# A numerical solution based on vortex distribution and quadratic boundary elements for the problem of the compressible fluid flow around obstacles

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**Abstract.** In this paper we apply the Boundary Element Method (BEM) to solve the 2D problem of the compressible fluid flow around obstacles. Starting with the Singular Boundary Integral Equation (SBIE) of the problem deduced by an indirect approach with vortex distribution, a numerical procedure which uses quadratic boundary elements of Lagrangian type is proposed. A computer code based on this method is made, using Mathcad programming language, and numerical solutions are found for some particular cases. An analytical checking of the computer code is also made by comparing numerical solutions with exact ones, in cases when the latter exist. The good agreements between the exact and the numerical solutions validate the proposed approach.

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## 1 Introduction

It is well known that there are numerous numerical techniques that can be used to solve problems of fluid flow. Among these we can highlight the Boundary Element Method (BEM), which is very suitable for problems of fluid flows which usually imply unbounded domains. It consists in two big stages: first it is obtained an equivalent boundary integral formulation of the mathematical model of the problem, and then this boundary integral equation is solved in order to obtain the numerical solution of the problem. Many books dedicated to this method point out the large are of problems which can be solved appealing to this method, as for example [1], [2], [3]. There are two main techniques which can be applied when solving problems of fluid flow by the aid of BEM: the direct technique and the indirect technique, both of them bringing the real advantage of the BEM over the other numerical methods - the fact that it reduces the problem dimension by one.

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The integral formulations of the problems usually represent singular boundary integral equations (SBIE) and solving them represents a great challenge because they require the evaluation of singular and near singular integrals. Special techniques must be considered in order to evaluate them, because an improper evaluation of these integrals brings large errors in the numerical formulation and so they can influence the well behavior of the problem to be solved.

In this paper we find a numerical solution for the problem of the 2D inviscid compressible fluid flow around an obstacle, starting with a SBIE deduced with the indirect technique with vortex distribution, based on quadratic isoparametric boundary elements.

We shall briefly describe the problems to solve (for a more detailed presentation, we address the reader to the reference [4]).

We consider that the uniform, steady, potential motion of an ideal inviscid fluid of subsonic velocity $U_\infty \bar{i}$, pressure $p_\infty$ and density $\rho_\infty$is perturbed by the presence of a fixed body of a known boundary, assumed to be smooth and closed.The objective is to find the perturbed motion, and the fluid action on the body.

In the mentioned paper a SBIE formulated in velocity vector terms that uses the fundamental solution of vortex type is deduced and a collocation method to solve it can also be found. In paper [5] a numerical solution based on linear boundary elements is obtained.

## 2 Boundary integral equation-vortex distribution

The SBIE is deduced by considering that the boundary is approximated with a vortex distribution having the unknown intensity $g(\overline{x})$, a hőlderian function on $C$. First, the integral formulations for the perturbation velocity components, which stand for $\overline{x}_0 \in C$ , a regular point on the boundary, are obtained:

$$u(\overline{x}_0) = \frac{1}{2} g(\overline{x}_0) n_y^0 + \frac{1}{2\pi}{}' \oint_C g(\overline{x}) \frac{y - y_0}{\|\overline{x} - \overline{x}_0\|^2} ds, \; v(\overline{x}_0) = -\frac{1}{2} g(\overline{x}_0) n_x^0 - \frac{1}{2\pi}{}' \oint_C g(\overline{x}) \frac{x - x_0}{\|\overline{x} - \overline{x}_0\|^2} ds, \tag{2.1}$$

where $n_x^0, n_y^0$ are the components of the normal unit vector outward the fluid in point $\overline{x}_0$, $\beta = \sqrt{1 - M^2}$ (for the subsonic flow, $M$= Mach number for unperturbed motion) and the sign " ' " denotes the Cauchy Principal Value of the integral.

The relations (2.1) and the boundary condition lead to the singular boundary equation which has the form:

$$-M^2 g(\overline{x}_0) n_x^0 n_y^0 + \frac{1}{\pi}{}' \int_C g(\overline{x}) \frac{\beta^2 (x - x_0) n_y^0 - (y - y_0) n_x^0}{\|\overline{x} - \overline{x}_0\|^2} ds = 2\beta n_x^0, \tag{2.2}$$

with the same notations as before.

Solving the SBIE (2.2) we get the vortex distribution and after that we can calculate the perturbation velocity and the local pressure coefficient.

We solve the SBIE (2.2) by using quadratic isoparametric boundary elements, similar as in case of solving the SBIE with sources distribution for the same problem, see [6].

The boundary is divided into $N$ unidimensional quadratic boundary elements, each of them with three nodes: two extreme nodes and an interior one. There are used 2N nodes for the boundary discretization. The approximation function is continuous on the boundary and local has a quadratic variation.

The quadratic isoparametric boundary element, see [1], uses the same set of basis functions, noted $N_1, N_2, N_3$, for describing the geometry and the unknown function. In the intrinsic system of coordinates, these functions have the expressions: $N_1(\xi) = \frac{\xi(\xi-1)}{2}, N_2(\xi) = 1-\xi^2, N_3(\xi) = \frac{\xi(\xi+1)}{2}, \xi \in [-1,1]$.

Local, on a boundary element we have: $g(\overline{x}) = [N]\{g\}$ ,where $[N] = (N_1\ N_2\ N_3)$ and $\{g\}$is the column matrix made with the nodal values of the unknown function.

Considering that the discrete equation is satisfied in every node, we reduce the problem to a system of linear equations:

$$E_j g(\overline{x}_j) + \sum_{i=1}^{N}\left(\sum_{l=1}^{3} a_{ij}^l g_l^i\right) = 2\pi\beta n_x^j, \tag{2.3}$$

where $a_{ij}^l = \int\limits_{-1}^{1} N_l \frac{\beta^2([N]\{x^i\}-x_j)n_y^j - ([N]\{y^i\}-y_j)n_x^j}{\|[N]\{\overline{x}\}-\overline{x}_j\|^2} J(\xi)\,d\xi$, $E_j = -M^2\pi n_x^j n_y^j, \{x^i\}$ , $\{y^i\}$ are the column matrices made with the global coordinates of the nodes of the boundary element noted $L_i$.There are used two systems of notation: a global and a local one ( $g_l^i$ is the the unknown value for the node number $l$ of the element $i$). Returning to the global system of notations, the linear algebraic system of the problem reduced as:

$$A\{g\} = \{B\}, A \in M_{2N}(R), \{g\}, \{B\} \in R^{2N}, B_j = 2\pi\beta n_x^j. \tag{2.4}$$

For evaluating the coefficients of the system, we use the following notations:

$$m_i = x_1^i + x_3^i - 2x_2^i, \quad n_i = x_3^i - x_1^i, \ u_{ij} = x_2^i - x_j,$$

$$M_i = y_1^i + y_3^i - 2y_2^i, \ N_i = y_3^i - y_1^i, \quad U_{ij} = y_2^i - y_j,$$

$$a_i = \frac{m_i^2 + M_i^2}{4}, \quad aa_i = \frac{n_i^2 + N_i^2}{4}, \quad b_i = \frac{m_i n_i + M_i N_i}{2},$$

$$c_{ij} = aa_i + m_i u_{ij} + M_i U_{ij}, \quad d_{ij} = n_i u_{ij} + N_i U_{ij}, \ e_{ij} = u_{ij}^2 + U_{ij}^2$$

$$J(\xi) = \sqrt{4a_i\xi^2 + 2b_i\xi + aa_i}. \tag{2.5}$$

In terms of the variable $\xi$, the coefficients $a_{ij}^l$, $l = \overline{1,3}$ have the following expressions:

$$a_{ij}^1 = \frac{1}{4}\int\limits_{-1}^{1} \frac{A_{ij}^l\xi^4 + B_{ij}^l\xi^3 + C_{ij}^l\xi^2 + D_{ij}^l\xi}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\,d\xi, \tag{2.6}$$

where

$$A^l_{ij} = \beta^2 n^j_y m_i - M_i n^j_x,\ B^l_{ij} = \beta^2 (n_i - m_i) n^j_y - (N_i - M_i) n^j_x,$$

$$C^l_{ij} = \beta^2 (2u_{ij} - n_i) n^j_y - (2U_{ij} - N_i) n^j_x,\ D^l_{ij} = -2\beta^2 u_{ij} n^j_y + 2U_{ij} n^j_x, \tag{2.7}$$

$$a^2_{ij} = \int_{-1}^{1} \frac{A^2_{ij}\xi^4 + B^2_{ij}\xi^3 + C^2_{ij}\xi^2 - B^2_{ij}\xi + D^2_{ij}}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi, \tag{2.8}$$

with:

$$A^2_{ij} = \frac{-\beta^2 n^j_y m_i + M_i n^j_x}{2}, \quad B^2_{ij} = \frac{-\beta^2 n^j_y n_i + N_i n^j_x}{2},$$

$$C^2_{ij} = \frac{\beta^2 (m_i - 2u_{ij}) n^j_y - (M_i - 2U_{ij}) n^j_x,}{2}, D^2_{ij} = \beta^2 u_{ij} n^j_y - U_{ij} n^j_x, \tag{2.9}$$

$$a^3_{ij} = \frac{1}{4} \int_{-1}^{1} \frac{A^3_{ij}\xi^4 + B^3_{ij}\xi^3 + C^3_{ij}\xi^2 + D^2_{ij}}{a_i\xi^4 + b_i\xi^3 + c_{ij}\xi^2 + d_{ij}\xi + e_{ij}} J(\xi)\, d\xi, \tag{2.10}$$

and where:

$$A^3_{ij} = \beta^2 n^j_y m_i - M_i n^j_x,\ B^3_{ij} = \beta^2 (n_i + m_i) n^j_y - (N_i + M_i) n^j_x,$$

$$C^3_{ij} = \beta^2 (2u_{ij} + n_i) n^j_y - (2U_{ij} + N_i) n^j_x,\ D^3_{ij} = 2\beta^2 u_{ij} n^j_y - 2U_{ij} n^j_x. \tag{2.11}$$

All the coefficients which appear in system (2.4) depend only on the coordinates of the nodes used for the boundary discretization. Some of them represent usual integrals and can be very easily evaluated with a computer code, by ordinary rules, but some of them represent singular integrals and for their evaluation more elaborated numerical schemes must be applied.

# 3 Regularization method for integrals with singular kernels evaluation

For evaluating the singular integrals different methods exist, see [1], [7], [8], but as shown in paper [9] the regularization method offers the best results in case when the singular boundary integral equation based on sources distribution was considered for the same problem and higher order boundary elements were used to solve it.

The regularization method applied for the evaluation of singular integrals was inspired by the work of M. Bonnet (see [1]). Using Taylor series we can replace the basis functions by Taylor polynomials, and, after making some simplifications, we get the modified shape functions, which are in fact, new integrands, that have no, or only weak singularities. For their further evaluations we can use the computer and a math application, for example Mathcad.

Let $\overline{x}_j \in L_i$, $\overline{x}_j = \sum\limits_{i=1}^{3} N_l(\eta)\overline{x}_l^i$, $\eta \in [-1,1]$,where $\eta$ is the value of the local coordinate $\xi$, that makes $\overline{x} = \overline{x}_j$.

Using the Taylor series, we get: $N_l(\xi) - N_l(\eta) = (\xi-\eta)\widehat{N}_l(\xi,\eta)$,where $\widehat{N}_l$, depends on $\xi$ but also of $\eta$, and it is named the modified shape function associated with $N_l$. It is given by the following formula: $\widehat{N}_l(\xi,\eta) = N_l^{'}(\eta) + \frac{1}{2}N_l^{"}(\eta)(\xi-\eta)$.

Based on this technique we obtain the expressions for the modified shape functions. Denoting by $\rho = \xi - \eta, \rho[-1-\eta, 1-\eta]$ we have: $\widehat{N}_1(\rho,\eta) = \eta - \frac{1}{2} + \frac{1}{2}\rho$, $\widehat{N}_2(\rho,\eta) = -2\eta - \rho$, $\widehat{N}_3(\rho,\eta) = \eta + \frac{1}{2} + \frac{1}{2}\rho$, $\|\overline{x} - \overline{x}_j\|^2 = \rho^2 \left\| \sum\limits_{i=1}^{3} N_l(\rho,\eta)\overline{x}_l^i \right\|^2 = \rho^2 \widehat{N}_{ij}$, $J(\rho) = \sqrt{4a_i\rho^2 + 2(4a_i\eta + b_i)\rho + 4a_i\eta^2 + 2b_i\eta + aa_i}$ and the following expressions for the coefficients involved:

$$a_{ij}^l = \int\limits_{-1-\eta}^{1-\eta} N_l(\eta) \frac{\beta^2 \left( \sum\limits_{i=1}^{3} \widehat{N}_l x_l^i \right) n_y^j - \left( \sum\limits_{i=1}^{3} \widehat{N}_l y_l^i \right) n_x^j}{\rho \widehat{N}_{ij}} J(\rho)\, d\rho +$$

$$+ \int\limits_{-1-\eta}^{1-\eta} \widehat{N}_l(\rho,\eta) \frac{\beta^2 \left( \sum\limits_{i=1}^{3} \widehat{N}_l x_l^i \right) n_y^j - \left( \sum\limits_{i=1}^{3} \widehat{N}_l y_l^i \right) n_x^j}{\widehat{N}_{ij}} J(\rho)\, d\rho, \tag{3.1}$$

As one can see only the first of the last two integrals in (3.1) has a weak singularity, the second being a regular integral. So, by introducing the expressions of the modified shape functions in the above relations we can write the singular integral as a sum of two integrals, only one having a weak singularity.

If $L_i$ is the current boundary element, integrals with singular kernels appear when $\bar{x}$ coincides with one of its nodes, so when $j = 2i-1$, $j = 2i$, $j = 2i+1$.We further deduce the expressions of the singular integrals for each of the three nodes of the current boundary element.

1. For $j = 2i-1$, we have $\eta = -1$ and with the following notations: $p_i = \frac{4x_2^i - 3x_1^i - x_3^i}{2}$, $P_i = \frac{4y_2^i - 3y_1^i - y_3^i}{2}$ we get: $\widehat{N}_{ij} = \left(\frac{\rho}{2}m_i + p_i\right)^2 + \left(\frac{\rho}{2}M_i + P_i\right)^2$.

The singular coefficients in this case are as follows:

$$a_{ij}^1 = an_{ij}^1 + as_{ij}^1, \tag{3.2}$$

where $an_{ij}^1$ , $as_{ij}^1$ are the nonsingular part, respectively the singular one, with a weak singularity, given by

$$an_{ij}^1 = \int\limits_0^2 \frac{\beta^2 m_i n_y^j - M_i n_x^j}{2\widehat{N}_{ij}} J(\rho)\, d\rho + \int\limits_0^2 \frac{(\rho-3)\left[\beta^2\left(\frac{\rho}{2}m_i + p_i\right)n_y^j - \left(\frac{\rho}{2}M_i + P_i\right)n_x^j\right]}{2\widehat{N}_{ij}} J(\rho)\, d\rho,$$

$$as_{ij}^1 = \int\limits_0^2 \frac{\beta^2 p_i n_y^j - P_i n_x^j}{\rho \widehat{N}_{ij}} J(\rho)\, d\rho, \qquad a_{ij}^2 = \int\limits_0^2 \frac{(2-\rho)\left[\beta^2\left(\frac{\rho}{2}m_i + p_i\right)n_y^j - \left(\frac{\rho}{2}M_i + P_i\right)n_x^j\right]}{\widehat{N}_{ij}} J(\rho)\, d\rho,$$

$$a_{ij}^3 = \int\limits_0^2 \frac{(\rho-1)\left[\beta^2\left(\frac{\rho}{2}m_i + p_i\right)n_y^j - \left(\frac{\rho}{2}M_i + P_i\right)n_x^j\right]}{2\widehat{N}_{ij}} J(\rho)\, d\rho. \tag{3.3}$$

For the numerical evaluation of the integral with a weak singularity, we use a truncation method, according to the notion of Cauchy Principal Value of an integral. So, in this case we have: $as_{ij}^1 = \int\limits_{\varepsilon}^{2} \frac{\beta^2 p_i n_y^j - P_i n_x^j}{\rho \widehat{N}_{ij}} J(\rho)\, d\rho$,where $\varepsilon$ represents a very small positive number.

2. If $j = 2i$, we have $\eta = 0$ and denoting by $p_i = \frac{x_3^i - x_1^i}{2}$, $P_i = \frac{y_3^i - y_1^i}{2}$,we get: $\widehat{N}_{ij} = \left(\frac{\rho}{2} m_i + p_i\right)^2 + \left(\frac{\rho}{2} M_i + P_i\right)^2$, and the following expressions:

$$a_{ij}^1 = \int\limits_{-1}^{1} \frac{(\rho-1)\left[\beta^2\left(\frac{\rho}{2} m_i + p_i\right) n_y^j - \left(\frac{\rho}{2} M_i + P_i\right) n_x^j\right]}{2\widehat{N}_{ij}} J(\rho)\, d\rho$$

$$a_{ij}^2 = an_{ij}^2 + as_{ij}^2,$$

$$an_{ij}^2 = \int\limits_{-1}^{1} \frac{\beta^2 \frac{m_i}{2} n_y^j - \frac{M_i}{2} n_x^j}{\widehat{N}_{ij}} J(\rho)\, d\rho - \int\limits_{-1}^{1} \frac{\rho\left[\beta^2\left(\frac{\rho}{2} m_i + p_i\right) n_y^j - \left(\frac{\rho}{2} M_i + P_i\right) n_x^j\right]}{\widehat{N}_{ij}} J(\rho)\, d\rho,$$

$$as_{ij}^2 = \left(\int\limits_{-1}^{-\varepsilon} + \int\limits_{\varepsilon}^{1}\right)\left(\frac{\beta^2 p_i n_y^j - P_i n_x^j}{\rho \widehat{N}_{ij}} J(\rho)\, d\rho\right)$$

$$a_{ij}^3 = \int\limits_{-1}^{1} \frac{(\rho+1)\left[\beta^2\left(\frac{\rho}{2} m_i + p_i\right) n_y^j - \left(\frac{\rho}{2} M_i + P_i\right) n_x^j\right]}{2\widehat{N}_{ij}} J(\rho)\, d\rho. \tag{3.4}$$

3. If $j = 2i+1$, we have $\eta = 1$ and denoting $p_i = \frac{-4x_2^i + x_1^i + 3x_3^i}{2}$, $P_i = \frac{-4y_2^i + y_1^i + 3y_3^i}{2}$, we similarly get:

$$a_{ij}^1 = \int\limits_{-2}^{0} \frac{(\rho+1)\left[\beta^2\left(\frac{\rho}{2} m_i + p_i\right) n_y^j - \left(\frac{\rho}{2} M_i + P_i\right) n_x^j\right]}{2\widehat{N}_{ij}} J(\rho)\, d\rho,$$

$$a_{ij}^2 = \int\limits_{-2}^{0} \frac{-(2+\rho)\left[\beta^2\left(\frac{\rho}{2} m_i + p_i\right) n_y^j - \left(\frac{\rho}{2} M_i + P_i\right) n_x^j\right]}{\widehat{N}_{ij}} J(\rho)\, d\rho, \quad a_{ij}^3 = an_{ij}^3 + as_{ij}^3,$$

$$an_{ij}^3 = \int\limits_{-2}^{0} \frac{\beta^2 m_i n_y^j - M_i n_x^j}{2\widehat{N}_{ij}} J(\rho)\, d\rho + \int\limits_{-2}^{0} \frac{(\rho+3)\left[\beta^2\left(\frac{\rho}{2} m_i + p_i\right) n_y^j - \left(\frac{\rho}{2} M_i + P_i\right) n_x^j\right]}{2\widehat{N}_{ij}} J(\rho)\, d\rho,$$

$$as_{ij}^3 = \int\limits_{-2}^{-\varepsilon} \frac{\beta^2 p_i n_y^j - P_i n_x^j}{\rho \widehat{N}_{ij}} J(\rho)\, d\rho \tag{3.5}$$

So all the coefficients of the system (2.4) can now be numerically evaluated using a computer and a software application. Returning to the global system of notations, we get the following form of the equation:

$$E_j g_j + \frac{1}{\pi} \sum_{i=1}^{2N} a_{ij} g_i = 2\beta n_x^j \tag{3.6}$$

where $a_{ij} = \begin{cases} a_{pj}^2, & i = 2p, p = \overline{1, N}, \\ a_{pj}^1 + a_{(p-1)j}^3, & i = 2p-1, p = \overline{2, N}, \end{cases}$ , $a_{1j} = a_{1j}^1 + a_{Nj}^3$.

The final form of (3.6) is: $\sum_{i=1}^{2N} A_{ij} g_i = 2\pi\beta n_x^j$,where $A_{ij} = \left\{ \begin{array}{c} a_{ij}, i \neq j \\ a_{jj} + E_j, i = j \end{array} \right.$.

Denoting by $A = (A_{ij})_{l<i,j<2N}$, and $B_j = 2\pi\beta n_x^j$, we can write the above system as specified in relation (2.4): $A\{g\} = \{B\}$

After solving the system we get the nodal values of the unknown function and we can further evaluate other fields of interest, as for example the velocity field and the local pressure coefficient.

## 4 Velocity and local pressure coefficient evaluation

Denoting by $u(\overline{x}_j)$ and $v(\overline{x}_j)$ the components of the velocity field we deduce, from relations (2.1), the following expressions:

$$u(\overline{x}_j) = \frac{1}{2} g_j n_y^j + \frac{1}{2\pi} \sum_{i=1}^{N} \left( g_1^i c_{ij}^1 + g_2^i c_{ij}^2 + g_3^i c_{ij}^3 \right), \tag{4.1}$$

$$v(\overline{x}_j) = -\frac{1}{2} g_j n_x^j - \frac{1}{2\pi} \sum_{i=1}^{N} \left( g_1^i b_{ij}^1 + g_2^i b_{ij}^2 + g_3^i b_{ij}^3 \right)$$

where, for $l = \overline{1,3}, i = \overline{1,N}, j = \overline{1,2N}$

$$b_{ij}^l = \int_{-1}^{1} \frac{\beta^2 \left( [N]\left\{x^i\right\} - x_j \right) n_y^j}{\left\| [N]\left\{\overline{x}\right\} - \overline{x}_j \right\|^2} J(\xi)\, d\xi, \quad c_{ij}^l = \int_{-1}^{1} \frac{\left( [N]\left\{y^i\right\} - y_j \right) n_x^j}{\left\| [N]\left\{\overline{x}\right\} - \overline{x}_j \right\|^2} J(\xi)\, d\xi \tag{4.2}$$

These coefficients have similar expressions with the coefficients of system (2.4). Those given by usual integrals have the following expressions:

$$b_{ij}^1 = \frac{1}{4} \int_{-1}^{1} \frac{m_i \xi^4 + (n_i - m_i)\xi^3 + (2u_{ij} - n_i)\xi^2 - 2u_{ij}\xi}{a_i \xi^4 + b_i \xi^3 + c_{ij} \xi^2 + d_{ij} \xi + e_{ij}} J(\xi)\, d\xi,$$

$$c_{ij}^1 = \frac{1}{4} \int_{-1}^{1} \frac{M_i \xi^4 + (N_i - M_i)\xi^3 + (2U_{ij} - N_i)\xi^2 - 2U_{ij}\xi}{a_i \xi^4 + b_i \xi^3 + c_{ij} \xi^2 + d_{ij} \xi + e_{ij}} J(\xi)\, d\xi$$

$$b_{ij}^2 = -\frac{1}{2} \int_{-1}^{1} \frac{m_i \xi^4 + n_i \xi^3 + (2u_{ij} - m_i)\xi^2 - n_i \xi - 2u_{ij}}{a_i \xi^4 + b_i \xi^3 + c_{ij} \xi^2 + d_{ij} \xi + e_{ij}} J(\xi)\, d\xi,$$

$$c_{ij}^2 = -\frac{1}{2} \int_{-1}^{1} \frac{M_i \xi^4 + N_i \xi^3 + (2U_{ij} - M_i)\xi^2 - N_i \xi - 2U_{ij}}{a_i \xi^4 + b_i \xi^3 + c_{ij} \xi^2 + d_{ij} \xi + e_{ij}} J(\xi)\, d\xi,$$

$$b_{ij}^3 = \frac{1}{4} \int_{-1}^{1} \frac{m_i \xi^4 + (n_i + m_i)\xi^3 + (2u_{ij} + n_i)\xi^2 + 2u_{ij}\xi}{a_i \xi^4 + b_i \xi^3 + c_{ij} \xi^2 + d_{ij} \xi + e_{ij}} J(\xi)\, d\xi,$$

$$c_{ij}^{1}=\frac{1}{4}\int\limits_{-1}^{1}\frac{M_i\xi^4+(N_i+M_i)\,\xi^3+(2U_{ij}+N_i)\,\xi^2+2U_{ij}\xi}{a_i\xi^4+b_i\xi^3+c_{ij}\xi^2+d_{ij}\xi+e_{ij}}J\left(\xi\right)d\xi \tag{4.3}$$

The coefficients arising from integrals with singular kernels are evaluated in the same manner as the system coefficients, using modified shape functions.

They also get similar expressions:

$$b_{ij}^{l}=\beta^2\int\limits_{-1-\eta}^{1-\eta}\frac{\left(\sum\limits_{i=1}^{3}\widehat{N}_l x_l^i\right)n_y^j}{\rho\widehat{N}_{ij}}J\left(\rho\right)d\rho,\quad c_{ij}^{l}=\int\limits_{-1-\eta}^{1-\eta}\frac{\left(\sum\limits_{i=1}^{3}\widehat{N}_l y_l^i\right)n_x^j}{\widehat{\rho N}_{ij}}J\left(\rho\right)d\rho, \tag{4.4}$$

Analyzing all singularities that appear we obtain the following results:

For $j=2i-1$:

$$b_{ij}^{1}=bn_{ij}^{1}+bs_{ij}^{1},\ \ c_{ij}^{1}=cn_{ij}^{1}+cs_{ij}^{1}$$

$$bn_{ij}^{1}=\beta^2\int\limits_{0}^{2}\frac{m_in_y^j+(\rho-3)\left(\frac{\rho}{2}m_i+p_i\right)n_y^j}{2\widehat{N}_{ij}}J\left(\rho\right)d\rho,\ \ bs_{ij}^{1}=\beta^2\int\limits_{0}^{2}\frac{p_in_y^j}{\rho\widehat{N}_{ij}}J\left(\rho\right)d\rho$$

$$cn_{ij}^{1}=\int\limits_{0}^{2}\frac{M_in_x^j-(\rho-3)\left(\frac{\rho}{2}M_i+P_i\right)n_x^j}{2\widehat{N}_{ij}}J\left(\rho\right)d\rho,\ \ cs_{ij}^{1}=\int\limits_{0}^{2}\frac{P_in_x^j}{\rho\widehat{N}_{ij}}J\left(\rho\right)d\rho$$

$$b_{ij}^{2}=\beta^2\int\limits_{0}^{2}\frac{(2-\rho)\left(\frac{\rho}{2}m_i+p_i\right)n_y^j}{\widehat{N}_{ij}}J\left(\rho\right)d\rho,\quad b_{ij}^{3}=\beta^2\int\limits_{0}^{2}\frac{(\rho-1)\left(\frac{\rho}{2}m_i+p_i\right)n_y^j}{2\widehat{N}_{ij}}J\left(\rho\right)d\rho$$

$$c_{ij}^{2}=\int\limits_{0}^{2}\frac{(2-\rho)\left[-\left(\frac{\rho}{2}M_i+P_i\right)n_x^j\right]}{\widehat{N}_{ij}}J\left(\rho\right)d\rho, \tag{4.5}$$

$$c_{ij}^{3}=\int\limits_{0}^{2}\frac{(\rho-1)\left[-\left(\frac{\rho}{2}M_i+P_i\right)n_x^j\right]}{2\widehat{N}_{ij}}J\left(\rho\right)d\rho.$$

For $j=2i$,we get the following expressions:

$$b_{ij}^{1}=\beta^2\int\limits_{-1}^{1}\frac{(\rho-1)\left(\frac{\rho}{2}m_i+p_i\right)n_y^j}{2\widehat{N}_{ij}}J\left(\rho\right)d\rho,\ \ c_{ij}^{1}=\int\limits_{-1}^{1}\frac{(\rho-1)\left[-\left(\frac{\rho}{2}M_i+P_i\right)n_x^j\right]}{2\widehat{N}_{ij}}J\left(\rho\right)d\rho$$

$$b_{ij}^{2}=bn_{ij}^{2}+bs_{ij}^{2},$$

$$bn_{ij}^{2}=\beta^2\int\limits_{-1}^{1}\frac{\frac{m_i}{2}n_y^j-\rho\left(\frac{\rho}{2}m_i+p_i\right)n_y^j}{\widehat{N}_{ij}}J\left(\rho\right)d\rho,\ \ bs_{ij}^{2}=\beta^2\int\limits_{0}^{2}\frac{p_in_y^j}{\rho\widehat{N}_{ij}}J\left(\rho\right)d\rho$$

$$c_{ij}^2 = cn_{ij}^2 + cs_{ij}^2,$$

$$cn_{ij}^2 = \int_{-1}^{1} \frac{-\frac{M_i}{2} n_x^j + \rho\left(\frac{\rho}{2} M_i + P_i\right) n_x^j}{\widehat{N}_{ij}} J(\rho)\, d\rho,\ \ cs_{ij}^2 = \int_{0}^{2} \frac{-P_i n_x^j}{\rho \widehat{N}_{ij}} J(\rho)\, d\rho$$

$$b_{ij}^3 = \beta^2 \int_{-1}^{1} \frac{(\rho+1)\left(\frac{\rho}{2} m_i + p_i\right) n_y^j}{2\widehat{N}_{ij}} J(\rho)\, d\rho, \tag{4.6}$$

$$c_{ij}^3 = \int_{-1}^{1} \frac{(\rho+1)\left[-\left(\frac{\rho}{2} M_i + P_i\right) n_x^j\right]}{2\widehat{N}_{ij}} J(\rho)\, d\rho$$

If $j = 2i+1$, we obtain:

$$b_{ij}^1 = \beta^2 \int_{-2}^{0} \frac{(\rho+1)\left(\frac{\rho}{2} m_i + p_i\right) n_y^j}{2\widehat{N}_{ij}} J(\rho)\, d\rho,\ \ c_{ij}^1 = \int_{-2}^{0} \frac{(\rho+1)\left[-\left(\frac{\rho}{2} M_i + P_i\right) n_x^j\right]}{2\widehat{N}_{ij}} J(\rho)\, d\rho$$

$$b_{ij}^2 = \beta^2 \int_{-2}^{0} \frac{-(2+\rho)\left[\left(\frac{\rho}{2} m_i + p_i\right) n_y^j\right]}{\widehat{N}_{ij}} J(\rho)\, d\rho,\ \ c_{ij}^2 = \int_{-2}^{0} \frac{(2+\rho)\left(\frac{\rho}{2} M_i + P_i\right) n_x^j}{\widehat{N}_{ij}} J(\rho)\, d\rho$$

$$b_{ij}^3 = bn_{ij}^3 + bs_{ij}^3,\ \ c_{ij}^3 = cn_{ij}^3 + cs_{ij}^3,$$

$$bn_{ij}^3 = \beta^2 \int_{-2}^{0} \frac{m_i n_y^j + (\rho+3)\left(\frac{\rho}{2} m_i + p_i\right) n_y^j}{2\widehat{N}_{ij}} J(\rho)\, d\rho, bs_{ij}^3 = \beta^2 \int_{-2}^{0} \frac{p_i n_y^j}{\rho \widehat{N}_{ij}} J(\rho)\, d\rho$$

$$cn_{ij}^3 = \int_{-2}^{0} \frac{-M_i n_x^j - (\rho+3)\left(\frac{\rho}{2} M_i + P_i\right) n_x^j}{2\widehat{N}_{ij}} J(\rho)\, d\rho, \tag{4.7}$$

$$cs_{ij}^3 = \int_{-2}^{0} \frac{-P_i n_x^j}{\rho \widehat{N}_{ij}} J(\rho)\, d\rho$$

The above coefficients can be evaluated numerically with a computer code and so, the velocity components can be found in any point of the boundary and also in any point of the fluid domain.

# 5 Numerical results and conclusions

In the following paragraphs we test the technique we used for solving the proposed problem by making an analytical checking. Based on the method exposed we have made, using Mathcad programming tools, a computer code and we found the numerical solutions of the problem in two particular cases in which exact solutions exist.

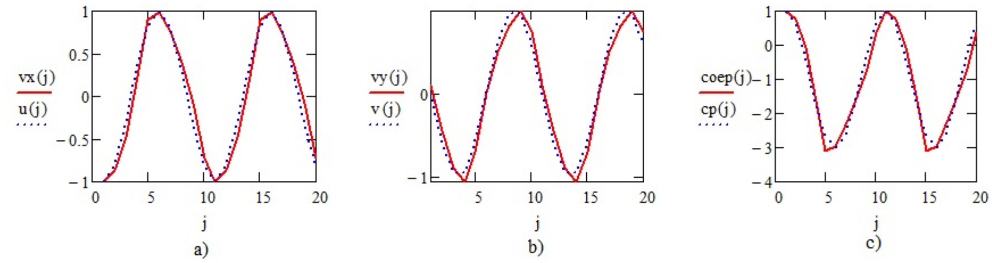


Figure 1: Circular obstacle - 20 nodes

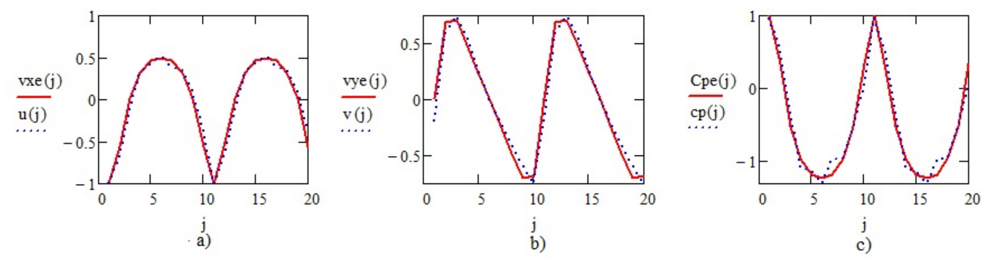


Figure 2: Elliptical obstacle - 20 nodes

The input data for the computer code are: the geometry of the obstacle, the number of nodes and the value of $\varepsilon$ which is used to evaluate the integrals with kernels having weak singularities.

The output data are: the nodal values for vortexes intensities, the values of velocity field components and the local pressure coefficient.

We have considered the problem of an incompressible ideal fluid flow ($M = 0$, or $\beta = 1$) around two different obstacles: a circular obstacle and an elliptical one. For each of these cases the exact solution of the problem can be found in paper [10]. Comparisons between the exact and the numerical values are made, for the components of the local pressure coefficients and for the components of the velocity.

For the circular obstacle the components of the velocity field are:

$u = -\cos 2\theta, v = -\sin 2\theta, cp = -1 + 2\cos 2\theta$,where $\theta$ represents the central angle.

In the figures, namely in Fig. 1, comparisons between the numerical and the exact solutions are performed for the case when 20 nodes are used for the boundary discretization and for $\varepsilon$ having the value 0.01. The components of the velocity along the $x-$axis are compared in Fig.1 a), those along the $y-$ in Fig.1 b), and in Fig.1 c) the numerical and the exact nodal values of the local pressure coefficient, are compared.

Figures 1 and 2 show that there exists a good agreement between the exact and the numerical solution in any of the comparisons made.

In case of an elliptical obstacle the exact solution of the problem, namely the dimensionless components of the velocity field are given by relations: $u(z) = \text{Re}\left(\frac{df}{dz}\right), v(z) = -\text{Im}\left(\frac{df}{dz}\right)$.For the particular case when $z = 2\cos\theta + i\sin\theta$ the exact solution is evaluated in the nodes chosen for the boundary discretization with a computer code made

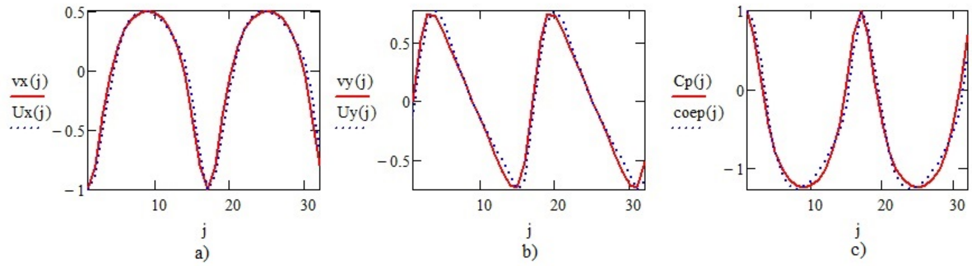


Figure 3: Elliptical obstacle - 32 nodes

also in Mathcad.

The comparisons between the numerical solution and the exact one are performed in Fig.2. for the case when 20 nodes are used for the boundary discretization and $\varepsilon = 0.01$ and in Fig.3. for 32 nodes and the same value for $\varepsilon$ (a - for the component of the velocity along Ox, b - for the component of the velocity along the $y-$axis, and c - for the local pressure coefficient).

As we can see all the figures show again a very good agreement between the two kinds of solutions, because the calculated and analytical values are very close, in both situations, even if only 20 or 32 nodes are used for the boundary discretization and a value not too small is considered for $\varepsilon$, namely 0.01.

Different number of nodes can be chosen for the boundary discretization and, as we notice by using a bigger number of nodes for the boundary discretization the numerical solution is improved.

The analytical checking validates both: the proposed approach and the computer code. As we can see from the herein paper if using quadratic boundary elements and modified shape functions for the singular integration procedure the accuracy of numerical solution is satisfactory even when we consider a small number of nodes for the boundary discretization, fact that shows the efficiency of the method proposed.

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