A new method for calculating the chromatic polynomial

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Abstract. Chromatic polynomials are widely used in graph theoretical or chemical applications in many areas. Birkhoff-Lewis theorem is the most important tool to find the chromatic polynomial of any given graph. Here we obtain several shortcut moves to calculate this polynomial covering all graphs.

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1 Introduction

One of the well-known applications of graph theory is the 4-colour problem. There are many notions related to colourings of graphs. A colouring of a graph $G(V,E)$ is a mapping $f : V \to C$, where C is the set of colours, with $f(u) \neq f(v)$ for $uv \in E$.

If there is a colouring of G with n colours, then G is said to be n-colourable. The smallest number n for which G is n-colourable is called the chromatic number and denoted by $\chi(G)$. Clearly a graph with at least one loop cannot have any colourings. As we shall be considering simple graphs, this case will be out of question. Similarly, edge colourings are defined.

Here we shall study another aspect related to colourings, the chromatic polynomial of a graph. The chromatic polynomial of G is defined to be a function $C_G(k)$ which expresses the number of distinct k-colourings possible for the graph G for each integer $k > 0$. This number was first used by Birkhoff in 1912. Chromatic polynomials are widely used in determining several properties of graphs. Although there is no known formula for the chromatic polynomial of any given graph, there are algorithms to do that. The most well-known algorithm is the Birkhoff-Lewis theorem stated as below:

Theorem 1.1 (Birkhoff-Lewis, 1946)**.** *The chromatic polynomial of a graph G can be found by the formula*

$$C_G(k) = C_{G-e}(k) - C_{G/e}(k),$$

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where $G - e$ is the graph obtained by deleting the edge e from G, and G/e is the graph obtained from G by removing e and identifying the end vertices of e and leaving only one copy of any resulting multiple edges (see Fig. 1.1 and 1.2).

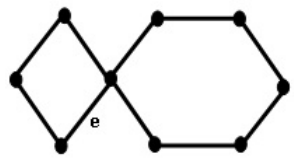

Figure 1.1 A graph G

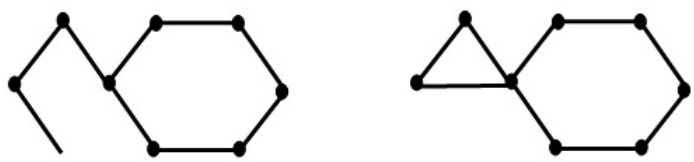

Figure 1.2 The graphs $G - e$ and G/e

As usual, we denote path, cycle, star, complete, complete bipartite and tadpole graphs by P_n, C_n, S_n, K_n, $K_{r,s}$ and $T_{r,s}$, respectively.

A bridge of a connected graph is an edge whose removal disconnects the graph. In another words, a bridge is an edge of a graph G whose removal increases the number of components of G, [5], p. 26. Also it is well known that an edge of a connected graph is a bridge iff it does not lie on any cycle. For the details of these and related notions, see [1], [7], [3], [4] and [6].

2 Calculating the chromatic polynomial by splitting the given graph

In this section, we want to obtain a new and alternative method to find the chromatic polynomial of any given graph. The idea behind this will be splitting the given graph to smaller subgraphs. We shall state results for the chromatic polynomial of a given graph G in terms of two smaller graphs G_1 and G_2. If necessary, these can be generalized to a finite number of subgraphs. We shall consider four different cases where a graph is splitted into two parts at one vertex or through an edge which is called a bridge.

In this paper we use $G_1 \cup G_2$ to denote the union of two graphs G_1 and G_2 which have one joint vertex.

One can think of starting by any graph and applying Theorem 1.1 to get to a point eventually. But these two theorems cannot be always enough for all graphs. Some more complex graphs need other splitting methods.

We first have the following useful result:

Theorem 2.1. *Let G_1 and G_2 be two graphs with no common vertex. Let also $G_1 \coprod G_2$ be the disjoint union of G_1 and G_2. Then*

$$C_G(k) = C_{G_1}(k) \cdot C_{G_2}(k).$$

Proof. As G_1 and G_2 are disjoint graphs, the colouring of G_1 is independent from the colouring of G_2. By the principal of counting, the result follows. □

This result can be generalized to the product of a finite number of graphs by mathmematical induction:

Corollary 2.2. *Let $G_1, G_2, ..., G_n$ be pairwise disjoint graphs and also let $G = \coprod_{i=1}^{n}$ be their disjoint union. Then*

$$C_G(k) = \prod_{i=1}^{n} C_{G_i}(k).$$

We now give a result which will be needed later:

Corollary 2.3. *Let G be a graph with two components, one being a unique vertex, see Figure 1.3.*

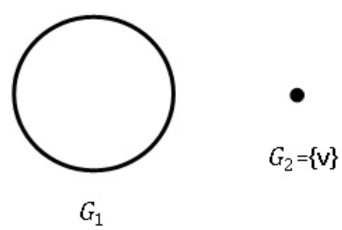

Figure 1.3 *A graph $G = G_1 \coprod \{v\}$*

Then

$$C_G(k) = k \cdot C_{G_1}(k).$$

Proof. As the colouring of $G_2 = \{v\}$ is independent from the colouring of G_1, the result follows. □

This easily can be generalized as follows:

Corollary 2.4. *Let G have r isolated vertices. That is*

$$G = G_1 \coprod \{v_1\} \coprod \cdots \coprod \{v_r\},$$

see Figure 1.4.

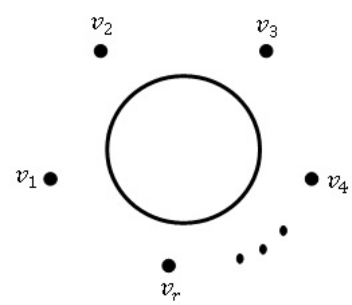

Figure 1.4 *A graph $G = G_1 \coprod \{v_1\} \coprod ... \coprod \{v_r\}$ with r isolated vertices*

Then

$$C_G(k) = k^r \cdot C_{G_1}(k).$$

Next, we consider the case where G has only one pendant vertex v. Let us term the edge connecting v to the rest of the graph by e, and let u be the vertex of G connected to v by e, see Figure 1.5.

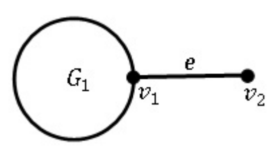

Figure 1.5 The graph G with one pendant vertex

Then we have the following relation between the chromatic polynomials of G and G_1:

Lemma 2.5. *Let G have only one pendant vertex v. Then*

$$C_G(k) = (k-1) \cdot C_{G_1}(k).$$

Proof. We shall use the Birkhoff-Lewis Theorem. Recall that

$$C_G(k) = C_{G-e}(k) - C_{G/e}(k),$$

see Figure 1.6.

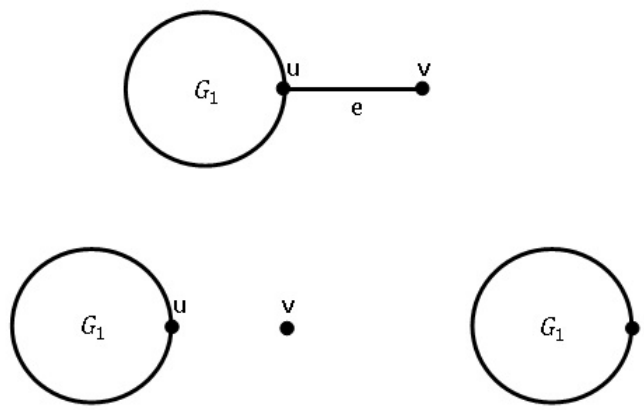

Figure 1.6 The graph G, $G - e$ and G/e

By Lemma 2.1, we can write

$$C_{G-e}(k) = k \cdot C_{G_1}(k).$$

Further

$$C_{G/e}(k) = C_{G_1}(k)$$

implies that

$$C_G(k) = k \cdot C_{G_1}(k) - C_{G_1}(k)$$

$$= (k-1) \cdot C_{G_1}(k).$$

□

Let G be a graph having r pendant vertices. See Figure 1.7.

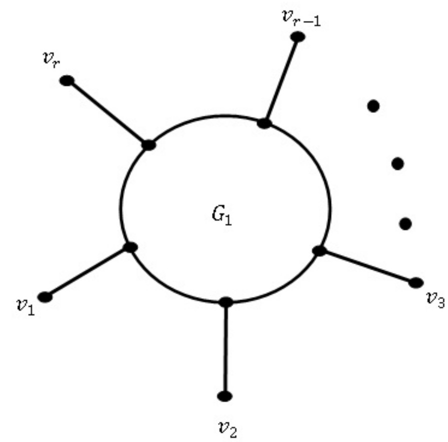

Figure 1.7 The graph G with r pendant vertices

Many graphs have pendant vertices and it is useful to be able to calculate the chromatic polynomial of the remaining graph G_1 in Fig. 1.7 after omitting all pendant vertices and edges connecting them to G_1. To achieve this, we have the following generalisation for a graph G with r pendant vertices:

Lemma 2.6. *Let G have r pendant vertices. Then*

$$C_G(k) = (k-1)^r \cdot C_{G_1}(k).$$

Proof. It follows from the proof of Lemma 2.5. □

Theorem 2.7. *Let G_1 be a graph and G be the graph obtained by joining G_1 with a path graph P_r as in Figure 1.8.*

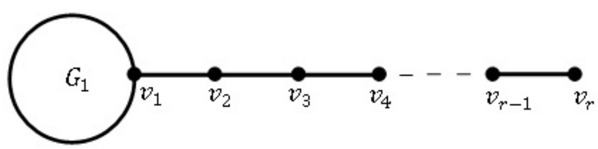

Figure 1.8 *The graph $G = G_1 \cup P_r$*

Then

$$C_G(k) = \frac{C_{G_1}(k) \cdot C_{P_r}(k)}{k}.$$

Example 2.1. If we choose the graphs P_4 and P_5 joined at a vertex v,

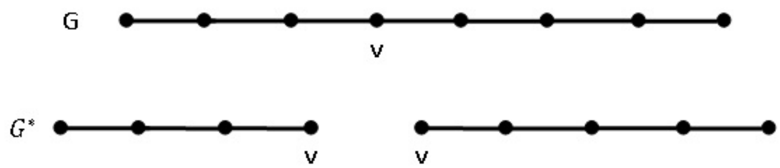

Figure 1.9 The graph $G = P_8$, $G^* = P_4 \coprod P_5$

We know that

$$C_G(k) = C_{P_8}(k) = k \cdot (k-1)^7.$$

Also by Theorem 2.1, we have

$$C_{G^*}(k) = C_{P_4}(k) \cdot C_{P_5}(k).$$

Therefore

$$\begin{aligned} C_{G^*}(k) &= k \cdot (k-1)^3 \cdot k \cdot (k-1)^4 \\ &= k^2 \cdot (k-1)^7. \end{aligned}$$

Then

$$\frac{C_G(k)}{C_{G^*}(k)} = \frac{k \cdot (k-1)^7}{k^2 \cdot (k-1)^7} = \frac{1}{k}$$

so

$$C_{G^*}(k) = \frac{C_{P_3} \cdot C_{P_4}}{k}.$$

Now we deal with the graphs which are connected at a cut vertex:

Lemma 2.8. *Let the graphs G_1 and G_2 be connected at a vertex v as in Figure 1.10. Then*

$$C_G(k) = \frac{C_{G_1}(k) \cdot C_{G_2}(k)}{k}.$$

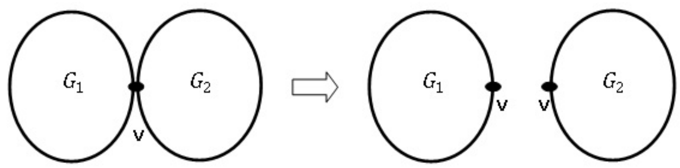

Figure 1.10 First way of splitting a graph $G = G_1 \cup G_2$

Theorem 2.9. *Let the graph G be the union of graphs G_1 and G_2 connected by a bridge e, as in Figure 1.11. Then*

$$C_G(k) = \frac{C_{G_1}(k) \cdot C_{G_2}(k)}{k \cdot (k-1)}.$$

Proof. Let the graph G be the union of graphs G_1 and G_2 connected by a bridge e. See Figure 1.11.

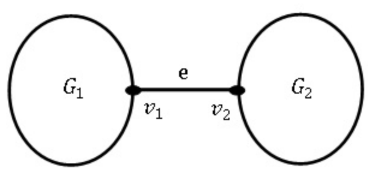

Figure 1.11 The graphs G_1 and G_2 connected by a bridge e

By Birkhoff-Lewis theorem we have

$$C_G(k) = C_{G-e}(k) - C_{G/e}(k).$$

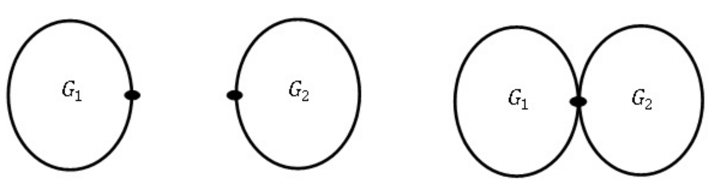

Figure 1.12 The graphs $G - e$ and G/e

By Theorem 2.1,

$$C_{G-e}(k) = C_{G_1}(k) \cdot C_{G_2}(k)$$

and by Lemma 2.3, we have

$$k \cdot C_{G/e}(k) = C_{G_1}(k) \cdot C_{G_2}(k).$$

Therefore we obtain

$$C_{G/e}(k) = \frac{C_{G_1}(k) \cdot C_{G_2}(k)}{k}.$$

If we substitute these in

$$C_G(k) = C_{G-e}(k) - C_{G/e}(k),$$

we obtain

$$\begin{aligned} C_G(k) &= C_{G_1}(k) \cdot C_{G_2}(k) - \frac{C_{G_1}(k) \cdot C_{G_2}(k)}{k} \\ &= C_{G_1}(k) \cdot C_{G_2}(k) \cdot (1 - \frac{1}{k}) \\ &= \frac{C_{G_1}(k) \cdot C_{G_2}(k) \cdot (k-1)}{k} \end{aligned}$$

which is the required result. □

The following result gives an important case:

Corollary 2.10. *Let a graph G be the union of a graph G_1 and a path P_r connected by a bridge e. Then*

$$C_G(k) = C_{G_1(k)} \cdot (k-1)^r.$$

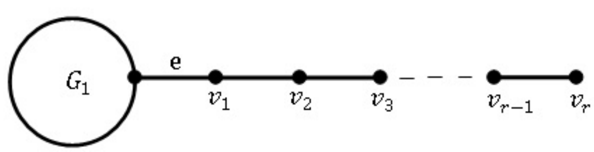

Figure 1.13 *A graph G_1 and a path P_r connected by a bridge e*

Proof. By Lemma 2.3, we have

$$C_G(k) = \frac{C_{G_1}(k) \cdot C_{P_r}(k) \cdot (k-1)}{k}.$$

Also as

$$C_{P_r}(k) = k \cdot (k-1)^{r-1},$$

we obtain

$$C_G(k) = C_{G_1}(k) \cdot (k-1)^r.$$

□

Now we deal with another case where we split a graph into two subgraphs through a common edge as in Figure 1.14.

Lemma 2.11. *Let G be a graph which can be splitted into two subgraphs G_1 and G_2 through a common edge $e = v_1v_2$. Then*

$$C_G(k) = \frac{C_{G_1}(k) \cdot C_{G_2}(k)}{k \cdot (k-1)}.$$

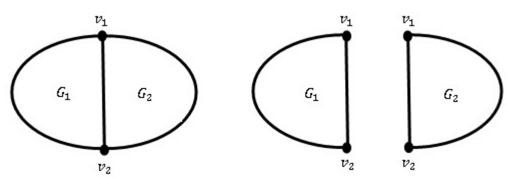

Figure 1.14 *A graph G splitted through an edge*

Proof. It follows by Birkhoff Lewis theorem. □

Now we deal with another common case where a graph G can be splitted into two cycle graphs C_m and C_n through a common edge e:

Theorem 2.12. *Let G be a graph which can be splitted into two cycle graphs C_m and C_n through a common edge e as in Figure 1.15. Then we have*

$$C_G(k) = \frac{C_{C_m}(k) \cdot C_{C_n}(k)}{k \cdot (k-1)}.$$

Proof. The given graph G is as in the following figure:

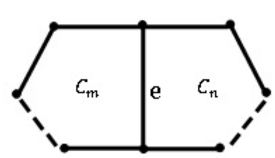

Figure 1.15 Splitting G into two cycle graphs C_m and C_n through a common edge e

By Birkhoff-Lewis theorem,

Figure 1.16 The graphs $G - e$ and G/e

$$G - e = C_{m+n-2}$$

and by Lemma 2.3, we have

$$C_{G/e}(k) = \frac{C_{C_{m-1}}(k) \cdot C_{C_{n-1}}(k)}{k}$$

$$\begin{aligned}
C_G(k) &= (k-1)^{m+n-2} + (-1)^{m+n-2} \cdot (k-1) \\
&\quad - \frac{[(k-1)^{m-1} + (-1)^{m-1} \cdot (k-1)] \cdot [(k-1)^{n-1} + (-1)^{n-1} \cdot (k-1))]}{k} \\
&= \frac{k \cdot (k-1)^{m+n-2} + k \cdot (-1)^{m+n-2} \cdot (k-1) - (k-1)^{m+n-2}}{k} \\
&\quad - \frac{(-1)^{n-1} \cdot (k-1)^m + (-1)^{m-1} \cdot (k-1)^n + (-1)^{m+n-2} \cdot (k-1)^2}{k} \\
&= \frac{(k-1)^{m+n-2} \cdot (k-1) + (-1)^{m+n-2} \cdot (k-1) + (-1)^{n-1} \cdot (k-1)^m}{k} \\
&\quad - \frac{(-1)^{m-1} \cdot (k-1)^n}{k} \\
&= \frac{(k-1)^{m+n-1} + (-1)^{m+n-2} \cdot (k-1) + (-1)^{n-1} \cdot (k-1)^m}{k} \\
&\quad - \frac{(-1)^{m-1} \cdot (k-1)^n}{k}
\end{aligned}$$

and as a consequence, we get the required result:

$$C_{G(k)} = \frac{C_{C_m(k)} \cdot C_{C_n}(k)}{k \cdot (k-1)}.$$ □

Theorem 2.13. *Let $1 < r < s$. For a graph G which is obtained by overlapping one edge of two path graphs P_r ve P_{s-r+2}, we have*

$$C_G(k) = \frac{C_{P_r}(k) \cdot C_{P_{s-r+2}}(k)}{k \cdot (k-1)}.$$

Proof. Let G be the graph in the first line of Figure 1.17. Let us separate G into two paths P_r ve P_{s-r+2} at the edge $v_{r-1}v_r$ as in the second line of Figure 1.17.

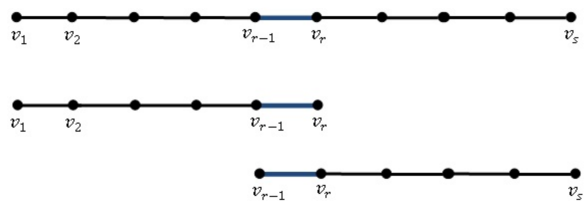

Figure 1.17 The graphs G, P_r, P_{s-r+2}

From the formula of chromatic polynomial of the path graph, we have

$$C_G(k) = C_{P_s}(k) = k \cdot (k-1)^{s-1}, \qquad C_{P_r}(k) = k \cdot (k-1)^{r-1},$$

$$C_{P_{s-r+2}}(k) = k \cdot (k-1)^{s-r+1},$$

and hence we conclude that

$$C_{P_r}(k) \cdot C_{P_{s-r+2}}(k) = k \cdot (k-1)^{r-1} \cdot k \cdot (k-1)^{s-r+1} = k^2 \cdot (k-1)^s.$$

This gives us the required result:

$$C_G(k) = \frac{C_{P_r}(k) \cdot C_{P_{s-r+2}}(k)}{k \cdot (k-1)}. \qquad \square$$

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