Efficient formulas for the exact integration of products of polynomials, exponentials and trigonometric functions

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Abstract. There are certain situations in which a definite integral where the integrand is a product of real polynomials, exponentials and trigonometric functions (sine or cosine) must be calculated. In this work we are going to consider a class of integrals of these types, and show that they can be computed by using integration by parts and complex variable manipulations. This leads to formulas given by finite summations expressed in terms of exponentials, trigonometric functions and the derivatives of the polynomial involved. Some examples have been presented to asses the efficiency of the proposed formulas.

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1 Introduction

In our investigations on adapted procedures for solving initial-value problems of second order ordinary differential equations (see [5]) there appeared certain coefficients expressed by definite integrals, namely

$$R^{+}_{jk} = \int_{-1}^{2\xi_j-1} e^{gh(2\xi_j-\alpha-1)/2} T_k(\alpha) \frac{1}{wh} \sin\left[w(\xi_j h - \frac{1}{2}h(\alpha+1))\right] d\alpha\,,$$

$$R^{-}_{jk} = \int_{-1}^{2\xi_j-1} e^{gh(2\xi_j+\alpha+1)/2} T_k(\alpha) \frac{1}{wh} \sin\left[w(\xi_j h - \frac{1}{2}h(\alpha+1))\right] d\alpha\,. \qquad (1.1)$$

where $\xi_j \in [0,1]$ and $g, h, w, \alpha \in \mathbb{R}$.

The efficiency of the resulting algorithm depends greatly on an adequate procedure to compute those coefficients. This was the main motivation for obtaining integral formulas for the evaluation of the following indefinite integrals

$$\int e^{ax} \sin(bx) P_n(x)\, dx \qquad \text{and} \qquad \int e^{ax} \cos(bx) P_n(x)\, dx\,,$$

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where x is a real variable, a, b are two real constants, and $P_n(x)$ is a polynomial of degree $n \in \mathbb{N}$.

These formulas are not obtained readily. The common procedure used in textbooks for its calculation is based on recurrence relations considered for integrals of these types involving only monomials (see [6], p. 326), namely

$$\int e^{ax}\sin(bx)\,x^n\,dx \qquad \text{and} \qquad \int e^{ax}\cos(bx)\,x^n\,dx\,,$$

which results in lengthy and cumbersome calculations.

Another possible approach that could be used to obtain the above integrals would make use of the Laplace transform theory. But the involved calculations would be also lengthy and cumbersome, even for the most simple cases. We will illustrate this procedure by calculating the integral

$$\int e^{ax}\sin(bx)\,x\,dx\,.$$

Considering an interval $[0, x_N]$, we have by the Fundamental Theorem of Calculus that the function

$$y(x) = \int_0^x e^{at}\sin(bt)\,t\,dt$$

is an integral of the given function on this interval. Thus, after taking derivatives with respect to x in the above equation, what we are looking for is the solution of the initial-value problem

$$y'(x) = e^{ax}\sin(bx)\,x\,, \quad y(0) = 0\,.$$

Let $Y(s) = \mathcal{L}(y(x))$ be the Laplace transform of $y(x)$. After applying the Laplace transform to the differential problem gives us

$$s\mathcal{L}(y(x)) - y(0) = \mathcal{L}(e^{ax}\sin(bx)\,x)\,.$$

By using the properties of the Laplace transform (see [1, 4]) this equation becomes

$$sY(s) = F(s-a)$$

where

$$F(s) = \mathcal{L}(\sin(bx)\,x) = -\frac{d}{ds}\left(\frac{b}{s^2+b^2}\right) = \frac{2bs}{\left(s^2+b^2\right)^2}\,.$$

Applying the inverse Laplace transform, after some calculus we get that

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\left(\frac{2b(s-a)}{s((s-a)^2+b^2)^2}\right) \\ &= \frac{-2ab + e^{ax}\left(a\left(a^2+b^2\right)x - a^2 + b^2\right)\sin(bx) - be^{ax}\left(\left(a^2+b^2\right)x - 2a\right)\cos(bx)}{\left(a^2+b^2\right)^2}\,. \end{aligned}$$

We can change the constant related with the initial value in the above formula by a general one, say k, and thus, we may write the integral in the more general form

$$y(x) = \frac{e^{ax}\left(a\left(a^2+b^2\right)x - a^2 + b^2\right)\sin(bx) - be^{ax}\left(\left(a^2+b^2\right)x - 2a\right)\cos(bx)}{\left(a^2+b^2\right)^2} + k\,.$$

In the next section we are going to present compact formulas expressed by means of some finite summations for the above indefinite integrals.

2 Derivation of the integral formulas

By using integration by parts it is straightforward to deduce the following formula (see [2], p. 106)

$$\begin{aligned}\int e^{kx} P_n(x)\,dx &= \frac{e^{kx}}{k}\left(P_n(x) - \frac{{P_n}'(x)}{k} + \frac{{P_n}''(x)}{k^2} - \ldots + (-1)^n \frac{P_n^{(n)}(x)}{k^n}\right) \\ &= \frac{e^{kx}}{k}\sum_{m=0}^{n}(-1)^m \frac{P_n^{(m)}(x)}{k^m} \qquad (2.1)\end{aligned}$$

where $P_n^{(m)}(x)$ is the m^{th} derivative of $P_n(x)$ with respect to x.

This formula remains valid if k is complex. Thus, taking $k = a + ib \in \mathbb{C}$ the above formula results in

$$\begin{aligned}&\int e^{(a+ib)x} P_n(x)\,dx \\ &\quad = \frac{e^{(a+ib)x}}{(a+ib)}\left(P_n(x) - \frac{{P_n}'(x)}{(a+ib)} + \frac{{P_n}''(x)}{(a+ib)^2} - \ldots + (-1)^n \frac{P_n^{(n)}(x)}{(a+ib)^n}\right) \\ &\quad = \frac{e^{(a+ib)x}}{(a+ib)}\sum_{m=0}^{n}(-1)^m \frac{P_n^{(m)}(x)}{(a+ib)^m}\,.\end{aligned}$$

By using the Euler's identity, we get readily that

$$\begin{aligned}&\int e^{ax}\cos(bx)P_n(x)\,dx + i\int e^{ax}\sin(bx)P_n(x)\,dx \\ &\quad = \frac{e^{(a+ib)x}}{a+ib}\left(\frac{(a+ib)^n P_n(x) - (a+ib)^{n-1}{P_n}'(x) + \ldots + (-1)^n P_n^{(n)}(x)}{(a+ib)^n}\right). \\ &\quad = \frac{e^{(a+ib)x}}{(a+ib)^{n+1}}\left((a+ib)^n P_n(x) - (a+ib)^{n-1}{P_n}'(x) + \ldots + (-1)^n P_n^{(n)}(x)\right) \\ &\quad = \frac{e^{(a+ib)x}}{(a+ib)^{n+1}}\sum_{m=0}^{n}(-1)^m (a+ib)^{n-m} P_n^{(m)}(x)\,. \qquad (2.2)\end{aligned}$$

In order to get the real and imaginary parts of the right hand side of the above formula, it is preferable to consider the complex number $a + ib$ in polar form, that is, we put $a+ib = re^{i\theta}$, where $r = \sqrt{a^2+b^2}$ is the modulus and $\theta \in [0, 2\pi)$ the argument.

Substituting $a + ib$ by $re^{i\theta}$, the above formula results in

$$
\begin{aligned}
&\int e^{ax}\cos(bx)P_n(x)\,dx + i\int e^{ax}\sin(bx)P_n(x)\,dx \\
&\quad= \frac{e^{(a+ib)x}}{(a+ib)^{n+1}}\Big((a+ib)^n P_n(x) - (a+ib)^{n-1}P_n{}'(x) + \ldots + (-1)^n P_n^{(n)}(x)\Big) \\
&\quad= \frac{e^{(a+ib)x}}{r^{n+1}e^{i(n+1)\theta}}\Big(r^n e^{in\theta}P_n(x) - r^{n-1}e^{i(n-1)\theta}P_n{}'(x) + \ldots + (-1)^n P_n^{(n)}(x)\Big) \\
&\quad= \frac{e^{ax}}{r^{n+1}}\Big(r^n e^{i(bx-\theta)}P_n(x) - r^{n-1}e^{i(bx-2\theta)}P_n{}'(x) + \ldots \\
&\qquad +(-1)^n e^{i(bx-(n+1)\theta)}P_n^{(n)}(x)\Big)\,. \qquad (2.3)
\end{aligned}
$$

Now, after equating the real and imaginary parts in the above equation we get the following two integral formulas

$$
\begin{aligned}
&\int e^{ax}\cos(bx)P_n(x)\,dx \\
&\qquad= \frac{e^{ax}}{r^{n+1}}\big(r^n\cos(bx-\theta)P_n(x) - r^{n-1}\cos(bx-2\theta)P_n{}'(x) + \ldots \\
&\qquad\quad +(-1)^n\cos(bx-(n+1)\theta)P_n^{(n)}(x)\big) \\
&\qquad= \frac{e^{ax}}{r^{n+1}}\sum_{j=0}^{n}(-1)^j r^{n-j}\cos(bx-(j+1)\theta)P_n^{(j)}(x) \\
&\qquad= C_n(a,b,x)\,. \qquad (2.4)
\end{aligned}
$$

$$
\begin{aligned}
&\int e^{ax}\sin(bx)P_n(x)\,dx \\
&\qquad= \frac{e^{ax}}{r^{n+1}}\big(r^n\sin(bx-\theta)P_n(x) - r^{n-1}\sin(bx-2\theta)P_n{}'(x) + \ldots \\
&\qquad\quad +(-1)^n\sin(bx-(n+1)\theta)P_n^{(n)}(x)\big) \\
&\qquad= \frac{e^{ax}}{r^{n+1}}\sum_{j=0}^{n}(-1)^j r^{n-j}\sin(bx-(j+1)\theta)P_n^{(j)}(x) \\
&\qquad= S_n(a,b,x)\,. \qquad (2.5)
\end{aligned}
$$

3 Computational experiments

In order to asses the performance of the above formulas we have considered the evaluation of different integrals, where the computations have been done using the system Mathematica 8.0.

3.1 Example 1.

We consider the integral

$$I(x) = \int e^{12x} \cos(\sqrt{5}x)(3x^5 - 2x^3 + 5x - 1)\, dx$$

The evaluation of this integral using the command `Integrate` in Mathematica 8.0 results in

$$I(x) \quad = \quad \frac{-e^{12x}}{10942526586601}\, Q(x)$$

where

$$\begin{aligned} Q(x) \quad = \quad & \big(-2643831926964x^5 + 1027663976085x^4 + 1455312314856x^3 \\ & -345364395054x^2 - 4353731396796x + 1219986365231\big) \sin\left(\sqrt{5}x\right) \\ & +\sqrt{5}\left(220319327247x^5 - 177438384360x^4 - 62129898118x^3 \right. \\ & \left. +44312840784x^2 + 355425476269x - 131284320125\right) \cos\left(\sqrt{5}x\right) \end{aligned}$$

taking a CPU time of 0.391 seconds.

The calculation of the same integral using the formula in (2.5) and the command `Simplify` gives the same value for $I(x)$ as before, and takes a CPU time of 0.000 seconds (Mathematica is unable to evaluate it).

3.2 Example 2.

As a second example we have considered the evaluation of the integrals

$$I_n(x) = \int e^{ax} \cos(bx) P_n(x)\, dx \qquad \text{and} \qquad J_n(x) = \int e^{ax} \sin(bx) P_n(x)\, dx\,,$$

taking $a = 40, b = \sqrt{13}, P_n(x) = 3x^n - 2x^3 + 5x - 1$ for $n = 4, 5, \ldots, 14$.

In Tables 1 and 2 we have included in the second and third columns the CPU time in seconds used for evaluating the indefinite integrals, the first one refers to the computation made by Mathematica, and the second one with the above formulas in (2.4) and (2.5). The two last columns refer to the evaluation of the obtained integrals in a specific value of the variable. We observe that Mathematica needs much more time than that needed by the proposed formulas. This was as expected, because Mathematica has to do the calculations for getting the indefinite integrals while the proposed formulas are just summations, which require a minimum effort. Once the formulas have been calculated, the evaluation in a specific value required the same computational cost in both cases.

We note that the values provided by Mathematica have a residual term which is an imaginary part, and this is not the case for the provided formulas, because all these computations are with real numbers, while Mathematica uses complex arithmetic.

In Table 1 we see that as n is increasing the computations of $I_n(x)$ need more computational time. The presence of the residual term in the evaluations of $I_n(0.6)$ is also present for $n \geq 8$. Similar comments are valid for the computations of $J_n(x)$

n	CPU $I_n(x)$	CPU $C_n(a,b,x)$	$I_n(0.6)$	$C_n(a,b,0.6)$
4	0.343	0	-5.86634×10^8	-5.86634×10^8
5	0.297	0	-5.41443×10^8	-5.41443×10^8
6	0.469	0	-5.15188×10^8	-5.15188×10^8
7	0.515	0	-4.99919×10^8	-4.99919×10^8
8	0.609	0	$-4.91031 \times 10^8 - 1.21945 \times 10^{-8} i$	-4.91031×10^8
9	0.672	0	$-4.85852 \times 10^8 + 3.30328 \times 10^{-9} i$	-4.85852×10^8
10	0.718	0	$-4.82833 \times 10^8 - 1.06083 \times 10^{-8} i$	-4.82833×10^8
11	0.781	0	$-4.81071 \times 10^8 + 5.18048 \times 10^{-9} i$	-4.81071×10^8
12	0.844	0	$-4.80042 \times 10^8 - 5.67524 \times 10^{-10} i$	-4.80042×10^8
13	0.937	0	$-4.79440 \times 10^8 - 1.41736 \times 10^{-8} i$	-4.79440×10^8
14	0.969	0	$-4.79089 \times 10^8 - 2.12458 \times 10^{-9} i$	-4.79089×10^8

Table 1: Data for $I_n(x) = \int e^{ax} \cos(bx) P_n(x)\, dx$.

n	CPU $I_n(x)$	CPU $C_n(a,b,x)$	$I_n(0.6)$	$C_n(a,b,0.6)$
4	0.297	0	1.05121×10^9	1.05121×10^9
5	0.312	0	9.70894×10^8	9.70894×10^8
6	0.485	0	9.24557×10^8	9.24557×10^8
7	0.546	0	8.97783×10^8	8.97783×10^8
8	0.625	0	$8.82291 \times 10^8 + 6.66186 \times 10^{-8} i$	8.82291×10^8
9	0.688	0	$8.73315 \times 10^8 + 1.25146 \times 10^{-9} i$	8.73315×10^8
10	0.734	0	$8.68108 \times 10^8 - 2.27009 \times 10^{-9} i$	8.68108×10^8
11	0.813	0	$8.65084 \times 10^8 - 2.62516 \times 10^{-8} i$	8.65084×10^8
12	0.875	0	$8.63326 \times 10^8 - 2.74449 \times 10^{-8} i$	8.63326×10^8
13	0.922	0	$8.62303 \times 10^8 + 2.13331 \times 10^{-8} i$	8.62303×10^8
14	0.954	0	$8.61707 \times 10^8 - 1.05210 \times 10^{-8} i$	8.61707×10^8

Table 2: Data for $J_n(x) = \int e^{ax} \sin(bx) P_n(x)\, dx$.

as can be seen in Table 2.

4 Conclusions

Two formulas expressed by summations in terms of exponentials, trigonometric functions and the derivatives of the polynomial $P_n(x)$ have been developed for the exact integration of

$$I_n(x) = \int e^{ax} \cos(bx) P_n(x)\, dx \qquad \text{and} \qquad J_n(x) = \int e^{ax} \sin(bx) P_n(x)\, dx\,.$$

These formulas are very effective, as has been shown using the Mathematica program to compute them in some particular examples.

The procedure used here may be extended to other integral formulas, after using trigonometric and hyperbolic identities. For example, using the above procedure we could obtain exact formulas for the following integrals:

$$\begin{array}{ll} \int e^{ax}\sin(bx)\sin(cx)P_n(x)\,dx, & \int e^{ax}\cos(bx)\cos(cx)P_n(x)\,dx, \\ \int e^{ax}\sin(bx)\cos(cx)P_n(x)\,dx, & \int e^{ax}\sin^m(bx)P_n(x)\,dx, \\ \int e^{ax}\cos^m(bx)P_n(x)\,dx, & \int \sinh(ax)\sin(bx)P_n(x)\,dx, \\ \int \sinh(ax)\cos(bx)P_n(x)\,dx, & \int \cosh(ax)\sin(bx)P_n(x)\,dx, \end{array},$$

$$\int \cosh(ax)\cos(bx)P_n(x)\,dx$$

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