Quasi-boundary value method for an ill-posed problem for the homogeneous biharmonic equation

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Abstract. In this paper, we propose a quasi boundary value method for an homogeneous biharmonic equation in a rectangular domain. It is known that the problem for the Biharmonic equations is severely ill-posed in the sense of Hadamard [12], i.e., the solution does not depend continuously on the given data. Convergence estimates for the regularized solutions are obtained under a priori and a posteriori bound assumptions for the exact solution. Some numerical results are given to show the effectiveness of the proposed method.

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1 Introduction

Let $\Omega = \{(x,t) : 0 < x < \pi,\ 0 < t < T\}$ and $A = \Delta^2$ be a bi-Laplace operator. For $T > 0$, $h \in L^2(0,\pi)$, consider the problem of solving the biharmonic equation, denoted briefly as (BHP),

$$Au \equiv \Delta^2 u = u_{tttt}(x,t) + 2u_{ttxx}(x,t) + u_{xxxx}(x,t) = 0, \quad (x,t) \in \Omega, \tag{1.1}$$

satisfying boundary conditions in the spatial variable x, namely

$$u|_{x=0} = 0,\ \Delta u|_{x=0} = 0, \quad u|_{x=\pi} = 0,\ \Delta u|_{x=\pi} = 0, \tag{1.2}$$

and the following boundary conditions in the variable t :

$$u|_{t=0} = h(x),\ \frac{\partial u}{\partial t}|_{t=0} = 0, \quad 0 \le x \le \pi, \tag{1.3}$$

$$\Delta u|_{t=0} = 0,\ \frac{\partial \Delta u}{\partial t}|_{t=T} = 0, \quad 0 \le x \le \pi. \tag{1.4}$$

The theory of boundary value problems for second order elliptic operators on Lipschitz domains is a well-developed subject. It has received a great deal of study in the

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past decades and while some important open questions remain, well-posedness of the Dirichlet, Neumann, and regularity problems in L^p and other function spaces has been extensively studied in the full generality of divergence form operators $-divA\nabla$ with bounded measurable coefficients.

The theory for elliptic equations of order greater than two is much less well developed [11]. Such equations are common in physics and in engineering design, they naturally appear in many areas of mathematics too, including conformal geometry [6], and non-linear elasticity [7], [1]. The prototypical example of a higher-order elliptic operator, well known from the theory of elasticity, is the bi-Laplacian $\Delta^2 = \Delta(\Delta)$; a more general example is the polyharmonic operator Δ^m, $m \geq 2$. The biharmonic equation arises in many engineering applications such as the deformation of thin plates, the motion of fluids, free boundary problems, and non-linear elasticity see [19, 1]. For historical details we refer to [22, 3, 11, 5]. For a more elaborate history of the biharmonic problem and the relation with elasticity from an engineering point of view one may consult a survey of Meleshko [19].

The well-posed biharmonic problem has attracted a great interest in the past years, and there is an extensive mathematical literature on studies of the biharmonic operator acting in $L^2(\Omega)$ for particular regions $\Omega \subset \mathbb{R}^2$. On the contrary, much less is known on the ill-posed biharmonic problem in a rectangular domain. For example in [13] the authors obtained a necessary and sufficient condition for well-posedness for the biharmonic equation in a rectangular domain in the space $L^2(\Omega)$. There was considered [17] a non-local boundary value problem for the biharmonic equation in a disk. However, these authors do not investigate the error estimates.

The method of non-local boundary value problems for the second order elliptic equations has been used by several authors, such as [24, 8, 20, 9, 25, 4, 2]. However, the use of this method remains quite scarce in the case of biharmonic problems. We note also that this method has been successfully applied to ill-posed parabolic problems.

In the present paper we will present a non-local boundary value problem method to construct stable approximate solutions to the problems (1.1), when we replace the initial conditions $u(x, 0) = h(x)$ in (1.3) with

$$u(x, 0) + \alpha u(x, T) = h(x),$$

where $\alpha > 0$ plays a role of regularization parameter. We suggest a priori and a posteriori strategies for choosing the parameter α in our regularization methods for (BHP).

The paper is organized as follows. In the next Section 2 we present some preliminaries and basic results. In Section 3, the quasi-boundary value method will be given and convergence estimates are established under a priori and a posteriori regularity assumptions on the problem data. Numerical results are shown in Section 4. Finally, conclusions are presented in Section 5.

2 Preliminaries and basic results

Definition 2.1. We call a function $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ satisfying equation (1.1) and the boundary conditions (1.2)-(1.4) a classical solution to problem (BHP).

For notational convenience and simplicity, we set

$$\mathcal{C}_{p,q}=\left\{h\in H:\ \|h\|_{p,q}^2=\sum_{k=1}^{\infty}\frac{e^{2pkT}}{k^q}c_k^2(h)<+\infty\right\}\quad p\geq 0,\ q\geq 0. \tag{2.1}$$

By the definition of $\mathcal{C}_{p,q}$ we have the following topological inclusions

$$\mathcal{C}_{p_2,q}\subseteq\mathcal{C}_{p_1,q},\quad p_2\geq p_1.$$

The following technical lemmas play the key role in our analysis and calculations.

Lemma 2.1. *Let*

$$[1,+\infty[\ni z\mapsto\mathcal{R}(z)=\frac{1}{\alpha z^r+2e^{-zT}}, \tag{2.2}$$

where $\alpha>0,\ T>0,$ *and* $r\geq 1$. *Then one has*

$$\mathcal{R}(z)\leq\frac{1}{\alpha}\left(\frac{\ell_1}{\ln(\ell_2(1/\alpha))}\right)^r, \tag{2.3}$$

where $\ell_1=rT$, *and* $\ell_2=2(T)^r/r$.

Proof. Differentiating the function $\mathcal{R}(z)$ with respect to z yields

$$\mathcal{R}'(z)=\frac{-1}{(\alpha z^r+2e^{-zT})^2}(\alpha rz^{r-1}-2Te^{-zT}).$$

Thus $\mathcal{R}'(z)=0$ when

$$\widehat{z}=\left\{z\mapsto\alpha rz^{r-1}\right\}\cap\left\{z\mapsto 2Te^{-zT}\right\}. \tag{2.4}$$

Therefore

$$\mathcal{R}(z)\leq\mathcal{R}(\widehat{z})\leq\frac{1}{\alpha\widehat{z}^r+2e^{-\widehat{z}T}}\leq\frac{1}{\alpha\widehat{z}^r}. \tag{2.5}$$

Now we have

$$\alpha r\widehat{z}^{r-1}-2Te^{-\widehat{z}T}\Leftrightarrow\frac{2T}{\alpha r}=\widehat{z}^{r-1}e^{\widehat{z}T}.$$

By using the inequality ($e^s\geq s,\ s\geq 0$), then for $s=\widehat{z}T$, we obtain $e^{\widehat{z}T}\geq\widehat{z}T$ and we can write

$$\frac{2T}{\alpha r}\leq e^{\widehat{z}T}\left(\frac{e^{\widehat{z}T}}{T}\right)^{r-1}\leq T^{1-r}e^{r\widehat{z}T},$$

which implies that

$$\widehat{z}\geq\frac{1}{rT}\ln\left(\frac{2T^r}{\alpha r}\right). \tag{2.6}$$

Hence, we obtain

$$\mathcal{R}(z)\leq\frac{1}{\alpha}\left(\frac{\ell_1}{\ln(\ell_2(1/\alpha))}\right)^r, \tag{2.7}$$

where $\ell_1=rT$, and $\ell_2=2(T)^r/r$. □

The following lemma is required in our proof and its proof can be found in [23] or [13].

Lemma 2.2. *By A we denote an operator which is a closure in $L^2(\Omega)$ of the operator given by the differential expression*

$$Au \equiv u_{tttt}(x,t) + 2u_{ttxx}(x,t) + u_{xxxx}(x,t), \quad (x,t) \in \Omega, \tag{2.8}$$

on the linear manifold of functions $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ satisfying the boundary conditions (1.2-1.4). Then the operator A is symmetric and positive.

2.1 Instability of the solution

We will construct solution $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ of the equation (1.1) satisfying the boundary conditions (1.3)-(1.4), in the form

$$u(x,t) = \sum_{k=1}^{\infty} \omega_k(x)\varphi_k(t) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \varphi_k(t)\sin(kx), \tag{2.9}$$

where $\omega_k(x) = \sqrt{\frac{2}{\pi}}\sin(kx)$, $k = 1, 2, ...$ denote an orthonormal basis in $L^2(0,\pi)$.

We construct a formal solution to (BHP). Substituting (2.9) to equation (1.1) and to the boundary conditions (1.2)-(1.4), for $\varphi_k(t)$ we obtain the problems

$$\varphi_k^{(4)}(t) - 2k^2\varphi_k^{''}(t) + k^4\varphi_k(t) = 0, \quad 0 < t < T, \tag{2.10}$$

$$\varphi_k(0) = c_k(h), \quad \varphi_k^{'}(0) = 0, \tag{2.11}$$

$$\varphi_k''(0) - k^2\varphi_k(0) = 0, \quad \varphi_k^{(3)}(T) - k^2\varphi_k^{'}(T) = 0. \tag{2.12}$$

where $c_k(h)$ is the Fourier coefficient of the expansion according to the orthonormal basis $\{\omega_k(x)\}_{k=1}^{\infty}$ of the function h.

$$h(x) = \sum_{k=1}^{\infty} c_k(h)\omega_k(x) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} c_k(h)\sin(kx).$$

Equation (2.10) has a general solution

$$\varphi_k(t) = (C_1 t + C_2)e^{kt} + (C_3 t + C_4)e^{-kt}.$$

We satisfy this solution to the boundary conditions (2.11), (2.12). Then we get the system of linear equations

$$\begin{cases} C_2 + C_4 & = c_k(h) \\ C_1 + kC_2 + C_3 - kC_4 & = 0 \\ C_1 - C_3 & = 0 \\ C_1 e^{kT} + C_3 e^{-kT} & = 0. \end{cases} \tag{2.13}$$

In fact, by direct calculation it is easy to make sure that a function

$$u(x,t)=\sqrt{\frac{2}{\pi}}\sum_{k=1}^{\infty}\cosh(kt)c_k(h)\sin(kx), \tag{2.14}$$

is the solution to (BHP). It is easy to see that the boundary data $h(x)=\sqrt{\frac{2}{\pi}}\frac{\sin(kx)}{k}$ tend to zero at $k\to\infty$ but the solution $u(x,t)$ does not tend to zero in any norm. Consequently, the considered (BHP) is ill-posed in the sense of Hadamard [12].

2.2 A criterion of existence of a solution to problem (BHP)

The most suitable notion for demonstrating conditions of stability is the notion of a strong solution.

Definition 2.2. [13] We call a function $u\in L^2(\Omega)$ a strong solution to problem (BHP) if there exists a sequence of functions $u_m\in C^4(\Omega)\cap C^3(\overline{\Omega})$ satisfying the boundary conditions (1.2)-(1.4) such that u_m converge in the norm $L^2(\Omega)$ to u.

As the required sequence u_m, we choose a sequence of partial sums of the Fourier series:

$$u_m(x,t)=\sum_{k=1}^{m}\omega_k(x)\varphi_k(t). \tag{2.15}$$

If $h\in L^2(0,\pi)$, then the existence of the strong solution to problem (BHP) is equivalent to the convergence of the sequence u_m in $L^2(\Omega)$. By virtue of the Parseval equality, the convergence of the sequence u_m in $L^2(\Omega)$ is equivalent to the convergence of the numerical series

$$\sum_{k=1}^{\infty}\|\varphi_k(t)\|^2_{L^2(0,T)}<\infty. \tag{2.16}$$

The following lemma is required in our proof and its proof can be found in [13].

Lemma 2.3. *The (BHP) has a strong solution if and only if*

$$h\in\mathcal{C}_{1,1}\Leftrightarrow\sum_{k=1}^{\infty}\frac{e^{2kT}}{k}|c_k(h)|^2<\infty,$$

where $c_k(h)$ is the Fourier coefficient of the function h and its unique strong solution is given by (2.14).

3 Regularization method and convergence estimates

The main idea of the original non-local boundary value problem method [24, 8],(also called quasi-boundary value method) is to approach the considered ill-posed problem by a family of well-posed problems depending on a (small) regularization parameter.

In our work we shall regularize (BHP) by the following non-local boundary value problem

$$(P_\alpha^\delta) \quad \begin{cases} \Delta^2 u_\alpha^\delta(x,t) = 0, & (x,t) \in \Omega, \\ u_\alpha^\delta(0,t) = \Delta u_\alpha^\delta(0,t) = 0, & 0 \le t \le T, \\ u_\alpha^\delta(\pi,t) = \Delta u_\alpha^\delta(\pi,t) = 0, & 0 \le t \le T, \\ u_\alpha^\delta(x,0) + \alpha u_\alpha^\delta(x,T) = h^\delta(x), \quad \dfrac{\partial u_\alpha(x,0)}{\partial t} = 0, & 0 \le x \le \pi, \\ \Delta u_\alpha^\delta(x,0) = 0, \quad \dfrac{\partial \Delta u_\alpha^\delta(x,T)}{\partial t} = 0 & 0 \le x \le \pi, \end{cases}$$

where the initial condition $u_\alpha^\delta(0,t) = h^\delta(x)$ is replaced by the nonlocal condition

$$u_\alpha^\delta(x,0) + \alpha u_\alpha^\delta(x,T) = h^\delta(x),$$

where $\alpha > 0$ is a regularization parameter, and the measured data $h^\delta \in L^2(0,\pi)$, satisfies

$$\|h^\delta - h\|_{L^2(0,\pi)} \le \delta, \tag{3.1}$$

in which the constant $\delta > 0$ is called an error level.

We suggest a priori and a posteriori strategies for choosing the parameter α in our regularization methods for (BHP), and we show that

$$\left\|u_\alpha^\delta(.,0) - h\right\| \to 0, \quad \text{as } \alpha \to 0, \tag{3.2}$$

$$\|u_\alpha^\delta(.,t) - u(.,t)\| \to 0, \quad \text{as } \alpha \to 0. \tag{3.3}$$

Now we are ready to state and prove the main results of this paper.

3.1 A priori parameter choice rule

Theorem 3.1. *For all $h \in L^2(0,\pi)$, the function*

$$u_\alpha^\delta(x,t) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \frac{\cosh(kt)}{1 + \alpha\cosh(kT)} c_k^\delta(h) \sin(kx), \tag{3.4}$$

is the unique solution of (P_α^δ) and it depends continuously on h.

Proof. Let $u_{\alpha 1}^\delta$ and $u_{\alpha 2}^\delta$ be the solutions of problem (P_α^δ) corresponding to the data h_1^δ and h_2^δ, respectively, then we have

$$\begin{aligned} \left\|u_{\alpha 1}^\delta(.,t) - u_{\alpha 2}^\delta(.,t)\right\|^2 &= \left\|\sum_{k=1}^{\infty} \omega_k(x) \left(\frac{\cosh(kt)}{1+\alpha\cosh(kT)}\right)(c_{k1}^\delta(h) - c_{k2}^\delta(h))\right\|^2 \\ &\le \sum_{k=1}^{\infty} \left(\frac{\cosh(kt)}{1+\alpha\cosh(kT)}\right)^2 \left|c_{k1}^\delta(h) - c_{k2}^\delta(h)\right|^2 \\ &\le \sum_{k=1}^{\infty} \left(\frac{\cosh(kT)}{1+\alpha\cosh(kT)}\right)^2 \left|c_{k1}^\delta(h) - c_{k2}^\delta(h)\right|^2 \\ &\le \frac{1}{\alpha^2}\|h_1^\delta - h_2^\delta\|^2 \end{aligned} \tag{3.5}$$

□

Theorem 3.2. *For all h in $L_2(0,\pi)$, $\|u_\alpha(.,0)-h\|$ tends to zero as α tends to zero. That is $u_\alpha(.,0)$ converges to h in $L_2(0,\pi)$.*

Proof. If $h(x)=\sum_{k=1}^{\infty}\omega_k(x)c_k(h)$, then

$$\begin{aligned}\|u_\alpha(.,0)-h\|^2 &= \left\|\sum_{k=1}^{\infty}\frac{\alpha\cosh(kT)}{1+\alpha\cosh(kT)}c_k(h)\omega_k(x)\right\|^2\\ &\le \sum_{k=1}^{\infty}\frac{\alpha^2\cosh^2(kT)}{(1+\alpha\cosh(kT))^2}|c_k(h)|^2.\end{aligned}$$

Fix $\epsilon>0$. Choose N so that $\sum_{k=N+1}^{\infty}|c_k(h)|^2<\frac{\epsilon}{2}$.

$$\begin{aligned}\|u_\alpha(.,0)-h\|^2 &\le \sum_{k=1}^{N}\frac{\alpha^2\cosh^2(kT)}{(1+\alpha\cosh(kT))^2}|c_k(h)|^2\\ &+ \sum_{k=N+1}^{\infty}\frac{\alpha^2\cosh^2(kT)}{(1+\alpha\cosh(kT))^2}|c_k(h)|^2\\ &\le \alpha^2\sum_{k=1}^{N}\cosh^2(kT)|c_k(h)|^2+\sum_{k=N+1}^{\infty}|c_k(h)|^2\\ &\le \alpha^2\sum_{k=1}^{N}\cosh^2(kT)|c_k(h)|^2+\frac{\epsilon}{2}.\end{aligned}$$

Now let α be such that $\alpha^2<\epsilon\left(2\sum_{k=1}^{N}\cosh^2(kT)|c_k(h)|^2\right)^{-1}$, and we are done. □

Theorem 3.3. *If $h=\sum_{k=1}^{\infty}c_k(h)\omega_k(x)\in L^2(0,\pi)$, and $\|u(x,T)\|\le E_2$. Then, we have that $u_\alpha(x,T)$ converges to $u(x,T)$ as α tends to zero.*

Proof. Let α, $\beta>0$. Then

$$\begin{aligned}\|u_\alpha(x,T)-u_\beta(x,T)\|^2 &= \sum_{k=1}^{\infty}\left\|\frac{(\beta-\alpha)\cosh^2(kT)}{(1+\alpha\cosh(kT))(1+\beta\cosh(kT))}c_k(h)\right\|^2\\ &= \sum_{k=1}^{\infty}\frac{(\beta-\alpha)^2\cosh^4(kT)|c_k(h)|^2}{\left[1+(\alpha+\beta)\cosh(kT)+\alpha\beta\cosh^2(kT)\right]^2}\\ &= \sum_{k=1}^{\infty}\frac{(\beta-\alpha)^2|c_k(h)|^2}{\left[\cosh^{-2}(kT)+(\alpha+\beta)\cosh^{-1}(kT)+\alpha\beta\right]^2}\\ &= \sum_{k=1}^{N}\frac{(\beta-\alpha)^2|c_k(h)|^2}{\left[\cosh^{-2}(kT)+(\alpha+\beta)\cosh^{-1}(kT)+\alpha\beta\right]^2}\\ &+ \sum_{k=N+1}^{\infty}\frac{(\beta-\alpha)^2|c_k(h)|^2}{\left[\cosh^{-2}(kT)+(\alpha+\beta)\cosh^{-1}(kT)+\alpha\beta\right]^2}\\ &= I_1+I_2. \qquad (3.6)\end{aligned}$$

Let $\epsilon > 0$ and $h(x) = \sum_{k=1}^{\infty} \omega_k(x) c_k(h)$. We have that

$$\|u(x,T)\|^2 = \sum_{k=1}^{\infty} \cosh^2(kT)|c_k(h)|^2 \le E_2^2.$$

Choose N so that $\sum_{k=N+1}^{\infty} \cosh^2(kT)|c_k(h)|^2 < \frac{\epsilon}{2}$. Then

$$\begin{aligned} I_1 + I_2 &\le \sum_{k=1}^{N} (\beta-\alpha)^2 \cosh^4(kT)|c_k(h)|^2 \\ &+ \sum_{k=N+1}^{\infty} \left(\frac{\beta-\alpha}{\beta+\alpha}\right)^2 \cosh^2(kT)|c_k(h)|^2 \\ &\le \sum_{k=1}^{N} (\beta-\alpha)^2 \cosh^4(kT)|c_k(h)|^2 + \frac{\epsilon}{2}. \end{aligned} \tag{3.7}$$

Now if we choose $\gamma > 0$ so that $\gamma^2 < \epsilon \left(2 \sum_{k=1}^{N} (\beta-\alpha)^2 \cosh^4(kT)|c_k(h)|^2\right)^{-1}$ and require that α and β be less than γ, we have that

$$\|u_\alpha(x,T) - u_\beta(x,T)\|^2 < \epsilon,$$

and we are done. □

In the following Theorem 3.4, we will prove that the regularized solution u_α^δ given by (3.4) is a stable approximation to the exact solution u given by (2.9).

Theorem 3.4. *Suppose that u is the solution of (BHP) and u_α^δ is the solution of problem (P_α^δ). Let the measured data h^δ satisfy $\left\|h^\delta - h\right\| \le \delta$, and the exact solution u satisfy $\sum_{k=1}^{\infty} k^{2r} \cosh^2(kT) c_k^2(h) \le E_1^2$, $(r > 0)$. The regularization parameter α is chosen as $\alpha = \delta^\theta$, $(0 < \theta < 1)$, then for fixed $0 < t \le T$, we have the following convergence estimate*

$$\|u_\alpha^\delta(.,t) - u(.,t)\| \le \delta^{1-\theta} + 2\left(\frac{\ell_1}{\ln\left(\ell_2(1/(\delta^\theta))\right)}\right)^r E_1, \tag{3.8}$$

where $\ell_1 = rT$, and $\ell_2 = 2(T)^r/r$.

Proof. Denote u_α as the solution of problem (P_α^δ) corresponding to the exact data h. It is clear

$$\|u_\alpha^\delta - u\| \le \|u_\alpha^\delta - u_\alpha\| + \|u_\alpha - u\| \tag{3.9}$$

from Theorem 3.1, we know

$$\|u_\alpha^\delta(.,t) - u_\alpha(.,t)\|_{L^2(0,\pi)}^2 \le \frac{1}{\alpha^2}\|h^\delta - h\|^2. \tag{3.10}$$

Then,

$$\begin{aligned}
\|u(.,t)-u_\alpha(.,t)\| &= \left\|\sum_{k=1}^{\infty}\left(\cosh(kt)-\frac{\cosh(kt)}{1+\alpha\cosh(kT)}\right)c_k(h)\omega_k(x)\right\|^2 \\
&\le \sum_{k=1}^{\infty}\left(\frac{\alpha\cosh(kt)}{1+\alpha\cosh(kT)}\right)^2\cosh^2(kT)c_k^2(h) \\
&\le \sum_{k=1}^{\infty}\frac{\alpha^2e^{2kT}}{(1+\frac{\alpha}{2}e^{kT})^2}\cosh^2(kT)c_k^2(h) \\
&= 4\alpha^2\sum_{k=1}^{\infty}\frac{1}{(\alpha+2e^{-kT})^2}\frac{k^{2r}}{k^{2r}}\cosh^2(kT)c_k^2(h) \\
&\le 4\alpha^2\sum_{k=1}^{\infty}\frac{1}{(\alpha k^r+2e^{-kT})^2}k^{2r}\cosh^2(kT)c_k^2(h) \\
&= 4\alpha^2\sum_{k=1}^{\infty}\mathcal{R}^2(k)k^{2r}\cosh^2(kT)c_k^2(h) \\
&\le 4\alpha^2\frac{1}{\alpha^2}\left(\frac{\ell_1}{\ln(\ell_2(1/\alpha))}\right)^{2r}\sum_{k=1}^{\infty}k^{2r}\cosh^2(kT)c_k^2(h). \\
&\le 4\left(\frac{\ell_1}{\ln(\ell_2(1/\alpha))}\right)^{2r}E_1^2 \qquad (3.11)
\end{aligned}$$

Using (2.3), (3.9), (3.10) and (3.11), the estimate (3.8) can be obtained. □

Theorem 3.5. *If* $h(x)=\sum_{k=1}^{\infty}\omega_k(x)c_k(h)$, *and there exists an* $r>0$ *so that* $\sum_{k=1}^{\infty}k^{2r}|c_k(h)|^2\le E_2$. *The regularization parameter* α *is chosen as* $\alpha=\delta^\theta$, $(0<\theta<1)$, *then one has the following error estimate*

$$\left\|u_\alpha^\delta(.,0)-h\right\|\le\delta^{1-\theta}+2\left(\frac{\ell_1}{\ln(\ell_2(1/\delta^\theta))}\right)^r E_2. \qquad (3.12)$$

Proof. By using (3.9) and (3.10) one has

$$\left\|u_\alpha^\delta(.,0)-h\right\|\le\frac{1}{\alpha}\left\|h^\delta-h\right\|+\|u_\alpha(.,0)-h\|. \qquad (3.13)$$

$$
\begin{aligned}
\|h-u_\alpha(.,0)\| &= \left\|\sum_{k=1}^{\infty}\left(1-\frac{1}{1+\alpha\cosh(kT)}\right)c_k(h)\omega_k(x)\right\|^2 \\
&\le 4\alpha^2\sum_{k=1}^{\infty}\frac{1}{(\alpha+2e^{-kT})^2}\frac{k^{2r}}{k^{2r}}c_k^2(h) \\
&= 4\alpha^2\sum_{k=1}^{\infty}\mathcal{R}^2(k)k^{2r}c_k^2(h) \\
&\le 4\alpha^2\frac{1}{\alpha^2}\left(\frac{\ell_1}{\ln(\ell_2(1/\alpha))}\right)^{2r}\sum_{k=1}^{\infty}k^{2r}c_k^2(h). \\
&\le 4\left(\frac{\ell_1}{\ln(\ell_2(1/\alpha))}\right)^{2r}E_2^2. \qquad (3.14)
\end{aligned}
$$

Thus

$$
\left\|u_\alpha^\delta(.,0)-h\right\| \le \delta^{1-\theta}+2\left(\frac{\ell_1}{\ln(\ell_2(1/\delta^\theta))}\right)^{r}E_2.
$$

□

3.2 A posteriori parameter choice rule

Lemma 3.6. *Let $z(x,t)=u(x,t)-u_\alpha(x,t)$. Then we have*

$$
\|z(.,t)\| \le 2^{\frac{t}{T}}\|z(.,T)\|^{\frac{t}{T}}\|z(.,0)\|^{1-\frac{t}{T}}, \quad \forall t\in[0,T]. \qquad (3.15)
$$

Proof. Using Hölder's inequality, we get, for all $t\in[0,T]$,

$$
\begin{aligned}
\|z(.,t)\|^2 &= \sum_{k=1}^{\infty}\cosh^2(kt)(z(.,0),\omega_k)^2 \le \sum_{k=1}^{\infty}e^{2kt}(z(.,0),\omega_k)^2 \\
&= \sum_{k=1}^{\infty}e^{2kt}\left((z(.,0),\omega_k)^{2t/T}(z(.,0),\omega_k)^{2(1-t/T)}\right) \\
&\le \left(\sum_{k=1}^{\infty}\left(e^{2kt}\left((z(.,0),\omega_k)^{2t/T}\right)^{T/t}\right)\right)^{t/T}\left(\sum_{k=1}^{\infty}\left((z(.,0),\omega_k)^{2(1-t/T)}\right)^{T/(T-t)}\right)^{1-t/T} \\
&\le \left(\sum_{k=1}^{\infty}2\cosh^2(kT)\left((z(.,0),\omega_k)^2\right)\right)^{t/T}\left(\sum_{k=1}^{\infty}\left((z(.,0),\omega_k)^2\right)\right)^{1-t/T} \\
&= 4^{t/T}\|z(.,T)\|^{2t/T}\|z(.,0)\|^{2(1-t/T)}.
\end{aligned}
$$

Thus,

$$
\|z(.,t)\| \le 2^{\frac{t}{T}}\|z(.,T)\|^{\frac{t}{T}}\|z(.,0)\|^{1-\frac{t}{T}}, \quad \forall t\in[0,T].
$$

□

Theorem 3.7. *Suppose that $\delta<\|h\|$ and choose $\tau>1$ such that $0<\tau\delta<\|h\|$. Then there exists a unique number $\alpha(\delta)>0$ such that*

$$
\left\|u_{\alpha(\delta)}(.,0)-h(.)\right\| = \tau\delta. \qquad (3.16)
$$

Further, if u is the solution of (BHP) and u_α^δ is the solution of problem (P_α^δ). Let the measured data h^δ satisfy $\left\|h^\delta - h\right\| \leq \delta$, and the exact solution u satisfy $\|u(.,T)\| \leq E_3$. then we have the following convergence estimate

$$\|u_\alpha^{\alpha(\delta)}(.,t) - u(.,t)\| \leq 2^{1+\frac{t}{T}}\delta^{1-\frac{t}{T}}(1+\tau)\left(\frac{E_3}{\tau-1}\right)^{\frac{t}{T}}, \ \forall t \in [0,T]. \tag{3.17}$$

Proof. Set $z(.,t) = u(.,t) - u_{\alpha(\delta)}(.,t)$, $\forall t \in [0,T]$. Then

$$\|z(.,0)\| = \|u(.,0) - u_{\alpha(\delta)}(.,0)\| \leq \|u(.,0) - h\| + \|u_{\alpha(\delta)}(.,0) - h\| \leq (1+\tau)\delta, \tag{3.18}$$

$$\|z(.,T)\| \leq \|u(.,T)\| + \|u_{\alpha(\delta)}(.,T)\|. \tag{3.19}$$

From (3.16) and (3.4), we have

$$\begin{aligned}
\tau\delta &= \left\|u_{\alpha(\delta)}(.,0) - h\right\| = \left\|\alpha(\delta)u_{\alpha(\delta)}(.,T)\right\| \\
&= \left\|\sum_{k=1}^{\infty} \frac{\alpha(\delta)\cosh(kT)}{1+\alpha(\delta)\cosh(kT)})(h,\omega_k)\omega_k\right\| \\
&\leq \left\|\sum_{k=1}^{\infty} \frac{\alpha(\delta)\cosh(kT)}{1+\alpha(\delta)\cosh(kT)})(h-u(.,0),\omega_k)\omega_k\right\| \\
&+ \left\|\sum_{k=1}^{\infty} \frac{\alpha(\delta)\cosh(kT)}{1+\alpha(\delta)\cosh(kT)})(u(.,0),\omega_k)\omega_k\right\| \\
&\leq \|h-u(.,0)\| + \left\|\sum_{k=1}^{\infty} \frac{\alpha(\delta)}{\frac{\alpha(\delta)}{2}+e^{-kT}})(u(.,0),\omega_k)\omega_k\right\| \\
&\leq \delta + \left\|\sum_{k=1}^{\infty} \alpha(\delta)e^{kT}(u(.,0),\omega_k)\omega_k\right\| \\
&\leq \delta + 2\alpha(\delta)\left\|u_{\alpha(\delta)}(.,T)\right\|.
\end{aligned}$$

This implies that

$$\frac{\delta}{\alpha(\delta)} \leq \frac{2}{\tau-1}\left\|u_{\alpha(\delta)}(.,T)\right\|.$$

It follows that

$$\begin{aligned}
\left\|u_{\alpha(\delta)}(.,T)\right\| &\leq \left\|\sum_{k=1}^{\infty} \frac{\cosh(kT)}{1+\alpha(\delta)\cosh(kT)})(h-u(.,0),\omega_k)\omega_k\right\| \\
&+ \left\|\sum_{k=1}^{\infty} \frac{\cosh(kT)}{1+\alpha(\delta)\cosh(kT)})(u(.,0),\omega_k)\omega_k\right\| \\
&\leq \left\|\sum_{k=1}^{\infty} \frac{1}{\frac{\alpha(\delta)}{2}+e^{-kT}}(h-u(.,0),\omega_k)\omega_k\right\| \\
&+ \left\|\sum_{k=1}^{\infty} \cosh(kT)(u(.,0),\omega_k)\omega_k\right\| \\
&\leq \frac{2\delta}{\alpha(\delta)} + \|u(.,T)\| \leq \frac{3+\tau}{\tau-1}\|u(.,T)\|. \qquad (3.20)
\end{aligned}$$

From (3.19) and (3.20) we obtain

$$\left\|u(.,T)-u_{\alpha(\delta)}(.,T)\right\| \leq \frac{2(1+\tau)}{\tau-1}E_3. \tag{3.21}$$

It follows from (3.18), (3.20) and (holder inequality) that

$$\begin{aligned}\left\|u(.,t)-u_{\alpha(\delta)}(.,t)\right\| &\leq 2^{\frac{t}{T}}\left\|u(.,T)-u_{\alpha(\delta)}(.,T)\right\|^{\frac{t}{T}}\left\|u(.,0)-u_{\alpha(\delta)}(.,0)\right\|^{1-\frac{t}{T}}\\ &\leq 2^{1+\frac{t}{T}}\delta^{1-\frac{t}{T}}(1+\tau)\left(\frac{E_3}{\tau-1}\right)^{\frac{t}{T}}. \end{aligned} \tag{3.22}$$

□

4 Numerical illustrations

In this section, we use a numerical example to verify the stability of our proposed regularization method. Solving biharmonic problems numerically typically involves significant challenges with respect to the approximation of high-order derivatives and the imposition of double boundary conditions. For this reason we propose a semi-discrete finite difference method

We choose $T=1$, $h(x)=\sqrt{\frac{2}{\pi}}\sin(x)$ and the exact solution $u(x,t)$ is computed by solving the ill-posed problem for the biharmonic equation (BHP). It is not too hard to see that the exact solution is

$$u(x,t)=\sqrt{\frac{2}{\pi}}\cosh(t)\sin(x), \tag{4.1}$$

The equation (1.1) can be written as

$$\left(\frac{\partial^2}{\partial t^2}-L\right)^2 u(x,t)=0,$$

and it is easy to check that the operator

$$\begin{cases} \mathcal{D}(L)=\left\{u\in L^2(0,\pi)|\ u|_{x=0}=\Delta u|_{x=0}=0,\ u|_{x=\pi}=\Delta u|_{x=\pi}=0\right\}\\ L=-\frac{\partial^2}{\partial x^2}, \quad \mathcal{D}(L)\subset L^2(0,\pi), \end{cases} \tag{4.2}$$

is positive, self-adjoint with compact resolvent (L is diagonalizable). The eigenpairs (λ_k,φ_k) of L are $\lambda_k=k^2, \quad \varphi_k=\sqrt{\frac{2}{\pi}}\sin(kx),\ k\in\mathbb{N}^*$.

As proposed in section 3, instead of solving the ill-posed problem (BHP), we solve the well-posed problem (P_α^δ), where the regularization parameter α to be prescribed according to criterion (3.16).

By using the central difference with step length $\ell=\frac{\pi}{N+1}$ to approximate the first derivative u_x and the second derivative u_{xx}, we can get the following semi-discrete problem (system of ordinary differential equation):

$$(4.3)\quad \begin{cases} (\frac{d^2}{dt^2}-\mathbb{A}^N)^2 u_\alpha(x_i,t)=0, \quad x_i=ih,\ i=1,...,N,\ t\in(0,1),\\ u_\alpha(x_0=0,t)=u_\alpha(x_{N+1}=\pi,t)=0, \quad t\in(0,1),\\ \Delta u_\alpha(x_0=0,t)=\Delta u_\alpha(x_{N+1}=\pi,t)=0, \quad t\in(0,1),\\ u_\alpha(x_i,0)=h(x_i)=\sqrt{\frac{2}{\pi}}\sin(x_i),\ \frac{\partial u_\alpha}{\partial t}(x_i,0)=0, \quad x_i=ih,\ i=1,...,N,\\ \Delta u_\alpha(x_i,0)=\frac{\partial \Delta u_\alpha(x_i,T)}{\partial t}=0, \quad x_i=ih,\ i=1,...,N, \end{cases}$$

where $\mathbb{A}^N$ is the discretization matrix stemming from the operator $A=-\frac{\partial^2}{\partial x^2}$

$$\mathbb{A}^N=\frac{1}{h^2}Tridiag(-1,2,-1)\in\mathcal{M}_N(\mathbb{R}).$$

The final value of the solution of the problem (4.3) is

$$(4.4)\quad u_\alpha(x_i,1)=\frac{\cosh(\sqrt{\mathbb{A}^N})}{(\mathbb{I}_N+\alpha\cosh(\sqrt{\mathbb{A}^N}))}h(x_i),$$

where the vector $h=(h_1,h_2,...h_N)=(h(x_1),h(x_2),...h(x_N))$ represent the discrete form of the data function $h(x)$. Note that in practice, the data $h(x)$ is obtained by measurement and therefore it is inevitably contaminated by measurement error, some uniformly distributed random noises ϵ are added to h in our test examples, i.e.,

$$h^\delta=h+\epsilon randn(size(h)),$$

and ϵ indicates the noise level of the measurement data. The bound on the measurement error δ can be measured in the sense of Root Mean Square Error (RMSE) according to

$$\delta:=\|h^\delta-h\|_{l^2}=\left(\frac{1}{N+1}\sum_{i=1}^{N+1}\left(h_i-h_i^\delta\right)^2\right)^{1/2},$$

and it is easy to see that ϵ and δ posses the same order of magnitude. In order to investigate the algorithm, we evaluate the relative error Rer defined by

$$Rer=\frac{\|u_\alpha^\delta(.,1)-u(.,1)\|_{l^2}}{\|u(.,1)\|_{l^2}}.$$

For the choice of α we use the criterion (3.16) which gives the unique $\alpha(\delta)>0$ such that

$$(4.5)\quad \left\|u_{\alpha(\delta)}(.,0)-h^\delta(.)\right\|_{L^2(0,\pi)}=\alpha(\delta)\left\|u_{\alpha(\delta)}(.,T)\right\|_{L^2(0,\pi)}=\tau\delta,$$

where $\tau>1$ is a given fixed number and

$$(4.6)\quad \left\|h-h^\delta\right\|_{L^2(0,\pi)}\le\delta.$$

Table 1. The relative errors at $T=1$ for $N=31$, 36 and various noisy levels.

ϵ	$N = 31$	α	RE	ϵ	$N = 36$	α	RE
0.000		0.0162	6.4085×10^{-5}	0.000		0.0139	1.9879×10^{-5}
0.100		0.305	0.3966	0.100		0.255	0.3977
0.010		0.079	0.1272	0.010		0.075	0.1341
0.001		0.03	0.0337	0.001		0.0335	0.0374
0.0001		0.0173	0.0043	0.0001		0.015	0.0048

Table 2. The relative errors at $T = 1$ for $N = 41$, 45 and various noisy levels.

ϵ	$N = 41$	α	RE	ϵ	$N = 45$	α	RE
0.000		0.0122	3.8776×10^{-5}	0.000		0.011	9.3362×10^{-5}
0.100		0.33	0.4192	0.100		0.22	0.4859
0.010		0.12	0.1585	0.010		0.108	0.1547
0.001		0.03	0.0376	0.001		0.028	0.0385
0.0001		0.015	0.0072	0.0001		0.0142	0.0070

Numerical results are shown in figures 1,...,4, table 1 and table 2. The numerical results for exact Cauchy data $u(x, 1)$ are shown in figure 1. The numerical results for $u(., 1)$, $u_\alpha^\delta(., 1)$ and the relative errors between exact and regularization solutions with $\epsilon \in \{0.01, 0.001, 0.0001\}$ are shown in figures 2,3 and 4. Table 1 and table 2 gives the comparison of the errors between the exact and regularization solutions for $\epsilon \in \{0, 0.1, 0.01, 0.001, 0.0001\}$ at different values of N. We first observe that for the exact data $u(x, 1)$, the numerical results presented in figures 1 show a good approximation for the method. From figures. 1, 2, 3, and 4 and Tables 1 and 2, it can be observed that our proposed method is effective and stable. From Tables 1 and 2, we note that the more small ϵ is, the better the calculation effect is, which means that our proposed regularization method is convergent with respect to decreasing the noise level ϵ.

5 Conclusions

In this paper, we propose a nonlocal boundary value problem method to solve an elliptic problem for the biharmonic equation in the rectangle. The convergence and stability estimates for $0 < t \leq T$ have been obtained under a priori and a posteriori bound assumptions for the exact solution. Finally, some numerical tests show that our proposed regularization method is effective and stable.

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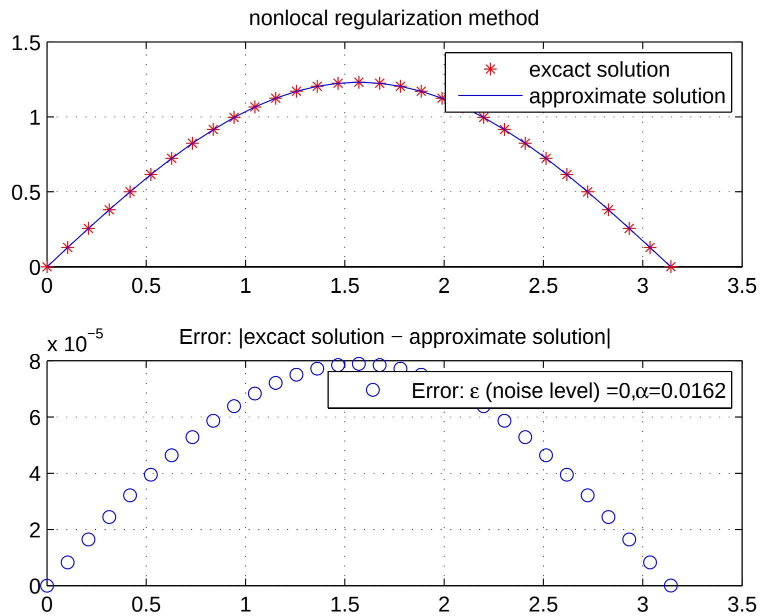
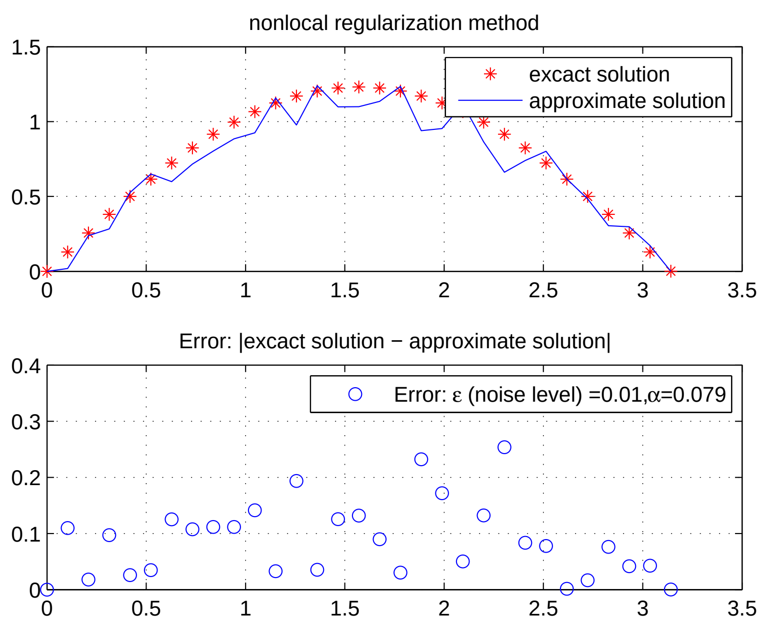


Figure 1:



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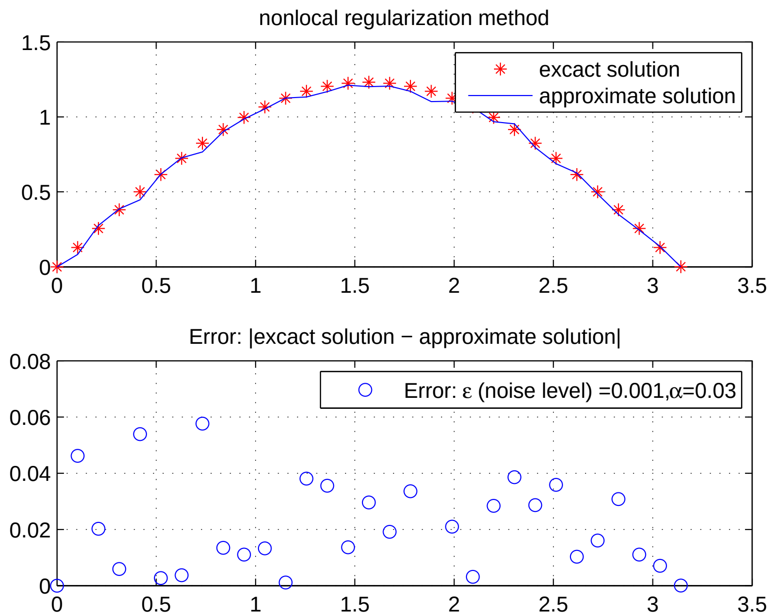

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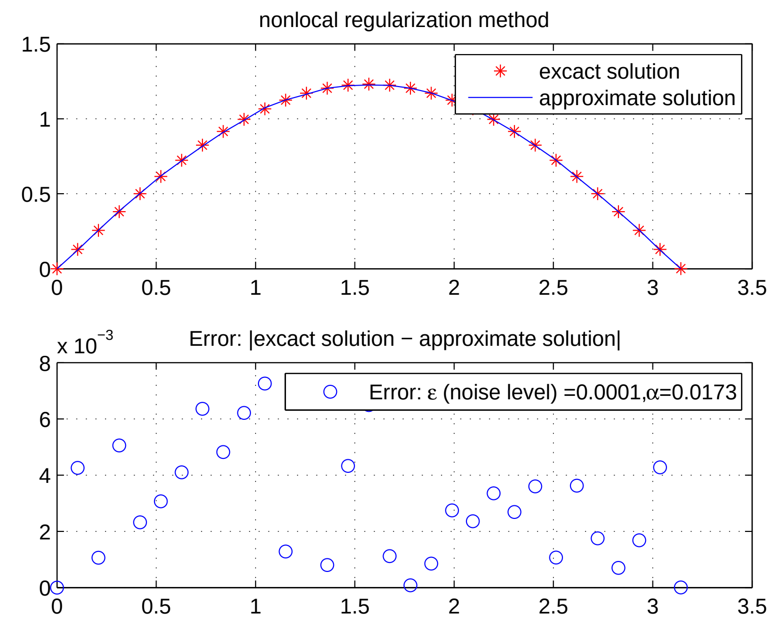

Figure 4: