On the displacement of two immiscible Stokes fluids in a 3D Hele-Shaw cell

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Abstract. In this paper we study the linear stability of the displacement of two incompressible Stokes fluids in a 3D Hele-Shaw cell. The corresponding growth constant contains two new terms, compared with the Saffman-Taylor formula. For large enough surface tension on the interface, we get an almost stable displacement, even if the displacing fluid is less viscous. Moreover, if the surface tension on the interface is zero, then our growth rate is bounded in terms of the wavenumbers of the perturbations. These results are in contradiction with the Saffman-Taylor stability criterion.

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Key words: 3D Hele-Shaw displacement, Hydrodynamic stability, Saffman-Taylor formula.

1 Introduction

Consider the flow of an incompressible Stokes fluid with viscosity μ in the small gap between two parallel plates. This technical device was introduced in [9]. The velocity orthogonal on the plates is neglected. The depth average procedure (across the plates) turns the 3D Stokes flow in a 2D particular flow. The averaged velocities verify a Darcy type equation for the flow in a porous medium with permeability $b^2/(12\mu)$, where b is the small distance between the plates.

An important application of the Hele-Shaw model is related with the displacement of two immisicble Stokes fluids with different viscosities. In fact, a very thin "mixed" region exists at the contact between the fluids, where we have a strong (continuous) variation of the viscosity. In the Hele-Shaw model, this mixed region is replaced by a sharp interface, where a surface tension can be considered. Therefore, in every point of the equivalent porous medium we have only one fluid. The Laplace law is assumed on this interface: the pressure jump is given by the surface tension multiplied with the interface curvature and the normal velocity is continuous. Moreover, this interface is a material one. On this way, we can visualize the displacement of oil by water in an (equivalent) porous medium, when the upper plate is transparent - see [1].

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Another model for displacement in porous media is the "saturation" model, where in every point of the medium we have both fluids and the capillary pressure appears.

Saffman and Taylor [13] obtained the well-known stability criterion: the displacement in a 2D Hele-Shaw cell is unstable if the displacing fluid is less viscous.

In this paper we study the linear stability of the displacement of two immiscible Stokes fluids in a 3D Hele-Shaw cell. We give a justification for neglecting the velocity component orthogonal on the plates, but we not use the average procedure in the flow equations. On the contrary, we consider a 3D interface between the fluids and we not neglect the meniscus curvature between the plates. On this 3D interface (which has two curvature radii) we assume the Laplace law. The average procedure is used only in the Laplace law, as a last step of the linear stability analysis procedure. We get a more general growth constant given by a ratio which contains two new terms, compared with the Saffman-Taylor formula.

Our main result is following: the displacement is *almost stable* even if the displacing fluid is less viscous *but* the surface tension γ on the air-fluid interface is large enough. Therefore the displacement stability is not decided only by the viscosities ratio of the displacing fluids (as in the Saffman-Taylor criterion) but also by the value of the surface tension on the fluid-fluid interface.

2 The growth constant formula

We consider a horizontal Hele-Shaw cell, with plates $z = 0$ and $z = b$ which are parallel with the fixed x_1Oy plane. The gravity is neglected. A Stokes fluid with viscosity μ_1 is displacing an immiscible Stokes fluid with viscosity μ_2. Both fluids are incompressible. A sharp interface $x_1 = I(y, z, t)$ exists between the two fluids. On I we suppose the Laplace's law: the pressure jump is given by the surface tension multiplied with the curvature and the normal velocity is continuous.

The velocity components are denoted by (u, v, w) and p is the pressure. The extra-stress tensor is denoted by τ. $\mathbf{L}$ is the 3×3 matrix containing the derivatives of the velocity components in terms of (x, y, z). The strain-rate tensor is $\mathbf{D} = (\mathbf{L} + \mathbf{L^T})$. The flow equations and the constitutive relations are:

$$p_{x_1} = \tau_{11,x_1} + \tau_{12,y} + \tau_{13,z};\ p_y = \tau_{21,x_1} + \tau_{22,y} + \tau_{23,z};\ p_z = \tau_{31,x_1} + \tau_{32,y} + \tau_{33,z};$$

$$\tau = \mu\mathbf{D}; \quad \mu = \mu_1,\ x_1 < I; \quad \mu = \mu_2,\ x_1 > I.$$

Here $\tau_{ij,x_1}, \tau_{ij,y}, \tau_{ij,z}$ are the partial derivatives of the component ij of the extra stress tensor τ and p_{x_1}, p_y, p_z are the partial derivatives of the pressure. We consider incompressible fluids, therefore

$$u_{x_1} + v_y + w_z = 0.$$

Two very thin layers of thickness c of the displaced fluid are left behind the interface, on the interior part of both Hele-Shaw plates. We consider that both fluids are moving with a small horizontal velocity e on the surface of the above thin layers.

Consider the *basic* flow given by the *constant* pressure gradient in the x direction; only the velocity component in the x_1 direction is not zero, depending only on z. The elements of this basic flow are denoted by the super-index ^{0}. The basic pressure, extra-stress tensor, strain-rate tensor and horizontal velocity in the fluid i are denoted by p^{0i}, τ^{0i}, u^{0i}, $\mathbf{D}^{0i}$, $i = 1, 2$. The pressure gradients and the viscosities of both fluids verify the important equation (2.3) below.

The basic constitutive relations are

$$\tau^{0i} = \mu_i \mathbf{D}^{0i}; \qquad i = 1, 2.$$

Since the basic velocity depends only on z, then the basic extra stress tensor in both fluids will depend only on z. From the above constitutive relations we get:

$$\tau^{0i}_{13} = \mu_i u^0_z, \quad \tau^{0i}_{11} = \tau^{0i}_{12} = \tau^{0i}_{22} = \tau^{0i}_{23} = \tau^{0i}_{33} = 0, \quad i = 1, 2.$$

Therefore the following basic flow equations hold

$$p^{0i}_{x_1} = \tau^{0i}_{11,x_1} + \tau^{0i}_{12,y} + \tau^{0i}_{13,z} = \tau^{0i}_{13,z}, \qquad p^{0i}_y = p^{0i}_z = 0$$

and from the two relations from above, we get

$$p^{0i}_x = \tau^{0i}_{13,z} = G_i = \mu_i u^{0i}_{zz}, \quad i = 1, 2,$$

where G_i are two negative constants. As we mentioned above, we suppose

$$u^{0i} = e \quad for \quad z = c, \quad z = b - c,$$

and then we obtain

$$u^{0i} = e + (G_i/2\mu_i)(z - c)(z - b + c), \quad i = 1, 2. \tag{2.1}$$

We use the average operator

$$< h >= [1/(b - 2c)] \int_c^{b-c} h(z)dz,$$

we neglect c^2 and bc, then from the above relation we obtain

$$< u^{0i} >= -\frac{b^2}{12\mu_i} G_i + e. \tag{2.2}$$

We introduce the moving reference frame xOy:

$$x = x_1 - et - (G_i/2\mu_i) < (z - c)(z - b + c) >,$$

and consider the basic (material) steady interface $x = 0$. The basic velocity must be continuous across the basic interface, then from (2.1) we get the important relation

$$\frac{G_1}{\mu_1} = \frac{G_2}{\mu_2}. \tag{2.3}$$

The perturbations of the basic velocity and pressure are denoted by u, v, w, p, τ. No confusion can exists, because from now on only the perturbations will be considered.

The perturbation of the basic interface is denoted by $\psi(y, z, t)$ and we have

$$\psi_t = u|_{x=\psi}. \tag{2.4}$$

We assume

$$u = v = w = 0 \quad for \quad z = c, \quad z = b - c, \tag{2.5}$$

and introduce the scalings

$$x' = x/l, \quad y' = y/l, \quad z' = z/b, \quad \epsilon = b/l.$$

From the free-divergence condition we obtain

$$(u_{x'} + v_{y'})/l + w_{z'}/b = 0, \quad \epsilon(u_{x'} + v_{y'}) + w_{z'} = 0,$$

and in the frame of asymptotic expansions we get

$$w_{z'} = 0, \quad u_{x'} + v_{y'} = 0 \Rightarrow w_z = 0 \quad and \quad u_x + v_y = 0.$$

The boundary conditions are giving us $w = 0$.

We insert the perturbations in the constitutive and flow equations, then

$$\tau^0 + \tau = \mu(\mathbf{D}^0 + \mathbf{D}), \quad -\nabla(p^0 + p) + \nabla \cdot (\tau^0 + \tau) = 0.$$

In the frame of linear stability, we get $\tau = \mu\mathbf{D}$, $\nabla p = \nabla \cdot \tau$, where

$$\mu = \mu_1, \quad for\ x < \psi; \qquad \mu = \mu_2, \quad for\ x > \psi. \tag{2.6}$$

The perturbed flow equations and the perturbed constitutive relations are

$$p_x = \tau_{11,x} + \tau_{12,y} + \tau_{13,z},\ p_y = \tau_{21,x} + \tau_{22,y} + \tau_{23,z},\ p_z = \tau_{31,x} + \tau_{32,y} + \tau_{33,z}$$

$$\tau_{11} = 2\mu u_x,\ \tau_{12} = \mu(u_y + v_x), \quad \tau_{13} = \mu u_z,\ \tau_{22} = 2\mu v_y, \quad \tau_{23} = \mu v_z,\ \tau_{33} = 0, \tag{2.7}$$

where μ verify the relation (2.6).

We consider the Fourier mode decomposition

$$u = f(z)\exp(-kx + \sigma t)\cos(ky), \quad v = f(z)\exp(-kx + \sigma t)\sin(ky), \quad \text{for } x \geq \psi;$$
$$u = f(z)\exp(kx + \sigma t)\cos(ky), \quad v = -f(z)\exp(kx + \sigma t)\sin(ky), \quad \text{for } x \leq \psi,$$

where $k \geq 0$ and the dimension of $f(z)$ is $(length/time)$.

The perturbations decay to zero far from the interface $x = \psi$ and $u_x + v_y = 0$. Moreover, we have

$$u_x = (-k)u,\ v_x = (-k)v,\ x > \psi; \qquad u_x = (-k)u,\ v_x = (-k)v,\ x < \psi,$$

$$(u_y + v_x)_x = 2k^2 f\exp(-kx + \sigma t)\sin(ky),\ v_{yy} = -k^2 f\exp(-kx + \sigma t)\sin(ky),\ x \geq \psi;$$
$$(u_y + v_x)_x = -2k^2 f\exp(kx + \sigma t)\sin(ky),\ v_{yy} = k^2 f\exp(-kx + \sigma t)\sin(ky), \quad x \leq \psi.$$

Consider a jump of viscosity in the point $x_0 = 0$. The (material) perturbed interface was denoted by $\psi(x, y, z, t)$ and we have $\psi_t = u$, then

$$\psi(x) = \frac{f(z)}{\sigma}\exp(\sigma t)\cos(ky), \quad \psi(x)^+_x(x_0) = -k\frac{f(z)}{\sigma}\exp(\sigma t)\cos(ky), \quad x > x_0;$$

$$\psi(x) = \frac{f(z)}{\sigma}\exp(\sigma t)\cos(ky), \quad \psi(x)_x^-(x_0) = k\frac{f(z)}{\sigma}\exp(\sigma t)\cos(ky), \quad x < x_0;$$

where upper indices $^+,^-$ denotes the lateral limit values.

We need the form of the amplitude $f(z)$. For this, we use (2.7) and get

$$p_z = \tau_{31,x} + \tau_{32,y} + \tau_{33,z} = \mu(u_z)_x + \mu(v_z)_y = \mu(u_x + v_y)_z \Rightarrow p_z = 0, \tag{2.8}$$

then we have also $(p_y)_z = 0$. The partial z derivative of p_y is giving

$$0 = (p_y)_z = \mu(u_y + v_x)_{xz} + 2\mu(v_y)_{yz} + \mu(v_z)_{zz}. \tag{2.9}$$

The viscosity μ appearing in the equations (2.8) - (2.9) verifies the relation (2.6). The above Fourier expansion and the free-divergence condition give us

$$\mu(u_y + v_x)_{xz} + 2\mu(v_y)_{yz} = 0$$

then from the last two above relations we obtain $v_{zzz} = 0, \quad f_{zzz} = 0$. The no-slip conditions (2.5) give us

$$f = (z - c)(z - b + c). \tag{2.10}$$

The normal stress to the interface is given by $T_{11} = p^0 + p - (\tau_{11}^0 + \tau_{11}) = p^0 + p - \tau_{11}$. We search for the limit values of T_{11} near the point $x_0 = 0$. As the basic pressure is not depending on z, we consider the Taylor first order expansion of the basic pressures p^{0i} near the averaged (across the plates) interface, then for $i = 1, 2$ we get:

$$p^{0i}(x_0 + \psi) = p^{0i}(x_0 + < \psi >) \approx p^{0i}(x_0) + p_x^{0i}(x_0) < \psi >= \tag{2.11}$$

$$p^{0i}(x_0) + G_i < \psi > .$$

We introduce the notations $p^i, \tau^i, \quad i = 1, 2$ for the pertrubations of the pressure and extra - stress tensor in both fluids, then we get (recall $\tau_{11}^{0i} = 0$ and ψ is continuous across the interface $x = x_0$)

$$T_{11}^- = p^{01}(x_0) + G_1 < \psi > + p^1 - \tau_{11}^1, \quad T_{11}^+ = p^{02}(x_0) + G_2 < \psi > + p^2 - \tau_{11}^2. \tag{2.12}$$

The dynamic Laplace law gives us

$$p^{02}(x_0) - p^{01}(x_0) + (G_2 - G_1) < \psi > + (p^2 - \tau_{11}^2) - (p^1 - \tau_{11}^1) =$$

$$\gamma(x_0)(x_{0yy} + x_{0zz} + \psi_{yy} + \psi_{zz}),$$

where $\gamma(x_0)$ is the surface tension acting on $x = x_0$ and the curvature of the perturbed interface is approximated by $(\psi_{yy} + \psi_{zz})$. The basic pressures verify the Laplace law on the basic interface $x = x_0$ and it follows

$$p^{02}(x_0) - p^{01}(x_0) = \gamma(x_0)(x_{0yy} + x_{0zz}).$$

Therefore we obtain

$$(G_2 - G_1) < \psi > + (p^2 - \tau_{11}^2) - (p^1 - \tau_{11}^1) = \gamma(x_0)(\psi_{yy} + \psi_{zz}). \tag{2.13}$$

As a particular case, we consider the displacing fluid is air, then $\mu_1 = 0$. We use (2.3) and get $G_1 = 3, p^1 = 3$ and $\tau^1_{11} = 3$. Therefore the formula (2.13) becomes

$$G_2 < \psi > +(p^2 - \tau^2_{11}) = \gamma(x_0)(\psi_{yy} + \psi_{zz}),$$

which is used in [14], but without the term $\gamma(x_0)\psi_{zz}$ in the right hand side.

The viscous stress term τ_{11} is neglected in some papers - see [13], [6]. However this term is appearing in general *dynamic* Laplace law described in [2] and [12]. The viscous stress is also considered in [14], related with an Oldroyd-B fluid displaced by air in a Hele-Shaw cell. The derivatives (u_x, u_y, v_x, v_y) can be very large for small x and large k - see the Fourier decomposition - and (u_x, u_y, v_x, v_y) can not be neglected in front of u_{zz}, v_{zz}.

The terms $(p^i_x - \tau^i_{11,x}), \quad i = 1, 2$ are computed by using the perturbed flow equations:

$$p^i_x - \tau^i_{11,x} = \tau^i_{12,y} + \tau^i_{13,z} = \mu_i(u_y + v_x)_y + \mu(u_z)_z,$$

$$p^2 - \tau^2_{11} = (-1/k)\mu_2\{-2k^2 f + f_{zz}\}\exp(-kx + \sigma t)\cos(ky), \quad x > x_0; \tag{2.14}$$

$$p^1 - \tau^1_{11} = (1/k)\mu_2\{-2k^2 f + f_{zz}\}\exp(kx + \sigma t)\cos(ky), \quad x < x_0. \tag{2.15}$$

From $\psi_t = u$ and (2.10) it follows, with $(+)$ for $x < x_0$ and $(-)$ for $x > x_0$:

$$\psi = \frac{f}{\sigma}\exp(\pm kx + \sigma t)\cos(ky); \quad \psi_{yy} = \frac{f}{\sigma}(-k^2)\exp(\pm kx + \sigma t)\cos(ky),$$

$$\psi_{zz} = \frac{2}{\sigma}\exp(\pm kx + \sigma t)\cos(ky).$$

The relations (2.13) - (2.15) give us

$$k(G_2 - G_1) < f(z) > /\sigma - (\mu_2 + \mu_1)[-2k^2 f(z) + 2] = (\gamma/\sigma)[-k^3 f(z) + 2k]. \tag{2.16}$$

We neglect c^2, bc and from (2.10) we have

$$< f >= \int_c^{b-c}[(z - c)(z - b + c)/(b - 2c)]dz = -b^2/6.$$

We perform the average in the formula (2.16), we use the relation (2.2) and get

$$\sigma = \frac{(\mu_2 - \mu_1)(< u^0 > -e)k - \gamma k - \gamma k^3 b^2/12}{(\mu_2 + \mu_1)(1 + k^2 b^2/6)}. \tag{2.17}$$

Consider the capillary number $Ca = (\mu_2 - \mu_1) < u^0 > /\gamma$, then the above relation becomes

$$\sigma = \frac{< u^0 > [(\mu_2 - \mu_1) - Ca^{-1}]k - (\mu_2 - \mu_1)e - \gamma(k^3 b^2/12)}{(\mu_2 + \mu_1)(1 + k^2 b^2/6)}. \tag{2.18}$$

The Saffman-Taylor formula is

$$\sigma_{ST} = \frac{< u^0 > k(\mu_2 - \mu_1) - \gamma(k^3 b^2/12)}{(\mu_2 + \mu_1)}. \tag{2.19}$$

When $Ca >> 1$ and $e = 0$, the numerator of (2.18) is quite similar to the numerator of the Saffman-Taylor growth constant (2.19); only the denominator of (2.18) contains the new factor $(1 + k^2b^2/6)$.

Recall the formula (2.11), where

$$< \psi >= (1/\sigma) < f(z) > \exp(\pm kx + \sigma t) \cos(ky), \quad x \approx x_0.$$

Then, near the averaged basic interface, p^{0i} given by (2.11) is depending on y. But our basic pressure must depend only on x. We prove that p^{0i}_y is very small if we avoid a small interval near the interface $x_0 = 0$. We have

$$< \psi >_y = \frac{1}{\sigma}(-k) f(z) \exp(-kx + \sigma t) \sin(ky), \quad x > 0,$$

$$< \psi >_y = \frac{1}{\sigma}(k) f(z) \exp(kx + \sigma t) \sin(ky), \quad x < 0.$$

In Figure 1 we plot the function $F(k) = -k \exp(-kx), \quad x = 0.001, \quad k > 0$ and see that for large (k, x) this function is *almost zero*. Only in the range $k < 7000$ we have $max|F| \approx 400$. We can multiply $f(z)$ with a very small coefficient, then we obtain a small value of $max| < \psi >_y |$ also in the range $k < 7000$, for small enough values of the time t.

In Figure 2, 3 we plot the functions $F(k) = -k \exp(-kx), \quad x = 0.01, \quad k > 0$ and $F(k) = -k \exp(-kx), \quad x = 0.1, \quad k > 0$. We see that the maximum value of $|F|$ is decreasing when x is increasing.

Therefore, with the above estimations, we can consider that our basic pressures p^{0i} are *almost* independent on y. As $b = O(10^{-2})$, our stability analysis holds if we avoid an interval of order $O(10b)$ near the moving interface $x = 0$. This phenomenon is also mentioned in [14].

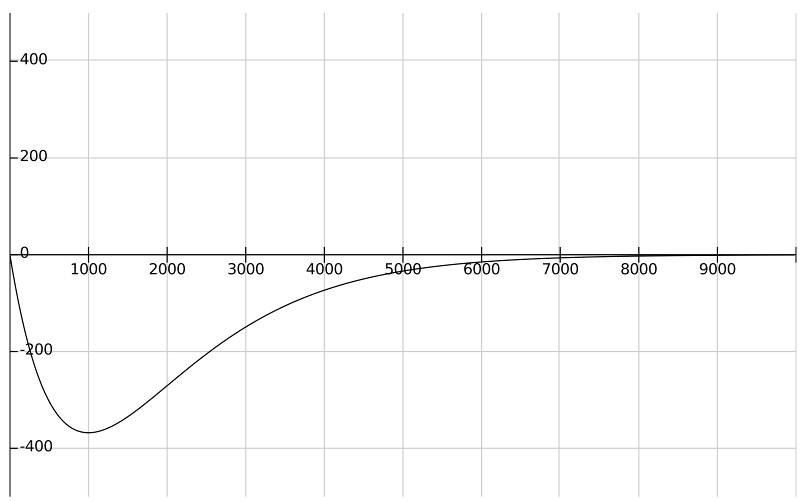

Figure 1. $F(k) = -k \exp(-kx), \quad x = 0.001, \quad k > 0$

$F(k)$ on the vertical axis, $\quad k$ on the horizontal axis

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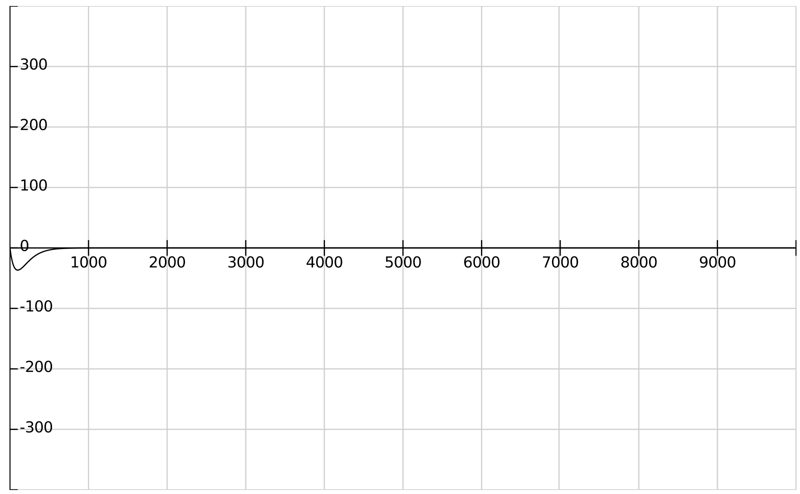

Figure 2. $F(k) = -k\exp(-kx), \quad x = 0.01, \quad k > 0$

$F(k)$ on the vertical axis, $\quad k$ on the horizontal axis

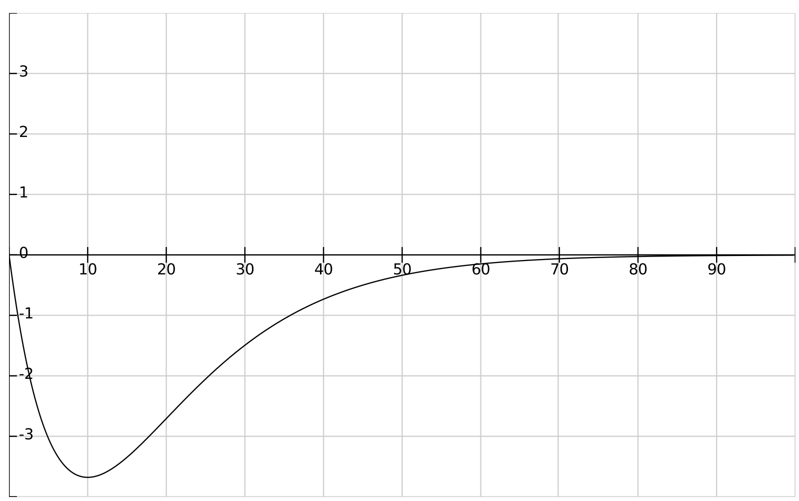

Figure 3. $F(k) = -k\exp(-kx), \quad x = 0.1, \quad k > 0$

$F(k)$ on the vertical axis, $\quad k$ on the horizontal axis

3 Conclusions

Two new important terms appear in the formula (2.17), compared with the Saffman - Taylor formula (2.19): i) $k^2b^2/6$ in the denominator; ii) $- < u^0 > Ca^{-1}k$ in the numerator. This terms appear from two reasons: a) we not neglected u_x, v_x, u_y, v_y in front of u_z, v_z in the first flow equation; b) we used the *total* curvature of the perturbed interface in the Laplace law (2.13).

We obtain the following results:

1) *Even if the surface tension γ on the interface is zero, the growth constant is bounded in terms of the wave number k.* In this case form (2.17) it follows the estimate

$$\sigma = M\frac{(< u^0 > -e)k}{(1 + k^2b^2/6)} < M(1 + \delta)\{\frac{< u^0 > -e}{b}\}\sqrt{3/2}, \quad \forall\delta > 0,$$

where

$$M = \frac{(\mu_2 - \mu_1)}{(\mu_2 + \mu_1)}.$$

Indeed, we have $B > 1/(2\sqrt{A}) \Rightarrow k/(1 + Ak^2) < B$. In our case, $A = b^2/6$, then we can consider $B = (\sqrt{3/2})(1 + \delta)/b$ and obtain the result.

2) If the surface tension γ on the interface is zero, then the growth constant tends to zero for very large wave numbers k.

Both above results 1) and 2) are in contradiction with the Saffman-Taylor formula (2.19), where $\gamma = 0$ is giving an unbounded growth constant in terms of k, as we mentioned before - see also Gorell and Homsy [6].

3) From the expression (2.17), we get

$$\sigma < \frac{k\{(\mu_2 - \mu_1)(< u^0 > -e) - \gamma\}}{(\mu_2 + \mu_1)(1 + k^2b^2/6)}.$$

Therefore if the surface tension γ is large enough, our growth constant is negative or zero, even if the air viscosity is zero, less than the viscosity μ of the displaced Stokes fluid. This is also in contradiction with the Saffman-Talor criterion. The relation (2.19) is always giving us a maximum *strictly positive* value of σ, in terms of $< u^0 >, \gamma, \mu_1, \mu_2$. We consider that this is the most important conclusion of our paper: the displacement stability is decided not only by the viscosities ratio of the displacing fluids, but also by the surface tension on the interface.The sufficient condition for the *almost stability* of the displacement is

$$\gamma > (\mu_2 - \mu_1) \cdot (< u^0 > -e), \quad \mu_2 > \mu_1$$

A different contradiction of the Saffman and Taylor stability criterion was observed in [3] [4], [5], [7], [8], [10]. All these papers are related with the displacement of air by a fluid with surfactant properties in a Hele - Shaw cell with preexisting surfactant layers on the plates; it is pointed out that a more viscous displacing fluid can give us an *unstable* air-fluid interface. The experiments and the numerical results are in good agreement - but also in a 3D frame. We can consider that our result is a complementary one, compared with the above experiments with surfactants fluids and Hele-Shaw cells. We proved that for a large enough surface tension, even if the displacing fluid (air) is less viscous, the interface air-fluid is *almost stable*.

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