{m}AB})$ being a 6×6 matrix. Given that Q_3 corresponds to $\bar{\sigma}_3$, the commutation relations are

$$[Q_{\bar{8}}, Q_A] = f'_{8AB} Q_B \qquad\qquad [Q_3, Q_m] = C_{mn} Q_n.$$

Based on the embedding of the $U(1)$ charges Z' and Z''

(7.4)
$$Q^{\alpha\beta}_{Z'} = \begin{pmatrix} \sqrt{3}k' f'_{8AB} & 0 & 0 \\ 0 & \ell'\epsilon_{mn} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$Q^{\alpha\beta}_{Z''} = \begin{pmatrix} \sqrt{3}k'' f''_{8AB} & 0 & 0 \\ 0 & \ell''\epsilon_{mn} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These matrices can be embedded in the SO(8) Clifford algebra through the relation

(7.5)
$$Q_{\bar{\gamma}} = -\frac{1}{4} Q^{\alpha\beta}_{H_0} \tau_{\alpha\beta}$$

where $\{\tau_{\alpha\beta} \ \ \alpha, \beta = 1, ..., 8\}$ is the set of generators of that algebra. From the structure constants [64], $c_{\bar{1}45} = c_{\bar{1}36} = c_{\bar{2}64} = c_{\bar{2}35} = c_{\bar{3}43} = c_{\bar{4}15} = c_{\bar{4}62} =$
$c_{\bar{5}61} = c_{\bar{5}52} = c_{\bar{6}31} = c_{\bar{6}42} = c_{\bar{7}41} = c_{\bar{7}23} = \frac{1}{2}$, $c_{\bar{8}12} = 2c_{\bar{8}43} = 2c_{\bar{8}56} = -\frac{1}{\sqrt{3}}$,

$$Q_{\bar{1}} = -\frac{1}{4} \cdot 2(Q^{45}_{\bar{1}}\gamma_{45} + Q^{36}_{\bar{1}}\gamma_{36}) = -\frac{1}{2}(c_{\bar{1}45}\gamma_{45} + c_{\bar{1}36}\gamma_{36}) = -\frac{1}{4}(\gamma_{45} + \gamma_{36})$$
$$Q_{\bar{2}} = -\frac{1}{4}(\gamma_{64} + \gamma_{35})$$
$$Q_{\bar{3}} = -\frac{1}{4}(\gamma_{43} + \gamma_{65})$$
$$Q_{\bar{4}} = -\frac{1}{4}(\gamma_{15} + \gamma_{62})$$
$$Q_{\bar{5}} = -\frac{1}{4}(\gamma_{61} + \gamma_{52})$$
$$Q_{\bar{6}} = -\frac{1}{4}(\gamma_{31} + \gamma_{41})$$
$$Q_{\bar{7}} = -\frac{1}{4}(\gamma_{41} + \gamma_{23})$$
$$Q_{\bar{8}} = \frac{1}{2\sqrt{3}}\left(\gamma_{12} + \frac{1}{2}\gamma_{43} + \frac{1}{2}\gamma_{56}\right)$$

where $\gamma_1, ..., \gamma_8$ are the gamma matrices in eight dimensions. Similarly

(7.6)
$$Q_{Z'} = \frac{k'}{2}\left(\gamma_{12} + \frac{1}{2}\gamma_{43} + \frac{1}{2}\gamma_{56}\right) + \ell'\epsilon_{mn}\gamma_{mn}$$
$$Q_{Z''} = \frac{k''}{2}\left(\gamma_{12} + \frac{1}{2}\gamma_{43} + \frac{1}{2}\gamma_{56}\right) + \ell''\epsilon_{mn}\gamma_{mn}.$$

Let $\tau_m = -i\mathbb{I}_{(4)} \otimes \sigma_m, m = 1, 2,\ \tau_3 = U_8 \otimes \sigma_3,\ \tau_A = U_A \otimes \sigma_3,\ A = 4, 5, 6, 7$ with

$$
\begin{aligned}
U_4 &= -i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \\
U_5 &= -i \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \\
U_6 &= i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \\
U_7 &= \begin{pmatrix} 0 & -\mathbb{I}_{(2)} \\ \mathbb{I}_{(2)} & 0 \end{pmatrix} \\
U_8 &= \begin{pmatrix} -i\mathbb{I}_{(2)} & 0 \\ 0 & i\mathbb{I}_{(2)} \end{pmatrix}.
\end{aligned}
\tag{7.7}
$$

Then $\gamma_m = -iI_{(8)} \otimes \sigma_m,\ m = 1, 2, \gamma_3 = \tau_1 \otimes \sigma_3,\ \gamma_A = \tau_A \otimes \sigma_3,\ A = 4, 5, 6, 7,$ $\gamma_8 = \tau_2 \otimes \sigma_3$ and

$$
Q_{\bar{1}} = -\frac{1}{4} \begin{pmatrix}
-i\sigma_3 & & -iU_6 & & & & & \\
& -i\sigma_3 & & & & & & \\
iU_6 & & -i\sigma_3 & & & & & \\
& & & -i\sigma_3 & & & & \\
& & & & -i\sigma_3 & & iU_6 & \\
& & & & & -i\sigma_3 & & \\
& & & & -iU_6 & & -i\sigma_3 & \\
& & & & & & & -i\sigma_3
\end{pmatrix}
$$

$$
Q_{\bar{2}} = -\frac{1}{4} \begin{pmatrix}
i\sigma_2 & & -iU_5 & & & & & \\
& i\sigma_2 & & & & & & \\
iU_5 & & i\sigma_2 & & & & & \\
& & & i\sigma_2 & & & & \\
& & & & i\sigma_2 & & iU_5 & \\
& & & & & i\sigma_2 & & \\
& & & & -iU_5 & & i\sigma_2 & \\
& & & & & & & i\sigma_2
\end{pmatrix}
$$

$$
Q_{\bar{3}} = -\frac{1}{4} \begin{pmatrix}
-i\sigma_1 & & iU_4 & & & & & \\
& -i\sigma_1 & & & & & & \\
-iU_4 & & -i\sigma_1 & & & & & \\
& & & -i\sigma_1 & & & & \\
& & & & -i\sigma_1 & & -iU_4 & \\
& & & & & -i\sigma_1 & & \\
& & & & iU_4 & & -i\sigma_1 & \\
& & & & & & & -i\sigma_1
\end{pmatrix}
$$

$$
\text{(7.8)} \qquad Q_{\bar{4}} = -\frac{1}{4}\begin{pmatrix} & & U_6 - iU_5 & \\ & & & -U_6 + iU_5 \\ U_6 + iU_5 & & & \\ & -U_6 - iU_5 & & \end{pmatrix}
$$

$$
Q_{\bar{5}} = -\frac{1}{4}\begin{pmatrix} & & U_5 + iU_6 & \\ & & & U_5 - iU_6 \\ U_5 - iU_6 & & & \\ & -U_5 + iU_6 & & \end{pmatrix}
$$

$$
Q_{\bar{6}} = -\frac{1}{4}\begin{pmatrix} & & U_4 & -\mathbb{I}_{(4)} \\ & & -\mathbb{I}_{(4)} & -U_4 \\ U_4 & \mathbb{I}_{(4)} & & \\ \mathbb{I}_{(4)} & -U_4 & & \end{pmatrix}
$$

$$
Q_{\bar{7}} = -\frac{1}{4}\begin{pmatrix} & & iU_4 & -i\mathbb{I}_{(4)} \\ & & -i\mathbb{I}_{(4)} & -iU_4 \\ -iU_4 & -i\mathbb{I}_{(4)} & & \\ -i\mathbb{I}_{(4)} & iU_4 & & \end{pmatrix}
$$

$$
\text{(7.9)} \qquad Q_{\bar{8}} = -\frac{1}{2\sqrt{3}}\begin{pmatrix} Q_{\bar{8}}^{(4)1} & 0 \\ 0 & Q_{\bar{8}}^{(4)2} \end{pmatrix}
$$

$$
Q_{\bar{8}}^{(4)1} = \begin{pmatrix} i(\mathbb{I}_{(2)} + \sigma_1) & & iU_4 & \\ & i(\mathbb{I}_{(2)} + \sigma_1) & & \\ -iU_4 & & i(\mathbb{I}_{(2)} + \sigma_1) & \\ & & & i(\mathbb{I}_{(2)} + \sigma_1) \end{pmatrix}
$$

$$
Q_{\bar{8}}^{(4)2} = \begin{pmatrix} i(-\mathbb{I}_{(2)} + \sigma_1) & & -iU_4 \\ & i(-\mathbb{I}_{(2)} + \sigma_1) & \\ iU_4 & & i(-\mathbb{I}_{(2)} + \sigma_1) \\ & & i(-\mathbb{I}_{(2)} + \sigma_1) \end{pmatrix}
$$

$$
Q_{Z'} = \begin{pmatrix} Q_{Z'}^{(4)1} & 0 \\ 0 & Q_{Z'}^{(4)2} \end{pmatrix} \qquad Q_{Z''} = \begin{pmatrix} Q_{Z''}^{(4)2} & 0 \\ 0 & Q_{Z''}^{(4)2} \end{pmatrix}.
$$

where $Q^{(4)1}_{Z'}$, $Q^{(4)2}_{Z'}$, $Q^{(4)1}_{Z''}$, $Q^{(4)2}_{Z''}$ have diagonal blocks $i\left(\left(\frac{1}{2}k'-\ell'\right)\mathbb{I}_{(2)}+\frac{k'}{2}\sigma_1\right)$, $i\left(-\left(\frac{1}{2}k'-\ell'\right)\mathbb{I}_{(2)}+\frac{k'}{2}\sigma_1\right)$, $i\left(\left(\frac{1}{2}k''-\ell''\right)\mathbb{I}_{(2)}+\frac{k''}{2}\sigma_1\right)$ and $i\left(-\left(\frac{1}{2}k''-\ell''\right)\mathbb{I}_{(2)}+\frac{k''}{2}\sigma_1\right)$ respectively.

The 16-spinor of $SO(8)$ could be expanded in terms of spinors that belong to representations of $Q_{\bar{m}}$, $Q_{Z'}$ and $Q_{Z''}$. Converting the generators of $SU(3)\times U(1)'\times U(1)''$ to generators of $SO(8)$ through the embedding relation, the $SU(3)\times U(1)'\times U(1)''$ content may be described. It can verified that the values of k', ℓ', m' and k'', ℓ'' and m'' required for the quantum numbers of the quarks and leptons in the standard model are different from those of the space M^{klm}. For there to be a projection to the decomposition of the 8-spinor of the tangent space of M^{klm}, the generators in $su(3)/su(2)$ should be included in the tangent space. This is not possible for the coset space $\frac{G_2\times SU(2)\times U(1)}{SU(2)\times U(1)'\times U(1)''}$ because there is only one $SU(3)$ subgroup of G_2. Nevertheless, if the structure constants of the commutators of λ_8 with the $su(2)$ generators occur in the matrices $(Q^{\alpha\beta}_{\bar{m}})$, $(Q^{\alpha\beta}_{Z'})$ and $(Q^{\alpha\beta}_{Z''})$, a contraction of G_2 to $SU(3)$, $SU(3)$ to $SU(2)$ and the $SO(8)$ Clifford algebra to the $SO(7)$ Clifford algebra would yield the same conditions on k', ℓ', m', k'', ℓ'', m'' as the relations derived from the spinor decomposition for the tangent space of M^{klm}.

If the form (6.4) is selected for Z, while Z' and Z'' are given by (6.1) and (6.2), the occurrence of S^7, together with its projection to S^3, in the description of the vector bosons of the strong interactions may be explained. If the $U(1)$ charges Z' and Z'' have a form similar to $Z^{(2)}$ in (6.4), conversion of the generators of $SU(3)\times U(1)'\times U(1)''$ to generators of $SO(8)$ through the embedding relation yields the $SU(3)\times U(1)'\times U(1)''$ content of the 16-spinor. Since there are twice as many components in this spinor, a second copy of the particle multiplets arises. It is then necessary to list the G_2 quantum numbers of the quarks and leptons. Comparison of the G_2, $SU(2)$ and $U(1)$ charges with the expansion of the 16-spinor determines the appropriate values of k', ℓ', m', k'', ℓ'' and m''.

8 New elements in the description of particle interactions

While the spinor space of the standard model is isomorphic to the sum of three copies of $\mathbb{C}\otimes\mathbf{H}\otimes\mathbf{O}$, the octonion algebra is not as evident in the symmetry transformations of the spin-1 gauge fields because the absence of a Jacobi identity for the structure constants renders a pure Yang-Mills action based on the vector fields on S^7 non-invariant. A generalization of the transformation rule of the connection form for bundles, which have a structure group G but a fibre F that is not necessarily being a Lie group [19],

$$\omega(\sigma'_*\cdot\xi)=L^{-1}_{(y\cdot g)^*}\mathcal{V}(\sigma'_*\cdot\xi)=L^{-1}_{(y\cdot g)^*}\left[\mathcal{V}(R_{g^*}\sigma_*\cdot\xi)+\mathcal{V}(L_{(y\cdot g)^*}L_{g^{-1}*}g_*\cdot\xi\right] \tag{8.1}$$

where $\sigma(x)$ is a section of the bundle, $g\in G$, $\xi\in T_x(M)$ and $\mathcal{V}$ is the projection onto the vertical subspace of the tangent space to the bundle. Given that $\omega(\sigma_*\cdot\xi)$ is valued in the tangent space of F at the origin, it can be defined by parallel transport from $y \in F$ on a parallelizable fibre to be $L^{-1}_{y^*}\mathcal{V}(\sigma_* \cdot \xi)$ and

$$\omega(\sigma'_* \cdot \xi) = L^{-1}_{(y\cdot g)^*} R_{g^*} L_{y^*} \omega(\sigma_* \cdot \xi) + L^{-1}_{(y\cdot g)^*}[\mathcal{V}(L_{(y\cdot g)^*} L_{g^{-1}*} g_* \cdot \xi))]. \tag{8.2}$$

When $F = S^7$ and $G = SO(8)$, $L_{g^{-1}*}g_* \cdot \xi$ is an arbitrary element of the Lie algebra of SO(8), $(\lambda_{y'*} \cdot L_{g^{-1}*} \cdot g_* \cdot \xi) \cdot \iota_L(y'^{-1})^T$, where $\lambda_{y'*}$ is an isomorphism of vector spaces and $\iota_L : S^7 \to SO(8)$ is an embedding such that left multiplication of y is represented as right multiplication by $\iota_L(y)^T$, is independent of $y' = y \cdot g$ if 21 constraints are imposed on g, and the space of solutions is spanned by the generators $\{X_i\}$ corresponding to the vector fields on S^7. As $exp(t_i X_i) = \iota_R(\tilde{g})$, $\tilde{g} = cos\ t$, $\tilde{g}_i = \frac{t_i}{t} sin\ t$, $t = t_1^2 + ... + t_7^2$, $\tilde{g} \in S^7$, the independence with respect to y also is equivalent to

$$y' \iota_R(\tilde{g}) \iota_L(y'^{-1})^T = y' \iota_L(y'^{-1})^T \iota_R(\tilde{g}) = \tilde{g} \qquad\qquad y',\ \tilde{g} \in \mathbb{O} \tag{8.3}$$

by the associativity of the algebra of the octonions generated by two elements. The independence of $L^{-1}_{y\cdot g} R_{g^*} L_{y^*}$, the tangent mapping of $L^{-1}_{y\cdot g} R_g L_y$, which, when represented as right multiplication, is $\iota_L(y)^T R_g^T [\iota_L(y \cdot g)^{-1}]^T$. With the matrices

$$\iota_L(y)^T = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ -y_1 & y_0 & y_3 & -y_2 & y_5 & -y_4 & y_7 & -y_6 \\ -y_2 & -y_3 & y_0 & y_1 & y_6 & -y_7 & -y_4 & y_5 \\ -y_3 & y_2 & -y_1 & y_0 & -y_7 & -y_6 & y_5 & y_4 \\ -y_4 & -y_5 & -y_6 & y_7 & y_0 & y_1 & y_2 & y_3 \\ -y_5 & y_4 & y_7 & y_6 & -y_1 & y_0 & -y_3 & -y_2 \\ -y_6 & -y_7 & y_4 & -y_5 & -y_2 & y_3 & y_0 & -y_1 \\ -y_7 & y_6 & -y_5 & -y_4 & y_3 & -y_2 & y_1 & y_0 \end{pmatrix} \tag{8.4}$$

and $R_g^T = (c_{ij})$, this product equals $h(y, g)$, where $h(y, g) \in H$, the stability group of the origin (1 0 0 0 0 0 0 0). From the relations, $\sum_\ell c_{ik} c_{jk} = \delta_{ij}$, $h_{00} = 1$, $h_{i0} = h_{0i} = 0$ and

$$
\begin{aligned}
(8.5) \qquad h_{11} = & (y_0^2+y_1^2)(c_{00}c_{11}-c_{10}c_{01}) \\
& + (y_0y_3+y_1y_2)(c_{00}c_{21}-c_{10}c_{31}-c_{20}c_{01}+c_{30}c_{11}) \\
& + (y_0y_2-y_1y_3)(-c_{00}c_{31}-c_{10}c_{21}+c_{20}c_{11}+c_{30}c_{01}) \\
& + (y_0y_5+y_1y_4)(c_{00}c_{41}-c_{10}c_{51}-c_{40}c_{01}+c_{50}c_{11}) \\
& + (y_0y_4-y_1y_5)(-c_{00}c_{51}-c_{10}c_{41}+c_{40}c_{11}+c_{50}c_{01}) \\
& + (y_0y_7+y_1y_6)(c_{00}c_{61}-c_{10}c_{71}-c_{60}c_{01}+c_{70}c_{11}) \\
& + (y_0y_6-y_1y_7)(-c_{00}c_{71}-c_{10}c_{61}+c_{60}c_{11}+c_{70}c_{01})(8.5) \\
& + (y_2^2+y_3^2)(-c_{20}c_{31}+c_{30}c_{21}) \\
& + (y_3y_4-y_2y_5)(-c_{20}c_{41}-c_{30}c_{51}+c_{40}c_{21}+c_{50}c_{31}) \\
& + (y_2y_4+y_3y_5)(-c_{20}c_{51}+c_{30}c_{41}-c_{40}c_{31}+c_{50}c_{21}) \\
& + (y_3y_6-y_2y_7)(-c_{20}c_{61}-c_{30}c_{71}+c_{60}c_{21}+c_{70}c_{31}) \\
& + (y_3y_7+y_2y_6)(-c_{20}c_{71}+c_{30}c_{61}-c_{60}c_{31}+c_{70}c_{21}) \\
& + (y_4^2+y_5^2)(-c_{40}c_{51}+c_{50}c_{41}) \\
& + (y_5y_6-y_4y_7)(-c_{40}c_{61}-c_{50}c_{71}-c_{60}c_{41}+c_{70}c_{51}) \\
& + (y_5y_7+y_4y_6)(-c_{40}c_{71}+c_{50}c_{61}-c_{60}c_{51}+c_{70}c_{41}) \\
& + (y_6^2+y_7^2)(-c_{60}c_{71}+c_{70}c_{61}) \\
& - (terms\ with\ (c_{i0},c_{j1}) \leftrightarrow (c_{i2},c_{j3}),\ (c_{i4},c_{j5}),\ (c_{i6},c_{j7})).
\end{aligned}
$$

Thus, $h_{11} = h_{11}(c_{ij})$ requires that the elements of R_g^T satisfy 64 conditions, and similarly, $(64)(49) = 3136$ conditions must be satisfied for $h_{ij},\ i,j = 1,...,7$ to be independent of y. Although this represents an overdetermined system before reducing the number of conditions through dependent equations or the 36 defining relations of $SO(8)$, it is not infinite because the constraints on the gauge matrix are not entirely independent at each point of S^7 but connected through octonion multiplication [18]. It has been verified that the matrices $\{exp(t_1X_1),...,exp(t_7X_7)\}$ do not satisfy every condition, and there is no $SO(8)$ matrix which maintains the y-independence of the transformation rule of the connection form. More generally, the structure group is reduced to a subgroup determined by obstructions based on the homology of the fibres.

Although the generalized gauge transformation rule does not yield a fibre-coordinate independent potential when the fibre is S^7, there does exist a projection of the tangent vector fields on S^7 which can be used to define a nonlinear generalization of the $SU(2)\times SU(2)$ gauge theory. The quark and anti-quark fields have the products $u_\alpha u_\beta = \epsilon_{\alpha\beta\gamma}u_\gamma$ and $\tilde{u}_\alpha\tilde{u}_\beta = \epsilon_{\alpha\beta\gamma}\tilde{u}_\gamma$ [29]. One $SU(2)\times SU(2)$ Lagrangian would be

$$
\begin{aligned}
(8.6) \qquad L_{SU(2)\times SU(2)} = & -\frac{1}{4}Tr(F_{\mu\nu}F^{\mu\nu}) + \frac{i}{2}\sum_\alpha \bar{u}_\alpha\Gamma^i D_i u_\alpha + \frac{i}{2}\sum_\beta \bar{\tilde{u}}_\beta\Gamma^i D_i\tilde{u}_\beta \\
& - \frac{1}{2}\sum_\alpha \bar{u}_\alpha u_\alpha - \frac{1}{2}m_{\tilde{u}}\sum_\beta \bar{\tilde{u}}_\beta\tilde{u}_\beta
\end{aligned}
$$

where $\{\Gamma^i\}$ is a bases for the Clifford algebra in four dimensions. The Lie algebra generators are linear vector fields on $S^3 \times S^3$.

8.1 Field-theoretical projection from S^7 to $S^3 \times S^3$

The geometric projection from S^7 to S^3 yields the three linear vector fields on the three sphere, the orthonormal set in $S^3 \times S^3$ and a seventh vector field which can be expressed in terms of a normal vector and the first triplet [21]

$$\vec{v}_7 = (2y_7^2 - 1)\vec{n} - 2y_4 y_7 \vec{t}_1 - 2y_4 y_5 \vec{t}_2 - 2y_4 y_6 \vec{t}_3. \tag{8.7}$$

Based on all seven vector fields and an additional spinor field, the projection produces a Lagrangian of the form

$$\begin{aligned} L_{S^7 \to S^3 \times S^3} = &-\frac{1}{4} Tr(F_{\mu\nu} F^{\mu\nu})_{SU(2)\times SU(2)} - \frac{1}{4}\langle Tr(F_{\mu\nu 7} F^{\mu\nu 7})\rangle \\ &+ \frac{i}{2}\sum_\alpha \bar{u}_\alpha \Gamma^i D_i u_\alpha + \frac{i}{2}\sum_\beta \bar{\tilde{u}}_\beta \Gamma^i D_i \tilde{u}_\beta \\ &- \frac{1}{2} m_u \sum_\alpha \bar{u}_\alpha u_\alpha - \frac{1}{2} m_{\tilde{u}} \sum_\beta \bar{\tilde{u}}_\beta \tilde{u}_\beta + \frac{i}{2}\langle \bar{H}\Gamma^i D_i H\rangle \\ &- \frac{1}{2} m_H \bar{H} H. \end{aligned} \tag{8.8}$$

Both $\langle Tr(F_{\mu\nu 7} F^{\mu\nu 7})\rangle$ and $\langle \bar{H} H\rangle$ must transform as singlets under $SU(2)\times SU(2)$. Fibre coordinate dependence can be eliminated from the transformation rule of the connection if the commutators of the vectors in the vertical space have position-independent structure constants. While this property holds for the six of the vector fields, the commutator with the seventh vector field would be position-dependent. Since the seventh component of the connection form does not project to a potential taking values only in the base space, it is necessary to consider the vector field over that part of the fibre which is located in the future-directed cone. After integration, the average values are defined entirely on the base space. In counting the number of intermediate vector bosons, the average value then would be 6.25. This value has been verified by calculations of the pomeron Regge trajectory slope [24].

The projection from S^7 to S^3 was derived from the equivalence of an S^7 bundle over M_4 and an $SU(2)$ bundle over $M^4 \times S^4$ defined by the fibration $S^3 \to S^7 \to S^4$. Since there exist fifteen Milnor spheres, the action of the structure groups on the fibres of the exotic spheres shall be used to define a variant of the $SU(2)$ Yang-Mills theory. Consider the bundles $\tilde{\mathcal{M}}_{m,n} = \{\tilde{M}_{m,n}, \beta, S^3, S^4, SO(4)\}$, where the corresponding element of the homotopy group $\pi_3(SO(4))$ is chosen to be $m\rho + n\sigma$, with ρ and σ being the generators given by $\rho(u)v = uvu^{-1}$ and $\sigma(u)v = uv$, $\| u \| = \| v \|$, $u, v \in \mathbb{H}$ [41]. Since the Milnor sphere is $\tilde{M}_{m,1}$, the corresponding generator is $m\rho + \sigma$. It has been shown that the fibre-coordinate independence of the transformation rule of the connection form of a sphere bundle with fibre S^3 reduces the gauge symmetry group to $SU(2)$ [21].

The diffeomorphism type is determined by the Eells-Kuiper invariant

$$\mu(\tilde{M}_{m,n}) = \frac{1}{2^7\cdot 7}\left[p_1^2(\tilde{W}_{m,n}) - 4\tau(\tilde{W}_{m,n})\right] \qquad mod\ 1$$
$$= \frac{1}{2^5\cdot 7}[(n+2m)^2-1] \qquad mod\ 1 \tag{8.9}$$

where $\tilde{W}_{m,n}$ is the coboundary, p_1 is the Pontryagin class and τ is the signature [41]. Since

$$\mu(\tilde{M}_{m,1}) = \frac{1}{2^3\cdot 7}m(m+1) \qquad mod\ 1, \tag{8.10}$$

there are different values of $\mu(\tilde{M}_{m,1})$ for $m = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 10,\ 12,$ $13,\ 16,\ 17,\ 20,\ 24.$

To describe the action of S^3 on the bundle $\tilde{\mathcal{M}}_{m,1}$, suppose that u is a unit quaternion which is an element of the S^3 fibre. Then there are two bundles with transformations

$$u \to e^{ip_-\theta}(1+j)u$$
$$u \to e^{jp_+\theta}(1+i)u \tag{8.11}$$

and

$$u \to e^{iq_-\theta}(1+j)u$$
$$u \to e^{jq_+\theta}(1+i)u. \tag{8.12}$$

If $P_{k,\ell} \to S^4$ is the $S^3\times S^3$ bundle defined by the union of these two group actions, let $M_{k,\ell} = P_{k,\ell}\times_{S^3\times S^3} S^3$. As the action of $S^3\times S^3$ on S^3 is given by $u \to Q_1uQ_2^{-1}$ [41], it follows that (x, u_1, u_2, u_3) is identified with $(x, [e^{ip_-\theta}(1+j)+e^{jp_-+\theta}(1+i)]u_1, [e^{iq_-\theta}(1+j)+e^{jq_+\theta}(1+i)]u_2, [e^{ip_-\theta}(1+j)+e^{jp_+\theta}(1+i)]u_3[e^{iq_-\theta}(1+j)+e^{jq_+\theta}(1+i)]^{-1})$. The exotic spheres, which are homeomorphic to S^7 and have Euler number 1, are denoted by $M_{k,1-k}$ and can be identified with the Milnor spheres $\tilde{M}_{m,1}$ if $m = k-1$. The spaces $M_{k,\ell}$ are classified by $k = \frac{p_-^2-p_+^2}{8}$ and $\ell = -\frac{q_-^2-q_+^2}{8}$, and, for $M_{k,1-k}$, $k-1 = \frac{q_-^2-q_+^2}{8}$. As $M_{1,0} \simeq \tilde{M}_{0,1} \simeq S^7$, the spheres are exotic when $k \geq 2$, and there is a finite number of solutions for p_-, p_+, q_- and q_+. The S^3 action on $M_{k,1-k}$ is given by

$$(x, u_1, u_2, u_3) \to (x, vu_1, vu_2, vu_3) \qquad v \in \mathbb{H},\ \ \| v \| = 1. \tag{8.13}$$

Given the vector fields X_1, X_2, X_3 on S^3, with the commutation relations $[X_i, X_j] = \epsilon_{ijk} X_k$, the identifications in the Milnor sphere are

$$
\begin{aligned}
(8.14) \qquad & u_1 \to [e^{\theta p_- X_1}(1 + X_2) + e^{\theta p_+ X_2}(1 + X_1)]u_1 \\
& u_2 \to [e^{\theta q_- X_1}(1 + X_2) + e^{\theta q_+ X_2}(1 + X_1)]u_2 \\
& u_3 \to [e^{\theta p_- X_1}(1 + X_2) + e^{\theta p_+ X_2}(1 + X_1)]u_3 \\
& \qquad\qquad [e^{\theta q_- X_1}(1 + X_2) + e^{\theta q_+ X_2}(1 + X_1)]^{-1}.
\end{aligned}
$$

Since the fibre-coordinate dependence can be eliminated from the transformation rule of the connection form for each S^3 component of the fibre of the covering, the gauge transformation now has the form $A'_\mu = g A_\mu g^{-1} + g^{-1} \partial_\mu g$ modulo these identifications, where $g = e^{t_1 X_1 + t_2 X_2 + t_3 X_3}$. The Lagrangian $-\frac{1}{4} Tr(F_{\mu\nu} F^{\mu\nu})$ remains gauge invariant.

There are also transformations on different components of $S^3 \times S^3 \times S^3$ related to the substitution of u_i by X_i and multiplication replaced by commutators. For example, amongst the points (u_1, u_2, u_3) in this space is (i, j, k), which could be identified with (X_1, X_2, X_3). Permutations of (i, j, k) could be used as well. To $\mathcal{O}(\theta)$, the transformations are

(8.15)

$$
\begin{aligned}
& X_1 \to 2(\theta p_+ - 1)X_3 + 2\theta(p_+ - p_-)X_2 \\
& X_2 \to 2(1 - \theta q_-)X_3 + 2\theta(q_+ - q_-)X_1 \\
& X_3 \to -[1 + \theta(q_- p_- - q_+ p_+)]X_3 + \theta[2(p_- - p_+) - (q_+ - q_-)](X_1 + X_2) \\
& \qquad + \frac{1}{4}[X_1 - \theta p_+ X_1 + \theta(p_- - p_+)X_1^2 - (X_2 + \theta p_- X_2 + \theta(p_- - p_+)X_2^2) \\
& \qquad\quad ((1 + \theta q_-)X_1 + (1 + \theta q_+)X_2)^2 + ...]|_\theta
\end{aligned}
$$

where the terms in the last commutator are restricted to be linear in θ only. If the group elements have the form $exp(t_1 X_1)$, $exp(t_2 X_2)$, $exp(t_3 X_3)$, the substitution of the new form of the generators in the exponentials yields an equivalent gauge transformation in another S^3 component of $S^3 \times S^3 \times S^3$.

8.2 A Lagrangian for S^7-valued fields based on a new composition law

Before the field theoretic projection to $S^3 \times S^3$, one possible Lagrangian, based on the octonion algebra, could be defined with a new composition law. Consider first the vector space representation of the octonions. Since the space is normed, there is a real-valued function Q of the octonions which satisfies $Q(x) \geq 0$ and $Q(\lambda x) = |\lambda|^2 Q(x)$, and a scalar product $B(x, y)$ can be defined by

$$
(8.16) \qquad Q(x + y) = Q(x) + Q(y) + B(x, y).
$$

Consider the twenty-four dimensional space A given by the product of an eight-dimensional vector space M and two spinor spaces S_p and S_i, and let ρ be a representation of the Clifford algebra on the spinor spaces. There is a bilinear mapping $\beta : S \times S \to S$ which vanishes on $S_p \times S_i$ and $S_i \times S_p$ with the property [15]

$$\beta(\rho(z) \cdot u, \rho(z) \cdot v) = Q(z)\beta(u, v). \tag{8.17}$$

Let

$$\begin{aligned} \Lambda(x + u, x' + u') &= B(x, x') + \beta(u, u') \\ F(x + u + u') &= \beta(\rho(x) \cdot u, u') = \beta(u, \rho(x) \cdot u') \quad x \in M, u \in S_p, u' \in S_i. \end{aligned} \tag{8.18}$$

Then a trilinear form [15] can be defined on $A \times A \times A$ by

$$\begin{aligned} \Phi(\xi, \eta, \zeta) = F(\xi + \eta + \zeta) + F(\xi) + F(\eta) + F(\zeta) \\ - [F(\xi + \eta) + F(\eta + \zeta) + F(\zeta + \xi)] \end{aligned} \tag{8.19}$$

and a new law of composition through the relation

$$\Phi(\xi, \eta, \zeta) = \Lambda(\xi \circ \eta, \zeta). \tag{8.20}$$

By the principle of triality, there is an automorphism J of order 3 of A, $J : M \to S_p$, $J : S_p \to S_i$ and $J : S_i \to M$.

The Lagrangian

$$L^D_{S^7} = \frac{i}{2} Q(\eta_p \circ \Gamma^i D_i \eta_i) - \frac{1}{2} Q(\eta_p \circ \eta_i) \tag{8.21}$$

would be globally invariant under the transformations, with e_i being a unit octonion, and $\eta_i \to e_i \cdot \eta_i$, $\eta_p \to \eta_p \cdot e_i$, since, if $\eta_p = x \cdot u'_1$, $u'_1 \in S_p$, $\eta_i = x \cdot u_1$, $u_1 \in S_i$.

$$\begin{aligned} Q(\eta_p \circ e_i \circ (e_i \circ \eta_i)) = Q((x \cdot u'_1 \circ e_i) \circ (e_i \cdot x \circ u_1)) = Q(x * e_i * e_i * x) \\ = Q(x * x) = Q(\eta_p \circ \eta_i) \end{aligned} \tag{8.22}$$

where $*$ is octonion multiplication. If these transformations are now position-dependent

(8.23)
$$L_{S^7} = -\frac{1}{4}Q(F_{\mu\nu} * \bar{F}^{\mu\nu}) + \frac{i}{2}Q(\eta_p \circ \Gamma^i D_i \eta_i) - \frac{1}{2}Q(\eta_p \circ \eta_i)$$

is invariant under $F_{\mu\nu} \to e_i * F_{\mu\nu}$. However, given that the field strength is equal to the commutator of the covariant derivatives of the potential, there is no transformation of the potential which induces the above mapping of $F_{\mu\nu}$. It is known also that the exact formulation of theories with gauge invariance is given by loop integrals of the potentials rather than the field strength, and a reformulation of the model could possess invariances of a standard kind.

8.3 Wilson loop variables

The Wilson loop factor $exp\left(i\int A_\mu dx^\mu\right)$ is shifted to

(8.24)
$$exp\left(i\int A_\mu dx^\mu\right) \to exp\left(i\int A_\mu dx^\mu + i\int \partial_\mu \chi dx^\mu\right)$$

by an abelian gauge transformation. If a and b are endpoints of the contour of the integral, then

(8.25)
$$\int \partial_\mu dx^\mu = \chi(b) - \chi(a).$$

When $\chi(b) - \chi(a) = 2\pi n$ or $a = b$, $exp\left(i\int \partial_\mu \chi dx^\mu\right) = 1$ and the path-dependent factor is invariant. The invariance of this factor is related to the identification of the paths with open and closed strings.

The transformation of the Wilson loop factor $tr\ P\ exp\left(-i\lambda\int_{x_1}^{x_2} A_\mu(x)dx^\mu\right)$ in nonabelian gauge theories [34] follows from the relation

(8.26)
$$\begin{aligned} &P\ exp\left(-i\lambda\int_{x_1}^{x_2} A'_\mu(x)dx^\mu\right) \\ &= P\ exp\left(-i\lambda\int_{x_1}^{x_2} gA_\mu g^{-1}dx^\mu - i\lambda(i\lambda)^{-1}\int_{x_1}^{x_2} g\partial_\mu g^{-1}dx^\mu\right) \\ &= P\ exp\left(-i\lambda\int_{x_1}^{x_2} gA_\mu g^{-1}dx^\mu + \int_{x_1}^{x_2} \partial_\mu g g^{-1}dx^\mu\right). \end{aligned}$$

Further,

$$\int_{x_1}^{x_2} \partial_\mu g g^{-1} dx^\mu = \int_{g(x_1)}^{g(x_2)} g^{-1} dg = ln\ (g(x_2)g(x_1)^{-1}). \tag{8.27}$$

For a closed loop, $ln(g(x_2)g(x_1)^{-1}) = 0$. Then $P\ exp\left(-i\lambda \int_{x_1}^{x_2} A'_\mu dx^\mu\right)$ can be identified with $g(x_2)P\ exp\left(-i\lambda \int_{x_1}^{x_2} A_\mu(x)dx^\mu\right) g(x_1)^{-1}$ for a group element $g(x)$ equal to a constant g on the contour. When $g(x) = g + \epsilon(x) + \mathcal{O}(\epsilon^2)$, where $|\epsilon(x)_{\alpha\beta}| \ll |g_{\alpha\beta}|$,

$$\begin{aligned}
&P\ exp\left(-i\lambda \int_{x_1}^{x_2} g(x)A_\mu(x)g(x)^{-1}dx^\mu\right) \\
&\quad = P\ exp\left(-i\lambda \int_{x_1}^{x_2} (g+\epsilon(x)+\mathcal{O}(\epsilon^2))A_\mu(x)(g^{-1} - g^{-1}\epsilon(x)g^{-1} + \mathcal{O}(\epsilon^2))\ dx^\mu\right) \\
&\quad = P\ exp\bigg(-i\lambda \int_{x_1}^{x_2} gA_\mu(x)g^{-1}dx^\mu \\
&\qquad\qquad - i\lambda \int_{x_1}^{x_2} (\epsilon(x)A_\mu(x)g^{-1} - gA_\mu(x)\epsilon(x)g^{-1}\epsilon(x)g^{-1})dx^\mu + \mathcal{O}(\epsilon^2)\bigg) \\
&\quad = gP\ exp\left(-i\lambda A_\mu(x)dx^\mu\right) g^{-1} + \mathcal{O}(\epsilon).
\end{aligned} \tag{8.28}$$

By the cyclic property of the trace,

$$\begin{aligned}
&tr\ gPexp\left(-i\lambda \int_{x_1}^{x_2} A_\mu dx^\mu\right) g^{-1} + \mathcal{O}(\epsilon) \\
&\quad = tr\left[gP\ exp\left(-i\lambda \int_{x_1}^{x_2} A_\mu(x)dx^\mu\right) g^{-1}\right] + \mathcal{O}(\epsilon) \\
&\quad = tr\left[P\ exp\left(-i\lambda \int_{x_1}^{x_2} A_\mu(x)dx^\mu\right) g^{-1}g\right] + \mathcal{O}(\epsilon) \\
&\quad = tr\ P\ exp\left(-i\lambda \int_{x_1}^{x_2} A_\mu(x)dx^\mu\right) + \mathcal{O}(\epsilon)
\end{aligned} \tag{8.29}$$

and the closed Wilson loop factor is invariant to leading order under nearly constant gauge transformations. By Stokes' theorem, the path-ordered exponential equals $\mathbb{I} - \frac{i\lambda}{2} F_{\mu\nu}\sigma^{\mu\nu} + \mathcal{O}(|\sigma^{\mu\nu}|^2)$, where $\sigma^{\mu\nu}$ is the area element in the interior of the contour, and gauge covariance to first order for infinitesimal loops follows. Furthermore, it has been demonstrated that the span of the Wilson loop operators is dense in the space of gauge-invariant variables [31]. Gauge invariance of supersymmetric Wilson loop operator has been verified to first order [30].

The infinitesimal form of the gauge transformation $A_\mu \to gA_\mu g^{-1} + g^{-1}\partial_\mu g$, $g = e^{i\alpha_a T^a}$, with $A_\mu = A_{\mu a}T^a$, $[T^a, T^b] = f^{ab}{}_c T^c$, yields

$$exp\Big(i\int A_\mu dx^\mu\Big) \to exp\Big(i\int A_\mu dx^\mu + i\int \alpha_a f^{ab}{}_c A_{\mu b} T^c dx^\mu + i\int \partial_\mu \alpha_a T^a dx^\mu + \frac{i}{2}\int \alpha_a \partial_\mu \alpha_b f^{ab}{}_c T^c dx^\mu\Big)$$

Invariance requires

$$\int \alpha_a f^{ab}{}_c A^{\mu b} dx^\mu + \int \partial_\mu \alpha_c dx^\mu + \frac{1}{2}\int \alpha_a \partial_\mu \alpha_b f^{ab}{}_c dx^\mu = 0. \tag{8.30}$$

One solution is

(8.31)

$$D_\mu \alpha_a = 0 \qquad \int \alpha_a \partial_\mu \alpha_b dx^\mu = \int \alpha_b \partial_\mu \alpha_a dx^\mu.$$

If $\alpha_a = f_a^\mu \partial_\mu \phi$, then $D^\mu \partial_\mu \phi = 0$, implying that ϕ is a scalar field. This construction can be extended to S^7.

The reduction of the $E_8 \times E_8$ gauge symmetry to $E_6 \times E_8$ produces a Higgs potential which can be derived from the solution to the constraints for the coset components of the gauge fields for only one E_8 factor, while the second set of E_8 gauge connection remains identified with part of the spin connection. For ten-dimensional super-Yang-Mills theory, the solutions in terms of E_6 generators are $\phi^a = Q^a$, $\phi_\rho = \beta^i Q_{i\rho}$ and $\phi^{\bar\rho} = \beta_i Q^{i\bar\rho}$ [52]. The G_2 action on $\{\phi^a, \phi^\rho, \phi^{\bar\rho}\}$ can be expanded to a larger symmetry by the construction of the Higgs potential through E_6 invariants [52] by a method similar to that of the expression for the effective heterotic string Lagrangian in terms of superinvariants. There is a nontrivial extremum of the Higgs potential, and, if the Higgs field transforming under the $\underline{27}$ representation of E_6 is not fundamental, the potential defined by the invariants has been shown to have an extremum with $SO(8)$ symmetry. The $SO(8)$-invariant field occurs in the solution to the wave equation $D^M \partial_M \phi = D^\mu \partial_\mu \phi + D^a \partial_a \phi = 0$, which, yields a necessary condition for the invariance of a Wilson loop factor defined in an eleven-dimensional space-time containing S^7 as the compact space. The solution to the higher-dimensional equation, $\phi(x, y) = \varphi(x)\varpi(y)$, where $D^\mu \partial_\mu \varphi(x) = 0$ and $D^a \partial_a \varpi(y) = 0$ is $SO(8)$ invariant, can be projected to the scalar field on the base space $\varphi(x)$, yielding the Wilson loop variable in four dimensions. The curvature term on S^7 may be related to the mass of the Higgs field, where this equation has been derived from the invariance of the Wilson loop factor. The condition of a closed loop is not required for these operators and the description of strongly interacting particles through open strings is verified.

The fermions are known to transform under $SU(3)$ which results as a nonlinear realization of an $SU(4)$ gauge symmetry derived from $SO(1,9)$. It is known that the Lie algebra of E_6 is isomorphic to the set of generators of three-dimensional octonionic matrices of unit determinant [14][66], and the reduction from $E_6 \to SO(1,9) \to SU(4) \to SU(3)$ is applicable to the action of the gauge group on strongly interacting fermions. The distinction between the effective internal symmetry for the bosons and fermions in strong interactions has been described previously in §3. This derivation of the symmetries is indicative of a twistor formalism for superstring theory [5] with octonion coordinates.

While perturbative diagrams in open superstring theory include closed superstring states, a mechanism such as that provided by the open Wilson loop factor is necessary to provide the open string description of quark-antiquark pairs in the strong interactions in closed $E_8 \times E_8$ heterotic string theory. The elementary particles, which form representations of $G_2 \times SU(2) \times U(1)$ are modes of the heterotic string with coefficients that are point-particle fields. The amplitudes for reactions of specific elementary particles are given by projections of the scattering matrix of heterotic strings onto the corresponding states. Nevertheless, because the interactions are described by string diagrams, the elementary particles such as the electron would be described by one-dimensional strings having characteristics determined by the theory.

The $E_8 \times E_8$ and $SO(32)$ heterotic string and the $SO(32)$ Type I superstring theories have been shown to be anomaly-free, and the latter model consists of open superstrings. Some remarks shall be made, therefore, about the physical relevance of these theories. The absence of global gravitational anomalies follows from the vanishing of the change in the action under $f : M \to M$, which can be related to a topological index defined on the coboundary W of the mapping torus of M and f, $M_f = (M \times [0,l])/\sim$, with the equivalence relation defined to be $(x,0) \sim (f(x),1)$, $\mu(M_f) = \frac{1}{192}(4p_1p_2 - 3p_1^3)\ mod\ 1$, where p_1 and p_2 are the first and second Pontryagin classes of W [46]. It has been shown that $\mu(M_f)$ vanishes when $M = S^{10}$. Since there is a stereographic projection from the sphere to Euclidean space, π_{ster}, any diffeomorphism of the extended plane will have the form $\pi_{ster} \circ f \circ \pi_{ster}^{-1}$, where $f : S^{10} \to S^{10}$. Restricting these diffeomorphisms by $\pi_{ster} \circ f \circ \pi_{ster}^{-1}(\infty) = \infty$, such that the coordinate expansions of $\pi_{ster} \circ f \circ \pi_{ster}^{-1}$ contain positive powers of $|x^A|$, where $\{x^A,\ A = 1, ..., 10\}$ are Euclidean coordinates, it follows that $\mu(M_f)$ would be zero when M is ten-dimensional flat space. Whereas the value of $\mu(M_f)$ is not known for a general ten-dimensional manifold, there should be no global gravitational anomalies in a superstring or heterotic string theory in ten-dimensional Minkowski space-time, as a change of signature can be included through a Wick rotation of the coordinates. The restriction on the validity of string theory in ten-dimensional target space-times has been verified by the consistency of perturbation theory and the rescaling of energy levels such that the effect of any local curvature induced by the string is effectively removed and the scattering amplitudes can be computed entirely in a flat background.

While it might be useful to eliminate models based on constraints following from the absence of global anomalies, it may be observed that this does not necessarily imply the criteria that distinguishes between the various heterotic and superstring theories. For example, the condition $\mu(M_f) = 0$ for the vanishing of global gravitational anomalies is unchanged. Moreover, although there are more nonvanishing homotopy groups of BSO, in contrast to BE_8, cancellation of perturbative anomalies has been found to hold with the equality $p_1(V) = p_1(T)$, where V is the gauge bundle and T is the tangent bundle, and this relation is satisfied trivially by the identification of the $SO(32)$ connection with the spin connection [46]. Being a topological index, the first Pontryagin class is invariant under local diffeomorphisms, and invariance under global diffeomorphisms would be required equally for the $SO(32)$ and $E_8 \times E_8$ theories.

The viability of the $SO(32)$ open superstring theory remains to be established. Furthermore, there is no similar mechanism for producing open Wilson loop variables when the standard Higgs field identified in §3 arise from four-dimensional action (3.6) derived by dimensional reduction of the six-dimensional Yang-Mills theory.

It follows that, for the $E_8 \times E_8$ superstring theory, the elementary particles must be represented by closed strings. A rapid rotation of this one-dimensional string about an axis generates a spheroid. Over sufficiently large time intervals, the charge distribution would appeared to spread over this surface. By conservation of charge, this amount of charge is unchanged by a deformation of the surace to a sphere with uniform density. For the electron, this distribution of the charge is equivalent to that given by Lorentz model.

The occurrence of S^7 in the reduction of the superstring theory contained in the twelve-dimensional model can be understood further through the identification of null vectors in ten dimension and fermion bilinears as a result of the isomorphism $\tilde{SO}(1, \nu+1) \simeq SL(2; \mathbf{K}_\nu)$, where $\{K_\nu\}$ represents the sequence of associative normed division algebras over the real numbers for $\nu = 1, 2, 4$, and the equivalence between the Lie algebra of $SO(1, 9)$ and the direct sum of the set of generators of the two-dimensional octonionic matrices of unit determinant and G_2:

$$P^\mu \leftarrow P = \lambda\lambda^\dagger = \begin{pmatrix} \xi\xi^\dagger & \xi\eta^\dagger \\ \eta\xi^\dagger & \eta\eta^\dagger \end{pmatrix} \qquad \lambda = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{8.32}$$

Since $det\ P = P^\mu P_\mu = 0$, P^μ belongs to the forward light cone [13]. The space of light-like lines in ten dimensions is S^8. Transformations which leave $\lambda\lambda^\dagger$ invariant form the algebra S^7 [12] and the action is given by the Hopf fibration $S^{15} \to S^8$. This action has been extended to the Kac-Moody algebra $\hat{S}^7$ [14], which arises a symmetry algebra of the twistor string theory and the light-cone superstring.

9 The twelve-dimensional theory

While Type IIA theory has been derived from 11-dimensional M-theory, the IIB supergravity should follow from reduction of a twelve-dimensional action. The postulated F-theory [67], which would reduces to Type IIB theory on $\mathcal{M}_{10}$, is defined on the total space $\mathcal{V}_{12}$ of a bundle over $\mathcal{M}_{10}$ with the fibre $\mathcal{V}_2$ to be modular invariant.

The fibre is usually chosen to be a torus, which may admit either a Euclidean or Lorentzian signature. For the twelve-dimensional field theory formulated in a space-time of signature $(10, 2)$, compactification to four-dimensions dimensions produces a manifold of signature $(3, 1)$ or $(2, 2)$. It is known that Kleinian manifolds of signature (2,2) support $N = 2$ superstring theories in four dimensions [59]. The internal space then would have a positive-definite metric. If the Lorentzian signature is used, a rotation of the one of the coordinates on the eight-dimensional manifold is necessary to obtain the compact form.

The twelve dimensional supergravity theory contains a zwölfbein e_m^a, spin-$\frac{3}{2}$ field ψ_m^α, antisymmetric tensor B_{mn}, spin-$\frac{1}{2}$ field $\bar{\chi}_{\dot{\alpha}}$ and a dilaton Φ [57]. The space of scalar fields is $SU(1,1)/U(1)$, which has a metric of indefinite signature. Since there is an action of $SU(1,1)$ on the hyperbolic disk that is isomorphic to the two-dimensional real group of special linear transformations, which can be projected to the modular group upon the periodic identification of fields in string theory, this space may be selected to be $\mathcal{V}_2$. With the equivalent manifold of Euclidean signature, the fibre would be diffeomorphic to $SU(2)/U(1) \simeq S^2$. To include this fibration in the same theory, it would be necessary to formulate the theory such that the sphere, torus and other Riemann surfaces occur. This may be done by defining the space with the spherical fibration to be the classical limit of a quantum theory with the one-loop correction given by the toroidal fibration. Higher-order effects then could be described by spaces fibred over Riemann surfaces of higher genus.

For supergravity theories, the maximal dimension of Lorentzian space-times for the spins not to be larger than 2, is eleven, although chiral fermions are not present. However, superalgebras also have been constructed in spacetimes with a different signature in dimensions higher than 11. It has been demonstrated that there is a unification of the Type IIA, Type IIB and heterotic superalgebras in $D = (10,2)$ [61]:

$$\{Q_\alpha^i, Q_\beta^j\} = \tau_a^{ij}(\gamma_{\alpha\beta}^{\mu\nu} Z_{\mu\nu}^a + \gamma_{\alpha\beta}^{\mu_1\ldots\mu_6} Z_{\mu_1\ldots\mu_6}^a) + \epsilon^{ij}\left(C_{\alpha\beta}Z + \gamma_{\alpha\beta}^{\mu_1\ldots\mu_4} Z_{\mu_1\ldots\mu_4}\right) \tag{9.1}$$

where $\tau^a = (\sigma_3, \sigma_1, 1)$ and the Z-tensors commute.

The introduction of null vectors, which does not preserve Lorentz invariance, could be interpreted as an averaging procedure for quantities such as momentum without generating new fields. While a twelve-dimensional supergravity theory exists, a solution to the constraints yields the ten-dimensional action after a minimal condition is imposed on the spin-$\frac{1}{2}$ field and the spin-$\frac{3}{2}$ field strength, and therefore, it is essentially a reformulation of the theory in ten dimensions, becaues non-physical components of the higher-dimensional fields are eliminated through Bianchi identities. Chiral fermions occur in the twelve-dimensional supergravity theory, which is necessary for consistency with particle physics phenomenology, Lorentz invarance is restored upon reduction to ten dimensions.

The L^{klm} spaces admit $N = 2$ supersymmetry and have been defined by a limit of solutions to the equations of $N = 1$, $D = 11$ supergravity. The gravitational field equations are

(9.2)
$$R^A{}_B = \frac{3}{4}a^2\left[1 - \frac{a^2}{12c^2}(\ell'm'' - \ell''m')^2\right]\delta^A{}_B$$
$$R^m{}_n = \frac{b^2}{2}\left[1 - \frac{b^2}{2c^2}(k'm'' - k''m')^2\right]\delta^m{}_n$$
$$R^3{}_3 = \frac{b^2}{4c^2}(k'm'' - k''m')^2 + \frac{a^4}{8c^2}(\ell'm'' - \ell''m')^2$$
$$a = \frac{k'm'' - k''m'}{\ell'm'' - \ell''m'}\gamma\sqrt{6\alpha} \quad b = \gamma\sqrt{2\beta} \quad c = (k'm'' - k''m')\gamma.$$

For an Einstein manifold, the proportionality constant between the Ricci tensor and metric is equal for each coordinate, and

(9.3)
$$\frac{3}{2}a^2\left[1 - \frac{a^2}{12c^2}(\ell'm'' - \ell''m')^2\right] = \frac{b^2}{2}\left[1 - \frac{b^2}{2c^2}(k'm'' - k''m')^2\right]$$

implying that

(9.4)
$$\beta^2 - \beta + \frac{9}{2}\left(\frac{k'm'' - k''m'}{\ell'm'' - \ell''m'}\right)^2 \alpha\left[1 - \frac{\alpha}{2}\right] = 0.$$

Equality of the coefficients of the other diagonal Ricci tensor components with that of $R^3{}_3$ yields

(9.5)
$$\frac{3}{4}a^2 = \frac{3}{4}b^2 - \frac{b^2}{2c^2}(k'm'' - k''m')^2$$

and

(9.6)
$$\frac{9}{2}\left(\frac{k'm'' - k''m'}{\ell'm'' - \ell''m'}\right)^2 \alpha = \frac{3}{2}\beta - 2\beta^2.$$

Similarly,

(9.7)
$$\frac{9}{2}\left(\frac{k'm'' - k''m'}{\ell'm'' - \ell''m'}\right)^2 \frac{1}{2}\alpha^2 = \frac{1}{2}(\beta - 2\beta^2).$$

Therefore,

$$\beta^2 - \beta + \frac{3}{2}\beta - 2\beta^2 - \frac{1}{2}(\beta - 2\beta^2) = 0 \tag{9.8}$$

and there are no constraints on β from the field equations. However, by choosing, $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$, the rank of $C_{\alpha\beta}{}^{\gamma\delta}$ is reduced to 7 and the number of independent solutions to the holonomy equations equals 2 [10][69]. The L^{klm} spaces have these values, characterized by $N = 2$ supersymmetry, and the $U(1)'$ and $U(1)''$ factors are identified [69].

A twelve-dimensional supergravity with $N = 1$ supersymmetry [58] which reduces to the leading-order part of the heterotic string effective field theory in ten dimensions has been constructed, and if the number of supersymmetry invariances is increased to 2, Type IIB superstring field equations are found in ten dimensions [57]. The Maurer-Cartan equations for $G_2 \times SU(2) \times U(1)$ are

$$\begin{aligned} d\Omega^{\ddot{3}} &= -\frac{1}{2}\epsilon_{\ell m}\Omega^{\ell} \wedge \Omega^{m} \\ d\Omega^{\ell} - \epsilon^{\ell m}\Omega^{3} \wedge \Omega_{m} &= 0 \\ d\Omega^{\bar{3}} &= 0 \\ d\Omega_{\bar{\alpha}} + \frac{1}{2} f_{\bar{\alpha}\bar{\beta}\gamma}\Omega^{\bar{\beta}} \wedge \Omega^{\bar{\gamma}} + \frac{1}{2} f_{\bar{\alpha}\beta\gamma}\Omega^{\beta} \wedge \Omega^{\gamma} &= 0 \\ d\Omega_{\alpha} + f_{\alpha\bar{\beta}\gamma}\Omega^{\bar{\beta}} \wedge \Omega^{\gamma} + \frac{1}{2} f_{\alpha\beta\gamma}\Omega^{\beta} \wedge \Omega^{\gamma} &= 0. \end{aligned} \tag{9.9}$$

The coordinates of the coset space must be $(z^A, z^{\dot{1}}, z^{\dot{3}}, z^{3})$ instead of $(z^A, z^{\dot{1}}, z^{\dot{2}}, z^{\dot{3}})$. Setting

$$\begin{aligned} B^{\alpha} &= \frac{1}{a}\Omega^{\alpha} \\ \mathcal{B}^{m} &= \frac{1}{b}\Omega^{m} \\ \mathcal{B}^{\hat{3}} &= \frac{1}{c}\left[\frac{1}{\sqrt{3}}\frac{(\ell' m'' - \ell'' m')}{\Delta_S}\Omega^{6} - \frac{(k' m'' - k'' m')}{\Delta_S}\Omega^{\ddot{3}} + \Omega^{\bar{3}}\right] \end{aligned} \tag{9.10}$$

where $\Delta_S = k(\ell' m'' - \ell'' m') - \ell(k' m'' - k'' m') + m(k'\ell'' - k''\ell')$, the torsion constraints yield

(9.11)
$$\begin{aligned} B^{\alpha\beta} &= k_1 f^{\alpha\bar{\beta}\beta}\Omega_{\bar{\beta}} + k_2 f^{\alpha\beta\gamma}\Omega_\gamma + k_3 f^{6\alpha\beta}\mathcal{B}_{\hat{3}} \\ \mathcal{B}^{\alpha\ddot{1}} &= \mathcal{B}^{\alpha\ddot{3}} = 0 \\ \mathcal{B}^{\alpha\hat{3}} &= k_4 f^{6\alpha\beta}\mathcal{B}_\beta \end{aligned}$$

with $k_1 = 1$, $k_2 = -\frac{1}{2}$, $k_4 = k_3$ and

(9.12)
$$\begin{aligned} B^{\ddot{1}\hat{3}} &= k_5\mathcal{B}_{\ddot{3}} + k_5'\mathcal{B}_{\ddot{2}} \\ \mathcal{B}^{\ddot{2}\hat{3}} &= -k_5'\mathcal{B}_{\ddot{1}} + k_5''\mathcal{B}_{\ddot{3}} \\ \mathcal{B}^{\ddot{3}\hat{3}} &= -(k_5\mathcal{B}_{\ddot{1}} + k_5''\mathcal{B}_{\ddot{2}}) \\ k_5' &= -\frac{b^2}{2}\frac{k'm'' - k''m'}{c\Delta_S} \end{aligned}$$

and k_5, k_5' can be chosen arbitrarily. However, a choice that is SU(2) symmetric is $k_5 = -k_5'$ and $k_5'' = k_5'$.

The curvature two-forms

(9.13)
$$\begin{aligned} R^{\alpha\beta} &= d\mathcal{B}^{\alpha\beta} - \mathcal{B}^{\alpha\gamma}\wedge\mathcal{B}_\gamma{}^\beta - \mathcal{B}^{\alpha\ddot{m}}\wedge\mathcal{B}_{\ddot{m}}{}^\beta - \mathcal{B}^{\alpha\hat{3}}\wedge\mathcal{B}_{\hat{3}}{}^\beta - \mathcal{B}^{\alpha\bar{\beta}}\wedge\mathcal{B}_{\bar{\beta}}{}^\beta \\ \mathcal{R}^{\alpha\ddot{m}} &= d\mathcal{B}^{\alpha\ddot{m}} - \mathcal{B}^{\alpha\gamma}\wedge\mathcal{B}_\gamma{}^{\ddot{m}} - \mathcal{B}^{\alpha\ddot{n}}\wedge\mathcal{B}_{\ddot{n}}{}^{\ddot{m}} - \mathcal{B}^{\alpha\hat{3}}\wedge\mathcal{B}_{\hat{3}}{}^{\ddot{m}} - \mathcal{B}^{\alpha\bar{\beta}}\wedge\mathcal{B}_{\bar{\beta}}{}^{\ddot{m}} \\ \mathcal{R}^{\ddot{m}\ddot{n}} &= d\mathcal{B}^{\ddot{m}\ddot{n}} - \mathcal{B}^{\ddot{m}\gamma}\wedge\mathcal{B}_\gamma{}^{\ddot{n}} - \mathcal{B}^{\ddot{m}\ddot{\ell}}\wedge\mathcal{B}_{\ddot{\ell}}{}^{\ddot{n}} - \mathcal{B}^{\ddot{m}\hat{3}}\wedge\mathcal{B}_{\hat{3}}{}^{\ddot{n}} - \mathcal{B}^{\ddot{m}\bar{\beta}}\wedge\mathcal{B}_{\bar{\beta}}^{\ddot{n}} \end{aligned}$$

are given by

(9.14)

$$R^{\alpha\beta}\Big|_{\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}} = \Big[\frac{a^2}{4\sqrt{3}c}\frac{\ell'm''-\ell''m'}{\Delta_S}(f^{\alpha\gamma\delta}f^6{}_\gamma{}^\beta - f^{6\alpha\gamma}f_\gamma{}^{\beta\delta})\Omega_\delta\wedge\mathcal{B}_{\hat{3}} + \frac{a^2}{12c^2}\frac{(\ell'm''-\ell''m')^2}{\Delta_S^2}f^{6\alpha\gamma}f^{6\beta\delta}\Omega_\gamma\wedge\Omega_\delta - \frac{a^2}{2}f^{\alpha\bar{\beta}\beta}f^{\bar{\beta}}{}_{\delta\epsilon}\Omega^\delta\wedge\Omega^\epsilon - \mathcal{B}^{\alpha\bar{\beta}}\wedge\mathcal{B}_{\bar{\beta}}{}^\beta\Big]\Big|_{\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}}$$

$$\mathcal{R}^{\alpha\ddot{m}}\Big|_{\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}} = -\Big[\frac{a^2}{2\sqrt{3}c}\frac{\ell'm''-\ell''m'}{\Delta_S}f^{6\alpha\gamma}\mathcal{B}_\gamma\wedge\mathcal{B}_{\hat{3}}{}^{\ddot{m}} - \frac{b^2}{\sqrt{3}c}\frac{\ell'm''-\ell''m'}{\Delta_S}f^{6\alpha\bar{\beta}}\mathcal{B}_{\bar{\beta}}\wedge\mathcal{B}_{\hat{3}}{}^{\ddot{m}}\Big]\Big|_{\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}}$$

$$\mathcal{R}^{\ddot{1}\ddot{3}}\Big|_{\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}} = \Big[-\frac{1}{4}\Omega^{\ddot{3}}\wedge\Omega^{\ddot{1}} - \frac{k_5}{2\sqrt{3}c}\frac{\ell'm''-\ell'm''}{\Delta_S}f^6{}_{\beta\gamma}\Omega^\beta\wedge\Omega^\gamma + \frac{k'm''-k''m'}{c\Delta_S}\Omega^{\ddot{1}}\wedge\Omega^{\ddot{2}} + \frac{1}{2}(k_5'\Omega^{\ddot{1}} - k_5''\Omega^{\ddot{3}})\wedge\mathcal{B}^{\hat{3}}\Big]\Big|_{\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}}.$$

The components of the curvature tensor are

$$R^{\alpha\beta}{}_{\delta\epsilon} = a^2\Big[-\frac{1}{2}f^{\alpha\bar{\beta}\beta}f_{\bar{\beta}\delta\epsilon} + \frac{a^2}{12c^2}\frac{(\ell'm''-\ell''m')^2}{\Delta_S^2}f^{6\alpha\beta}f^6{}_{\delta\epsilon} + \frac{a^2}{24c^2}\frac{(\ell'm''-\ell''m')^2}{\Delta_S^2}(f^{6\alpha}{}_\delta f^{6\beta}{}_\epsilon - f^{6\alpha}{}_\epsilon f^{6\beta}{}_\delta)\Big] \quad \delta,\epsilon\neq 6$$

$$R^{\alpha\beta}{}_{\delta 6} = \frac{a^2}{2}\Big[-f^{\alpha\bar{\beta}\beta}f_{\bar{\beta}\delta 6} + \frac{a^2}{12c^2}\frac{(\ell'm''-\ell''m')^2}{\Delta_S^2}(f^{\alpha\gamma}{}_\delta f^\beta{}_{6\gamma} - f_6{}^{\alpha\gamma}f_\gamma{}^\beta{}_\delta) - \frac{a^2}{12c^2}\frac{(\ell'm''-\ell''m')^2}{\Delta_S^2}(f^{\alpha\gamma}{}_6 f_{\delta\gamma}{}^\beta - f_\delta{}^{\alpha\gamma}f_\gamma{}^\beta{}_6)\Big]$$

$$R^{\alpha\beta}{}_{\delta\ddot{3}} = \frac{a^2}{8\sqrt{3}c^2}\frac{(k'm''-k''m')(\ell'm''-\ell''m')}{\Delta_S^2}(f^{\alpha\gamma\delta}f^6{}_\gamma{}^\beta - f^{6\alpha\gamma}f_\gamma{}^{\beta\delta})$$

$$
\begin{aligned}
R^{\alpha\ddot{1}}{}_{\gamma\ddot{3}} &= \frac{k_5 a^2}{4\sqrt{3}c}\frac{\ell' m'' - \ell'' m'}{\Delta_S} f^{6\alpha}{}_{\gamma} \\
R^{\alpha\ddot{1}}{}_{\gamma\ddot{2}} &= \frac{k_5' a^2}{4\sqrt{3}c}\frac{\ell' m'' - \ell'' m'}{\Delta_S} f^{6\alpha}{}_{\gamma} \\
R^{\alpha\ddot{3}}{}_{\gamma\ddot{1}} &= -\frac{k_5 a^2}{4\sqrt{3}c}\frac{\ell' m'' - \ell'' m'}{\Delta_S} f^{6\alpha}{}_{\gamma} \\
R^{\alpha\ddot{3}}{}_{\gamma\ddot{2}} &= -\frac{k_5'' a^2}{4\sqrt{3}c}\frac{\ell' m'' - \ell'' m'}{\Delta_S} f^{6\alpha}{}_{\gamma} \\
R^{\ddot{1}\ddot{3}}{}_{\beta\gamma} &= -\frac{k_5}{4\sqrt{3}c}\frac{\ell' m'' - \ell'' m'}{\Delta_S} f^{6}{}_{\beta\gamma} \\
R^{\ddot{1}\ddot{3}}{}_{\ddot{1}\ddot{3}} &= \frac{1}{8} - \frac{1}{4}k_5'\frac{k' m'' - k'' m'}{c\Delta_S} \\
R^{\ddot{1}\ddot{3}}{}_{\ddot{1}6} &= \frac{k_5'}{4\sqrt{3}c}\frac{\ell' m'' - \ell'' m'}{\Delta_S} \\
R^{\ddot{1}\ddot{3}}{}_{\ddot{3}6} &= -\frac{k_5''}{4\sqrt{3}c}\frac{\ell' m'' - \ell'' m'}{\Delta_S}.
\end{aligned}
$$

Then

(9.15)

$$
\begin{aligned}
\sum_{\beta=1}^{5} R^{\alpha\beta}{}_{\delta\beta} = -\sum_{\beta=1}^{5}\frac{a^4}{24c^2}\frac{(\ell' m'' - \ell'' m')^2}{\Delta_S^2} f^{6\alpha\beta} f^{6\beta}{}_{\delta} - \frac{a^2}{2}\sum_{\beta=1}^{5} f^{\alpha\bar{\beta}\beta} f_{\bar{\beta}\delta\beta} \\
+ \frac{a^4}{12c^2}\sum_{\beta=1}^{5}\frac{(\ell' m'' - \ell'' m')^2}{\Delta_S^2} f^{6\alpha\beta} f^{6}{}_{\alpha\beta} \\
= -\frac{a^4}{8c^2}\sum_{\beta=1}^{5}\frac{(\ell' m'' - \ell'' m')^2}{\Delta_S^2} f^{6\alpha}{}_{\beta} f^{6\beta}{}_{\delta} - \frac{a^2}{2}\sum_{\beta=1}^{5} f^{\alpha\bar{\beta}\beta} f_{\bar{\beta}\delta\beta}
\end{aligned}
$$

and

(9.16)
$$R^{\alpha 6}{}_{\delta 6} = \frac{a^4}{24c^2}\frac{(\ell' m'' - \ell'' m')^2}{\Delta_S^2} f^{6}{}_{\alpha\gamma} f_{6\delta}{}^{\gamma} - \frac{a^2}{2} f^{\alpha\bar{\beta}6} f_{\bar{\beta}\delta 6}$$

such that

(9.17)
$$R^{\alpha}{}_{\delta} = \frac{a^2}{2} f^{\alpha\bar{\beta}\gamma} f_{\delta\bar{\beta}\gamma} - \frac{a^4}{6c^2}\frac{(\ell' m'' - \ell'' m')^2}{\Delta_S^2} f^{6\alpha}{}_{\gamma} f^{6\gamma}{}_{\delta}.$$

For a Freund-Rubin solution, the coefficients of the Ricci tensors are identical

(9.18)
$$-\frac{a^4}{6c^2}\frac{(\ell' m'' - \ell'' m')^2}{\Delta_S^2}\frac{f^{6\alpha}{}_{\gamma}f^{6\gamma}{}_{\alpha}}{6} + \frac{a^2}{2}\frac{f^{\alpha\bar{\beta}\gamma}f_{\alpha\bar{\beta}\gamma}}{6} = \frac{1}{8} - \frac{k_5'}{4}\frac{k'm'' - k''m'}{c\Delta_S}.$$

Cross terms in the block form of the Riemann curvature tensor are eliminated if $\ell' m'' - \ell'' m' = 0$. This condition becomes

$$k'm'' - k''m' = \frac{c}{b}\Delta_S\left[\frac{2}{3}a^2 f^{\alpha\bar{\beta}\gamma}f_{\alpha\bar{\beta}\gamma} - 1\right]^{\frac{1}{2}}. \tag{9.19}$$

Because $\frac{k'}{k''} \neq \frac{m'}{m''}$, the $U(1)'$ and $U(1)''$ factors are inequivalent and neither of these factors can be eliminated. The eight-dimensional space continues to be the internal symmetry manifold for the compactification of a twelve-dimensional theory.

The holonomy matrix would have the form

$$C_{\hat{A}\hat{B}} = C_{\hat{A}\hat{B}\hat{C}\hat{D}}\gamma^{\hat{C}\hat{D}} = (R_{\hat{A}\hat{B}\hat{C}\hat{D}} - \nu\delta_{\hat{A}\hat{B}\hat{C}\hat{D}})\gamma^{\hat{C}\hat{D}} \tag{9.20}$$

for some ν, based on the curvature tensor components of the coset space in (9.15), and the rank generally equals the dimension, such that no supersymmetry remains. A special embedding has been found to preserve $N = 1$ supersymmetry [23].

It is acceptable for the compact space in a solution not to have spinors satisfying the holonomy condition because supersymmetry provides a mechanism for the unification of the gauge couplings [22][63], the cancelation of divergences [37] and it is present for certain non-singular cosmological backgrounds during the inflationary epoch [20], but it has not been experimentally observed in conjunction with internal symmetries. The consequences of supersymmetry arise as a result of properties of a Lagrangian or field equations, but these constraints are not necessarily required of a phenomenologically realistic solution.

A solution to the leading-order heterotic string effective field equations is $\mathfrak{M}_4 \times G_2/SU(3)$, where $\mathfrak{M}_4$ is a four-dimensional Lorentzian space-time with a negative cosmological constant [35]. The compactification radius of the coset manifold $G_2/SU(3)$ has been calculated with the condition of three generations of fermions [23]. The formula for the number of generations is different from half of the Euler characteristic of the compact space when torsion is included. Adding two dimensions and removing a second timelike coordinate, there would be a ground state $\mathfrak{M}_4 \times G_2/SU(3) \times SU(2)/U(1)$, with the two extra coordinates describing S^2, which satisfy Freund-Rubin conditions that represent an abbreviated set of curvature equations of a higher-dimensional theory. The product $G_2/SU(3) \times SU(2)/U(1)$ is an example of the coset space $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$ with the embedding parameters of $U(1)$ and $U(1)''$ in the isometry group being equal. The compact coset space is the Euclidean section of $\frac{G_2\times SU(1,1)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$. Although $SU(1,1)/U(1)$ is the space of scalar fields in twelve-dimensional supergravity, its curvature is not positive, and the Freund-Rubin conditions cannot be satisfied for each of the eight coordinates. It may be verified that the gravitational equations of (10,2) supergravity including projection operators that remove matrix combinations related to the null vectors in the extra two dimensions while the constraints include vanishing of several Riemann tensor components along one of these null directions. A projection of the negative curvature of $SU(1,1)/U(1)$ from the field equations allows the Freund-Rubin conditions to be satisfied by the coordinates of $\mathcal{M}_4$ and the six-dimensional compact space $G_2/SU(3)$. Both twelve-dimensional spaces, with the extra coordinates describing $SU(1,1)/U(1)$ and $SU(2)/U(1)$, can be projected to $\mathcal{M}_4\times G_2/SU(3)$ as a ten-dimensional ground state for heterotic string theory.

The embedding of the heterotic superalgebra in the twelve-dimensional superalgebra allows the specialization of the vacuum to that of heterotic string theory. Because the eleven-dimensional superalgebra in the space-time of signature (10,1) is not contained in the twelve-dimensional superalgebra of signature (10,2), there is no direct relation between the corresponding supergravity theories. However, an indirect connection between solutions to the field equations might be established. Viewing the coset manifold as $G_2/SU(3)\times SU(2)/U(1)$, the solution with $N=1$ supersymmetry is found to have topology $S^6\times S^2$. The manifold L^{klm} is known to be a solution of a twelve-dimensional gravity theory that has not been completed to include fermionic fields, and yet, it admits two spinors solving the holonomy condition. It has a topology $M_5^k\times S^2\times S^1$, where $M_5^k\times S^2$ belongs to the class of manifolds $M_{k,\ell}=\frac{SU(3)\times SU(2)}{SU(2)\times U(1)}$, characterized by the Chern numbers of a $U(1)$ bundle over complex projective space components. There is an approximate interpolation between the two solutions since $M_{k,\ell}$ are S^1 orbit spaces of $S^5\times S^3$, whereas the suspension map can be used to produce the first manifold, $S(S^5\times S^3)/S^1=((S^1\wedge S^5)\times S^3)/S^1\simeq S^6\times S^2$ if the S^3 component is selected to be factored by S^1. Furthermore, it is possible to define the contraction of the groups in the coset spaces such that the particle content derived from $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$ is equivalent to that of the union of the spectrum found for $\frac{SU(3)\times SU(2)\times U(1)}{SU(2)\times U(1)'\times U(1)''}$ and the antiparticles.

The twelve-dimensional supergravity equations of motion also provide a background for the heterotic string in the Green-Schwarz formalism because κ-invariance can be generalized to higher dimensions. The constraint action, which is necessary for κ-invariance, yields conditions on the fields representing the supercoordinates of the string which are viewed as field equations because of the validity of the derivatives of the equations on the worldsheet. This last result suggests that solutions to the equations of the twelve-dimensional theory introduce two extra coordinates parameterizing a worldsheet.

Given that the supergravity action is the leading-order part of the field-theory limit of F-theory, there would be two types of quantum corrections. First, the effective field theory can be expanded to include higher-order curvature terms. Secondly, the classical solution would have an S^2 component, while higher-genus surfaces shall arise from quantum effects including string loop corrections.

The path integral over twelve-dimensional metrics contains fluctuations in two dimensions that would be confined to smooth Riemann surfaces only if the conformal invariance of the two-dimensional theory is preserved. It has been established that the vanishing of the β-function, necessary for the conformal invariance of the σ-model, yields the equations for each of the fields including the metric that can be derived from an effective action in the target space-time. Since this property follows from quantization in the two additional dimensions, it is equivalent to the feasibility of projecting the solutions to the equations of motion from twelve-dimensional to ten-dimensional effective field theory, even at higher orders in the string loop expansion.

10 The weighting of the compactified solution

The choice of the compact space would depend on the weighting factor in the path integral over all smooth ten-dimensional metrics after a reduction by two dimensions. A solution to the equations of motion extremizes the action, and the least action yields the greatest weighting factor. The string effective action may be used in the absence of a string field theory.

One form of the leading-order part of the heterotic string effective Lagrangian [40] is

$$L_{2pt} = \phi^{-3}\left[\frac{1}{2}R - \frac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho} + \frac{9}{2}(\phi^{-1}\partial_\mu\phi)(\phi^{-1}\partial^\mu\phi)\right] \tag{10.1}$$

Vanishing of the variation of the gravitino gives

$$[D_\mu, D_\nu]\epsilon = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\epsilon = \frac{1}{256}\gamma^{\mu\nu}H^2\epsilon \tag{10.2}$$

where the factor of $e^{4\phi}$ has been removed because of the choice of the variables in the effective action. Since H_{mnp} is an antisymmetric tensor and

$$\gamma^m\gamma^n\gamma^p\gamma^q\gamma^r\gamma^s = g^{pq}g^{nr}g^{ms} + g^{pq}g^{nr}g^{ms} + g^{pq}\gamma^m\gamma^{nr}\gamma^s + \gamma^m\gamma^n\gamma^{pq}\gamma^r\gamma^s$$

the component of H^2 proportional to the identity matrix is

$$H_{mnp}H_{qrs}g^{pq}g^{nr}g^{ms} = H_{mnp}H^{pnm} = -H_{mnp}H^{mnp} \tag{10.3}$$

For a compactification with the six-dimensional compact space $G_2/SU(3)$, $R = \frac{8}{R_0^2}$, where R_0 is the radius. The metric of the four-dimensional space-time is maximally symmetric with the curvature tensor being $R_{\mu\alpha\nu\beta} = \lambda(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\alpha\nu})$, [35], such that

$$\begin{aligned} \lambda &= -\frac{12A^2}{(6A+B)^2R_0^2} \\ A &= \frac{1}{32g_{10}^2}\left(\sqrt{2} - \frac{c}{8}\right) \\ B &= \frac{1}{32g_{10}^2}(-12\sqrt{2} + c) \\ c &= -24\sqrt{2} \end{aligned} \tag{10.4}$$

after setting $\kappa = 1$. It follows that $\lambda = -\frac{4}{3}\frac{1}{R_0^2}$ and $R_{AdS_4} = 12\lambda = -\frac{16}{R_0^2}$.

It has been established that

$$H_{\alpha\beta\gamma} = \frac{1}{8(6A+B)R_0}c_{\alpha\beta\gamma}, \tag{10.5}$$

where $c_{\alpha\beta\gamma}$ are the structure constants of G_2. Since $c_{\alpha\beta\gamma}c^{\alpha\beta\gamma} = 8$,

$$H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} = \frac{1}{2304R_0^2}. \tag{10.6}$$

Then

$$\begin{aligned} I_{G_2/SU(3)} &= \int d^{10}x\sqrt{-g^{(10)}(x)}\phi^{-3}\left[\frac{1}{2}\left(-\frac{16}{R_0^2}\right) + \frac{1}{2}\left(\frac{8}{R_0^2}\right) - \frac{3}{4}\frac{1}{2304R_0^2}\right] \\ &= -\frac{12289}{2304R_0^2}\int d^{10}x\sqrt{-g^{(10)}(x)}\phi^{-3}. \end{aligned} \tag{10.7}$$

As the action on M^{10} vanishes, there is a greater weighting for the compactified metric.

11 Conclusions

From formulations of the standard model and experimental measurements of the CKM matrix, the form of the spinor space may be deduced. The automorphism group of this direct sum of products of division algebras then yields the isometry group of an eight-dimensional coset space $\frac{G_2\times SU(2)\times U(1)}{SU(3)\times U(1)'\times U(1)''}$, which has been examined in connection with the unified field theory. It is consistent with the unification of supersymmetric string models in a fundamental twelve-dimensional theory. By contrast with most Calabi-Yau manifolds of non-zero genus, the coset space has a high degree of symmetry. One component of the solution to the effective field equations is topologically the sphere S^6, which is consistent with the $\frac{1}{4}$ diffeomorphism theorem in differential topology and the smoothing methods in kinetic theory. Furthermore, the phenomenologically viable gauge invariances are derived directly from the isometry group of the space, without introducing the large exceptional groups.

The metric is not as symmetric as that of the round six-sphere. The relevance of this compactification is verified by a computation of the weighting factor for the solution to the heterotic string effective field equations with $G_2/SU(3)$. Since the energy is found to be negative, the weighting factor in the Euclidean path integral be increased relative to other spaces. There also may be a transition from the coset space to the round sphere metric, which generates a truncation of the higher-order terms in the effective action which vanish for symmetric spaces. Finally, it has been found that the current particle physics phenomenology can be theoretically explained through this coset space and the reduction sequences.

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