Max and Min Poisson-Lomax power series distributions

B. Gh. Munteanu

Abstract. In the scientific literature, some distributions are used as models for the study of lifetime. This paper presents a new family of power series distributions based on the Poisson-Lomax distributions family recently introduced by Al-Zahrani and Sagor (2014, [11]). This distribution is called a Max Poisson Lomax power series (MaxPoiLPS) distribution, respectively Min Poisson Lomax power series (MinPoiLPS) distribution. This implies the fact that MaxPoiLPS and MinPoiLPS represent the distributions of the random variables $\max\{X_1, X_2, \ldots, X_Z\}$, respectively $\min\{X_1, X_2, \ldots, X_Z\}$, where $X_1, X_2, \ldots$ are independent random variables, Lomax distributed and Z is a random variable zero truncated Poisson distributed. The general framework of these maximum and minimum distributions was presented in the paper (2013,[14]). The main characteristics and properties of this distribution are analyzed and detailed.

M.S.C. 2010: 60K10, 62N05.
Key words: power series distribution; Lomax distribution; Poisson distribution; distribution of the maximum and minimum.

1 Preliminary results

The Lomax distribution, known as the *Pareto type II distribution* is a special case of the generalized Pareto distribution. It finds applications in business, economics, actuarial modeling, biological sciences, etc. The Lomax distribution has been used in the literature under various approaches. For example, has been extended to modeling reliability and lifetime (Balkema and de Haan, [4], 1974). Then it was studied as an alternative to the exponential distribution when the data are heavy tailed (Bryson, [6], 1974). Ahsanullah ([1], 1991), Balakrishnan and Ahsanullah ([3], 1994), Childs and others ([7], 2001) have analyzed the record values of the Lomax distribution, recurrences relationships between moments and record values or statistical orders for the random variables Lomax distributed, truncated to the right. Al Awadhi and Ghitany ([2], 2001) used the Lomax distribution as mixing Poisson parameters and

Applied Sciences, Vol.19, 2017, pp. 84-92.

obtained a discrete Poisson-Lomax distribution. Ghitany and al. ([8], 2007), introduced the *Marshall-Olkin extended Lomax distribution*. 2013 is the year when Zahrani and Harbi [10] estimated the parameters of the Lomax distribution by means of the General Progresive Censoring method. Statistical evaluation of Lomax Logarithmic distribution is investigated by the Zahrani and Sagor ([12], 2014). The same year, Zahrani and Sagor ([11], 2014) proposed a new distribution of lifetime, distribution with three parameters, which is obtained by composing zero truncated Poisson distribution and Lomax distribution. It comes by maximum distribution of a sequence $Y_1, Y_2, \ldots, Y_Z$, of independent and identically distributed random variables, with the probability density f, in an random number Z, while $X = \max\{Y_1, Y_2, \ldots, Y_Z\}$. It uses the probability density of the conditioned random variable $X\,|Z$ as being $f_{X|Z}(x\,|z) = zf(x)\,[F(x)]^{z-1}$, where $F(x)$ is the distribution function corresponding to the probability density $f(x)$. One year later, Zahrani ([13], 2015) obtained an extension of the Poisson Lomax distribution using shifted Poisson distribution. In 2015, the group of researchers from the University of Mansoura, Egypt analyzed the properties of the *Lomax exponential distribution* ([5]).

In this paper, two distributions are introduced: the Max Poisson-Lomax and Min Poisson-Lomax distributions which are treated and analyzed in a a unitary manner using the power series distributions (PSD) class. The techniques and working methods have been described in the paper Leahu, Munteanu and Cataranciuc ([14], 2013). We obtained the distribution of the maximum and minimum values of a random sample size Z which has power series distribution class ([9], 2005).

2 The Max Poisson-Lomax and Min Poisson-Lomax distributions

It is known that a random variable admits a Lomax distribution with the parameters μ and α, and we note $X \sim L(\mu, \alpha),\ \mu, \alpha > 0$, if the cumulative distribution function (cdf) is $F_L(x) = 1 - \left(\frac{\mu}{x+\mu}\right)^{\alpha},\ x \geq 0$, while the corresponding probability density function (pdf) $f_L(x) = \frac{\alpha\mu^{\alpha}}{(x+\mu)^{\alpha+1}},\ x \geq 0$.

Theoretically speaking, things are presented and treated like this: we consider the random variables $U_{PoiL} = \max\{X_1, X_2, \ldots, X_Z\}$ and $V_{PoiL} = \min\{X_1, X_2, \ldots, X_Z\}$, where $(X_i)_{i\geq 1}$ are independent and identically distributed random variables, $X_i \sim L(\mu, \alpha),\ \mu, \alpha > 0$ and $Z \sim Poisson^*(\lambda),\ \lambda > 0,\ Z \in PSD$ with $\mathbb{P}(Z = z) = \frac{a_z \Theta^{\Theta}}{A(\Theta)}$, $z = 1, 2, \ldots,\ A(\Theta) = e^{\Theta} - 1,\ \Theta \in (0, +\infty),\ \Theta = \lambda$, while the sequence of non-negative real numbers is $a_z = \frac{1}{z!}$. We mention that the random variables $(X_i)_{i\geq 1}$ are independent of the random variable Z, the distribution of the latter being part of a power series distribution.

According to the general techniques presented in the work of Leahu, Munteanu and Cataranciuc [14], it can be said that the random variables U_{PoiL} and V_{PoiL} follow the distributions $MaxPoiL$ and $MinPoiL$ respectively, with parameters λ, μ and α, namely $U_{PoiL} \sim MaxPoiL(\lambda, \mu, \alpha)$ and $V_{PoiL} \sim MinPoiL(\lambda, \mu, \alpha)$, if the distribution functions are characterized by the relations:

$$U_{PoiL}(x) = \frac{A\left[\Theta F_L(x)\right]}{A[\Theta]} = \frac{e^{\Theta F_L(x)-1}}{e^{\Theta}-1}$$

$$= \frac{e^{-\lambda\left(1+\frac{x}{\mu}\right)^{-\alpha}} - e^{-\lambda}}{1-e^{-\lambda}},\ x \geq 0 \tag{2.1}$$

and

$$V_{PoiL}(x) = 1 - \frac{A\left[\Theta\left(1-F_L(x)\right)\right]}{A[\Theta]} = \frac{e^{\Theta}-1-e^{\Theta(1-F_L(x))}+1}{e^{\Theta}-1}$$

$$= \frac{1-e^{-\lambda\left[1-\left(1+\frac{x}{\mu}\right)^{-\alpha}\right]}}{1-e^{-\lambda}},\ x \geq 0. \tag{2.2}$$

The probability densities are described by the following relation:

$$u_{PoiL}(x) = \frac{\Theta f_L(x)\frac{d}{d\Theta}\left\{A\left[\Theta F_L(x)\right]\right\}}{A[\Theta]} = \frac{\Theta f_L(x)e^{\Theta F_L(x)}}{A\left[\Theta\right]}$$

$$= \frac{\alpha\lambda\left(1+\frac{x}{\mu}\right)^{-(\alpha+1)} e^{-\lambda\left(1+\frac{x}{\mu}\right)^{-\alpha}}}{\mu\left(1-e^{-\lambda}\right)},\ x \geq 0 \tag{2.3}$$

and

$$v_{PoiL}(x) = \frac{\Theta f_L(x)\frac{d}{d\Theta}\left\{A\left[\Theta\left(1-F_L(x)\right)\right]\right\}}{A[\Theta]} = \frac{\Theta f_L(x)e^{\Theta(1-F_L(x))}}{A\left[\Theta\right]}$$

$$= \frac{\alpha\lambda\left(1+\frac{x}{\mu}\right)^{-(\alpha+1)} e^{-\lambda\left[1-\left(1+\frac{x}{\mu}\right)^{-\alpha}\right]}}{\mu\left(1-e^{-\lambda}\right)},\ x \geq 0. \tag{2.4}$$

By using (2.1) and (2.3), the survival function (reliability function) and the hazard rate (failure rate function) of the MaxPoiL distribution are:

$$S_{U_{PoiL}}(x) = 1 - U_{PoiL}(x) = \frac{1-e^{-\lambda\left(1+\frac{x}{\mu}\right)^{-\alpha}}}{1-e^{-\lambda}},\ x \geq 0, \tag{2.5}$$

respectively,

$$h_{U_{PoiL}}(x) = \frac{u_{PoiL}(x)}{S_{U_{PoiL}}(x)} = \frac{\alpha\lambda\left(1+\frac{x}{\mu}\right)^{-(\alpha+1)} e^{-\lambda\left(1+\frac{x}{\mu}\right)^{-\alpha}}}{\mu\left(1-e^{-\lambda\left(1+\frac{x}{\mu}\right)^{-\alpha}}\right)},\ x \geq 0. \tag{2.6}$$

Similarly, by (2.2) and (2.4), we obtain the survival function and hazard rate for the MinPoiL distribution :

$$S_{V_{PoiL}}(x) = 1 - V_{PoiL}(x) = \frac{e^{-\lambda\left[1-\left(1+\frac{x}{\mu}\right)^{-\alpha}\right]} - e^{-\lambda}}{1-e^{-\lambda}},\ x \geq 0, \tag{2.7}$$

respectively,

$$h_{V_{PoiL}}(x) = \frac{v_{PoiL}(x)}{S_{V_{PoiL}}(x)} = \frac{\alpha\lambda\left(1+\frac{x}{\mu}\right)^{-(\alpha+1)} e^{-\lambda\left[1-\left(1+\frac{x}{\mu}\right)^{-\alpha}\right]}}{\mu\left(e^{-\lambda\left[1-\left(1+\frac{x}{\mu}\right)^{-\alpha}\right]} - e^{-\lambda}\right)},\ x \geq 0. \tag{2.8}$$

Proposition 2.1. *If $(X_i)_{i\geq 1}$ is a sequence of independent random variables, Lomax distributed with parameters $\mu, \alpha > 0$, while $U_{PoiL} = \max\{X_1, X_2, \ldots, X_Z\}$, where $Z \sim Poisson^*(\lambda)$, $\lambda > 0$, $Z \in PSD$ with $\mathbb{P}(Z = z) = \frac{a_z\Theta^\Theta}{A(\Theta)}$, $z = 1, 2, \ldots$, $A(\Theta) = e^\Theta - 1$, $\Theta \in (0, +\infty)$, $\Theta = \lambda$ and $(a_z)_{z\geq 1} = 1/z!$, then:*

$$\lim_{\Theta\to 0^+} U_{PoiL}(x) = \left[1 - \left(\frac{\mu}{x+\mu}\right)^\alpha\right]^m,\ x \geq 0,$$

considering $m = \min\{n \in \mathbb{N}^*,\ a_n > 0\}$.

Proposition 2.2. *Under the conditions of the Proposition 2.1, when $\Theta \to 0^+$, we have that:*

$$\lim_{\Theta\to 0^+} V_{PoiL}(x) = 1 - \left(\frac{\mu}{x+\mu}\right)^{\alpha l},\ x \geq 0,$$

where $l = \min\{n \in \mathbb{N}^*,\ a_n > 0\}$.

Corollary 2.3. *The r^{th} moments, $r \in \mathbb{N}$, $r \geq 1$ of the random variables $U_{PoiL} \sim \text{MaxPoiL}(\lambda, \mu, \alpha)$ and $V_{PoiL} \sim \text{MinPoiL}(\lambda, \mu, \alpha)$ are given by :*

$$\mathbb{E}U^r_{PoiL} = \sum_{z\geq 1} \frac{a_z\Theta^z}{A(\Theta)} \mathbb{E}\left[\max\{X_1,\ X_2, \ldots, X_z\}\right]^r \tag{2.9}$$

and

$$\mathbb{E}V^r_{PoiL} = \sum_{z\geq 1} \frac{a_z\Theta^z}{A(\Theta)} \mathbb{E}\left[\min\{X_1,\ X_2, \ldots, X_z\}\right]^r, \tag{2.10}$$

where pdf' s of the random variables $\max\{X_1,\ X_2, \ldots, X_z\}$ *and* $\min\{X_1,\ X_2, \ldots, X_z\}$ *are* $f_{\max\{X_1,\ X_2,\ldots,X_z\}}(x) = zf_L(x)\left[F_L(x)\right]^{z-1}$ *and* $f_{\min\{X_1,\ X_2,\ldots,X_z\}}(x) = zf_L(x)\left[1 - F_L(x)\right]^{z-1}$.

3 Special cases

3.1 The Max Binomial-Lomax and Min Binomial-Lomax distributions

The Max Binomial-Lomax (MaxBL) and Min Binomial-Lomax (MinBL) distributions are defined by the distribution functions presented in a general framework in [14], where $A(\Theta) = (\Theta + 1)^n - 1$, with $\Theta = \frac{p}{1-p}$, $p \in (0, 1)$:

$$U_{BL}(x) = \frac{A\left[\Theta F_L(x)\right]}{A[\Theta]} = \frac{\left[\Theta F_L(x)+1\right]^n - 1}{(\Theta+1)^n - 1}$$

$$(3.1) \qquad = \frac{\left[1 - p\left(1+\frac{x}{\mu}\right)^{-\alpha}\right]^n - (1-p)^n}{1-(1-p)^n}, \; x \geq 0,$$

respectively,

$$V_{BL}(x) = 1 - \frac{A\left[\Theta\left(1-F_L(x)\right)\right]}{A[\Theta]} = 1 - \frac{\left[\Theta\left(1-F_L(x)\right)+1\right]^n - 1}{(\Theta+1)^n - 1}$$

$$(3.2) \qquad = \frac{1 - \left[1 - p + p\left(1+\frac{x}{\mu}\right)^{-\alpha}\right]^n}{1-(1-p)^n}, \; x \geq 0.$$

3.2 On the Poisson limit theorem

The next theorem shows that the MaxPoiL and MinPoiL distributions approximate the MaxBL, respectively MinBL distributions under certain conditions.

Theorem 3.1. (Poisson limit theorem). *The* MaxPoiL *and* MinPoiL *distributions can be obtained as limiting of the* MaxBL*, respectively* MinBL *distributions with distribution functions given by (3.1) and (3.2) if* $n\Theta \to \alpha > 0$ *when* $n \to \infty$ *and* $\Theta \to 0^+$.
In other words,

(i)

$$\lim_{\substack{n\to\infty \\ p\to 0^+}} V_{BL}(x) = V_{PoiL}(x), \; \forall\, x \geq 0,$$

where $V_{BL}(x)$*, respectively* $V_{PoiL}(x)$, $x \geq 0$ *are the distribution functions* $V_{BL} \sim MinBL(n,p,\mu,\alpha)$ *and respectively* $V_{PoiL} \sim MinPoiL(\lambda,\mu,\alpha)$*, defined by (3.2) and (2.2);*

(ii)

$$\lim_{\substack{n\to\infty \\ p\to 0^+}} U_{BL}(x) = U_{PoiL}(x), \; \forall\, x \geq 0,$$

where $U_{BL}(x)$*, respectively* $U_{PoiL}(x)$, $x \geq 0$ *are the distribution functions* $U_{BL} \sim MaxBL(n,p,\mu,\alpha)$*, respectively* $U_{PoiL} \sim MaxPoiL(\lambda,\mu,\alpha)$*, defined by (3.1) and (2.1).*

Figures 1 and 2 show the behavior of the cdf ′ s of MinBL(n,p,μ,α), MinPoiL(λ,μ,α), MaxBL(n,p,μ,α) and MaxPoiL(λ,μ,α) for some values of the parameters: $n = 40, p = 1/10, \lambda = 4, \mu = 9, \alpha \in \{1/2, 1, 2, 4\}$.

Figures 3 and 4 show the behavior of the pdf ′ s of MinBL(n,p,μ,α), MinPoiL(λ,μ,α), MaxBL(n,p,μ,α) and MaxPoiL(λ,μ,α) for the same parameters.

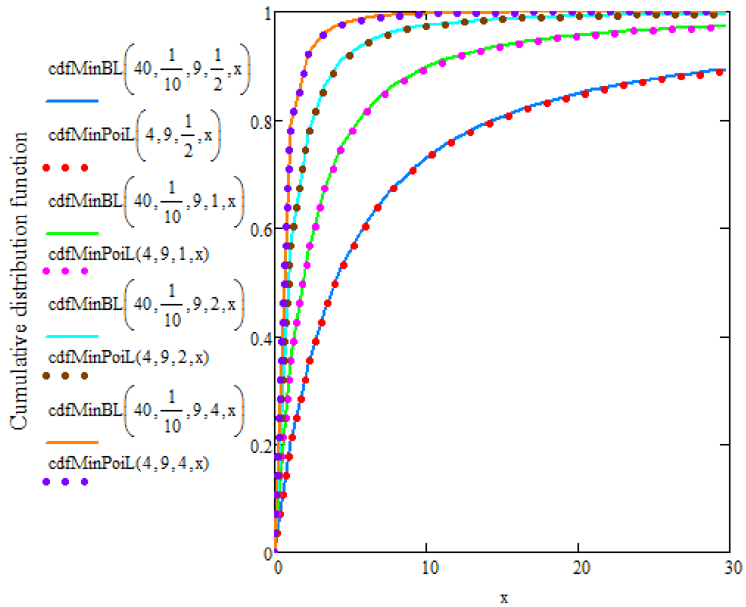

Figure 1: Cdf ′ s for the Min-Binomial-Lomax and Min-Poisson-Lomax distributions - graphical illustration of the Poisson Limit Theorem

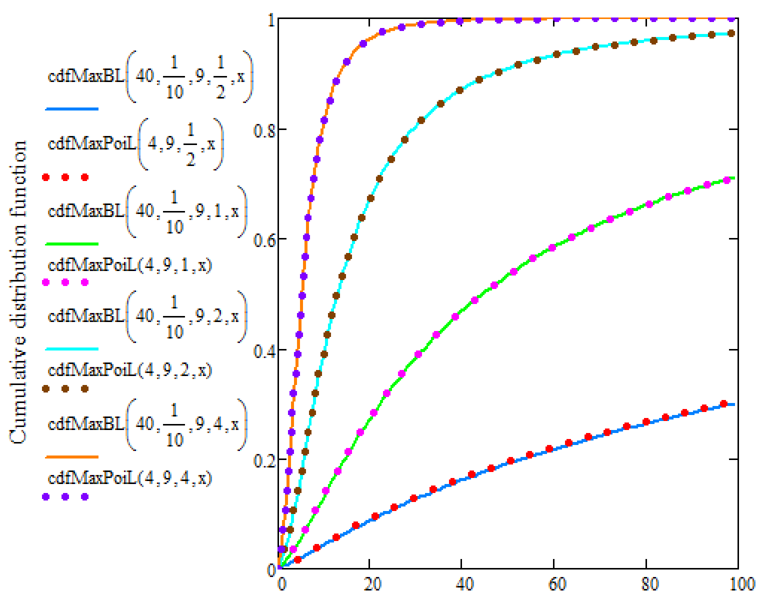

Figure 2: Cdf ′ s for the Max-Binomial-Lomax and Max-Poisson-Lomax distributions - graphical illustration of the Poisson Limit Theorem

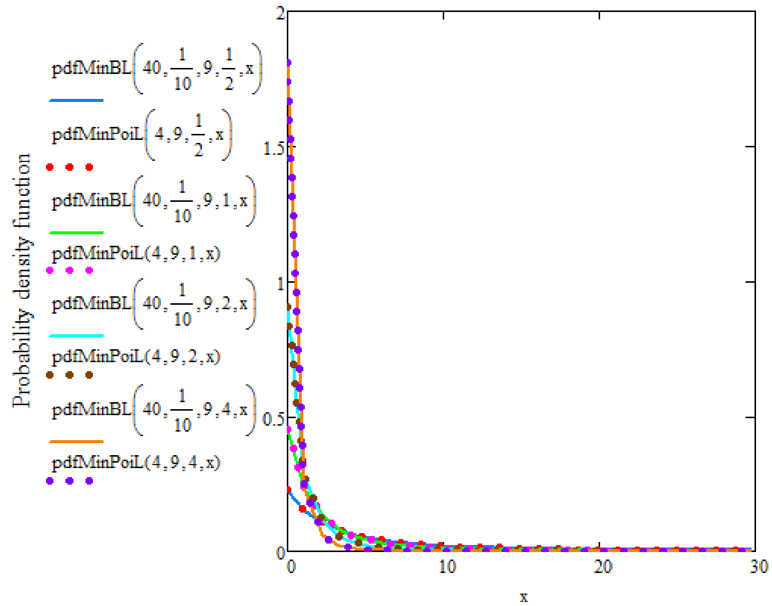

Figure 3: Pdf ′ s for the Min-Binomial-Lomax and Min-Poisson-Lomax distributions - graphical illustration of the Poisson Limit Theorem

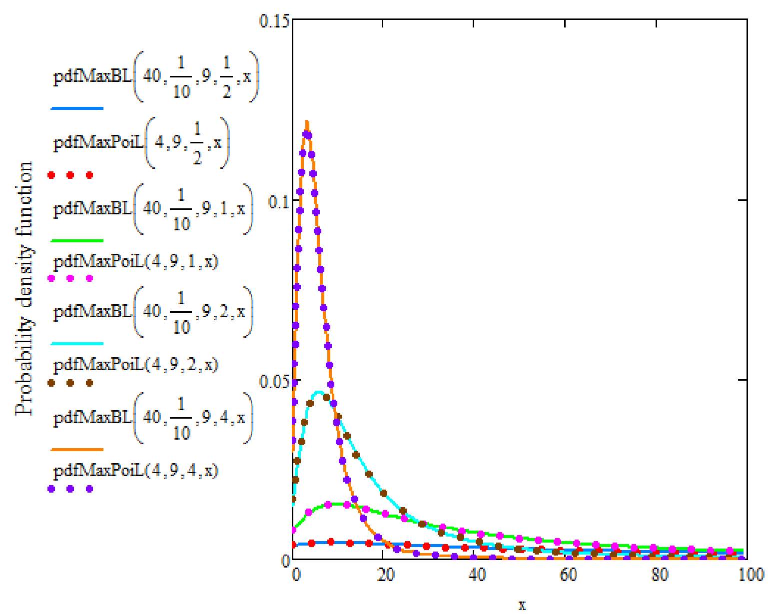

Figure 4: Pdf ′ s for the Max-Binomial-Lomax and Max-Poisson-Lomax distributions - graphical illustration of the Poisson Limit Theorem

4 Conclusions

The results revealed by this paper are related to the study of power series distributions type of a minimum or maximum of a sequence of independent and identically distributed random variables which are found in a random number.

The results extend the ones pioneered by Zahrani and Sagor [11] and later extended by Zahrani [13].

Also, it was presented in a unitary approach the distribution of a minimum or maximum number of independent and identically distributed random variables, Lomax distributed, through the PSD family, distribution characterised by the number of the random variables in the sequence.

The Poisson Limit Theorem has been formulated for the situations when the random variable number of the sum is a zero truncated binomial distribution and the limit distribution is a Poisson type distribution.

References

[1] M. Ahsanullah, *Record values of Lomax distribution*, Statistica Nederlandica, 41, 1(1991), 21-29.

[2] S. A. Al-Awadhi, M. E. Ghitany, *Statistical properties of Poisson- Lomax distribution and its application to repeated accidents data*, Journal of Applied Statistical Sciences, 10, 4(2001), 365-372.

[3] N. Balakrishnan, M. Ahsanullah, *Relations for single and product moments of record values from Lomax distribution*, Sankhya B, 56, 2(1994), 140-146.

[4] A. Balkema, L. de Haan, *Residual life time at great age*, Annals of Probability, 2, 5(1974), 792-804.

[5] A. H. El-Bassiouny, N. F. Abdo, H. S. Shahen, *Exponential Lomax Distribution*, International Journal of Computer Applications, 121, 13(2015), 24-29.

[6] M.C. Bryson, *Heavy-tailed distributions: Properties and tests*, Technometrics, 16, 1(1974), 61-68.

[7] A. Childs, N. Balakrishnan, M. Moshref *Order statistics from non-identical right truncated Lomax random variables with applications*, Statistical Papers, 42, 2(2001), 187-206.

[8] M. E. Ghitany, F. A. Al-Awadhi, L.A. Alkhalfan, *Marshall-Olkin extended Lomax distribution and its application to censored data*, Communications in Statistics-Theory and Methods, 36, 10(2007), 18551866.

[9] N. L. Johnson, A. W. Kemp, S. Kotz, *Univariate Discrete Distribution*, New Jersey, 2005.

[10] B. Al-Zahrani, M. Al-Harbi *On parameters estimation of Lomax distribution under general progressive censoring*, J. Qual. Reliabi. Engi., Article ID 431541 (2013), 7 pages, doi.org/10.1155/2013/431541.

[11] B. Al-Zahrani, H. Sagor, *The Poisson-Lomax distribution*, Revista Colombiana de Estadistica, 37 (2014), 223-243.

[12] B. Al-Zahrani, H. Sagor, *Statistical analysis of the Lomax-logarithmic distribution*, J. Stat. Comput. Sim., 85 (2014a), 1883-1901.

[13] B. Al-Zahrani, *An extended Poisson-Lomax distribution*, Advances in Mathematics: Scientific Journal 4, 2(2015), 79-89.

[14] A. Leahu, B. Gh. Munteanu, S. Cataranciuc, *On the lifetime as the maximum or minimum of the sample with power series distributed size*, Romai J. 9, 2(2013), 119-128.

Author's address:

Bogdan Gheorghe Munteanu
Department of Fundamental Sciences and Management,
Faculty of Aeronautical Management,
Henri Coanda Air Force Academy,
Mihai Viteazul 160, Brasov, 500187, Romania.
E-mail: munteanu.b@afahc.ro