# On the Ψ-boundedness of the solutions of a nonlinear Lyapunov matrix differential equation

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**Abstract.** It is proved sufficient conditions for Ψ–boundedness of the solutions of a nonlinear Lyapunov matrix differential equation.

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**Key words**: Ψ–boundedness; Ψ–uniform boundedness; nonlinear Lyapunov matrix differential equation.

## 1 Introduction

The Lyapunov matrix differential equations occur in many branches of control theory such as optimal control and stability analysis. Recent works for Ψ− boundedness, Ψ− stability, Ψ–instability, controllability, dichotomy and conditioning for Lyapunov matrix differential equations have been given in many papers. See [6] - [9], [13] - [15] and the references cited therein.

The purpose of present paper is to prove sufficient conditions for Ψ− boundedness of the solutions of the nonlinear Lyapunov matrix differential equation

$$Z' = A(t)Z + ZB(t) + F(t, Z). \tag{1.1}$$

and in particular, for

$$Z' = A(t)Z + F(t, Z). \tag{1.2}$$

Here, Ψ is a matrix function whose introduction permits to obtaining a mixed asymptotic behavior for the components of solutions.

The main tool used in this paper is the technique of Kronecker product of matrices, which has been successfully applied in various fields of matrix theory, group theory and particle physics. See, for example, the above cited papers and the references cited therein.

## 2 Preliminaries

In this section we present some basic notations, definitions, hypotheses and results which are useful later on.

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Let $\mathbb{R}^d$ be the Euclidean $d-$ dimensional space. For $x = (x_1, x_2, ..., x_d)^T \in \mathbb{R}^d$, let $\| x \| = \max\{| x_1|, | x_2|, ..., | x_d|\}$ be the norm of $x$ (here, $^T$ denotes transpose). Let $\mathbb{M}_{d\times d}$ be the linear space of all $d \times d$ real valued matrices.

For $A = (a_{ij}) \in \mathbb{M}_{d\times d}$, we define the norm $| A |$ by formula $| A | = \sup_{\|x\|\leq 1} \| Ax \|$. It is well-known that $| A | = \max_{1\leq i\leq d}\{\sum_{j=1}^{d} | a_{ij}|\}$.

By a solution of the equation (1.1) we mean a continuous differentiable $d \times d$ matrix function satisfying the equation (1.1) for all $t \in \mathbb{R}_+$.

In equation (1.1), we assume that $A$ and $B$ are continuous real $d \times d$ matrices on $\mathbb{R}_+ = [0, \infty)$ and $F : \mathbb{R}_+ \times \mathbb{M}_{d\times d} \longrightarrow \mathbb{M}_{d\times d}$ is continuous such that $F(t, O_d) = O_d$ (null matrix of order $d\times d$). It is well-known that these conditions ensure the local existence of a solution of (1.1) or (1.2) passing through any given point $(t_0, Z_0) \in \mathbb{R}_+ \times \mathbb{M}_{d\times d}$, but it not guarantee that the solution is unique or that it can be continued for large values of $t$.

Let $\Psi_i : \mathbb{R}_+ \longrightarrow (0, \infty)$, $i = 1, 2, ..., d$, be continuous functions and

$$\Psi = \text{diag } [\Psi_1, \Psi_2, \cdots \Psi_d].$$

**Definition 2.1.** *([8], [10]). A function $\varphi : R_+ \longrightarrow R^d$ is said to be $\Psi-$ bounded on $R_+$ if $\Psi(t)\varphi(t)$ is bounded on $R_+$ (i.e. there exists $m > 0$ such that $\| \Psi(t)\varphi(t) \| \leq m$, for all $t \in R_+$). Otherwise, is said that the function $\varphi$* is *$\Psi-$ unbounded on $R_+$.*

**Definition 2.2.** *([8], [9]) A matrix function $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{d\times d}$ is said to be $\Psi-$ bounded on $R_+$ if the matrix function $\Psi(t)M(t)$ is bounded on $R_+$ (i.e. there exists $m > 0$ such that $| \Psi(t)M(t) | \leq m$, for all $t \in R_+$). Otherwise, is said that the matrix function $M$* is *$\Psi-$ unbounded on $R_+$.*

**Definition 2.3.** *([10]). The solutions of differential system $z' = f(t,z)$ (where $z \in R^d$ and f is a continuous d vector function) are $\Psi-$ uniformly bounded on $R_+$ if for every $\alpha > 0$,* there exists *$H(\alpha) > 0$ such that any solution $z(t)$ of the system which satisfies the inequality $\| \Psi(t_0)z(t_0) \| < \alpha$ for some $t_0 \geq 0$, exists and satisfies the inequality $\| \Psi(t)z(t) \| < H(\alpha)$* for all $t \geq t_0$.

Now, we extend this definition for a matrix differential equation $Z' = F(t, Z)$, where $Z \in \mathbb{M}_{d\times d}$ and $F$ is a continuous $d \times d$ matrix function.

**Definition 2.4.** *The solutions of matrix differential equation $Z' = F(t, Z)$ is said to be $\Psi-$ uniformly bounded on $R_+$ if for every $\alpha > 0$,* there exists *$H(\alpha) > 0$ such that any solution $Z(t)$ of the equation which satisfies the inequality $| \Psi(t_0)Z(t_0) | < \alpha$ for some $t_0 \geq 0$, exists and satisfies the inequality $| \Psi(t)Z(t) | < H(\alpha)$* for all $t \geq t_0$.

**Remark 2.5.** 1. It is easy to see that if the solutions of $z' = f(t, z)$ or $Z' = F(t, Z)$ are $\Psi-$ uniformly bounded on $\mathbb{R}_+$, they are $\Psi-$ bounded on $\mathbb{R}_+$.
A simple example shows that the reverse implication is not true in general.
2. If we replace $\Psi$ with $\Psi^k$, $k \in \mathbb{Z} \setminus \{0, 1\}$, we generalize the notion of (uniform) boundedness of degree k with respect to a function $\varphi$ (see [2]).
3. For $\Psi = I_d$, one obtain the notions of classical (uniform) boundedness (see [3]).
4. It is easy to see that if $\Psi$ and $\Psi^{-1}$ are bounded on $\mathbb{R}_+$, then the $\Psi-$ (uniform) boundedness is equivalent with the classical (uniform) boundedness.

**Definition 2.6.** *([1]). Let $A = (a_{ij}) \in M_{m\times n}$ and $B = (b_{ij}) \in M_{p\times q}$. The Kronecker product of A and B, written $A \otimes B$, is defined to be the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

Obviously, $A \otimes B \in \mathbb{M}_{mp\times nq}$. The important rules of calculation of the Kronecker product are given in [1], [12] (Chapter 2) and Lemma 1, [6].

For the vectorization operator $\mathcal{V}ec$ see Definition 2, [6] and Lemmas 2, 3, 4, [6].

For "*the corresponding Kronecker product system associated with* (1.1)", *see Lemma 5, [6].* In addition, see Lemmas 6 and 8, [6].

The following Lemma plays a vital role in the proofs of main results of present paper.

**Lemma 2.1.** *a) A solution $Z(t)$ of (1.1) is $\Psi-$bounded on $R_+$ if and only if the corresponding solution $z(t) = \mathcal{V}ec(Z(t))$ of the differential system*

$$z' = (I_d \otimes A(t) + B^T(t) \otimes I_d)z + f(t, z), \tag{2.1}$$

*where $f(t, z) = \mathcal{V}ec(F(t, Z)), z = \mathcal{V}ec(Z)$, is $I_d \otimes \Psi-$bounded on $R_+$.*

*b) The solutions of (1.1) are $\Psi-$uniformly bounded on $R_+$if and only if the solutions of the differential system (2.1) are $I_d \otimes \Psi-$uniformly bounded on $R_+$.*

*Proof.* a) Let $Z(t)$ a $\Psi-$ bounded solution on $\mathbb{R}_+$ of (1.1). From Lemma 5, [6], Definition 2.2 and Lemma 6, [6], $z(t) = \mathcal{V}ec(Z(t))$ satisfies

$$\| (I_d \otimes \Psi(t))\, z(t) \|_{\mathbb{R}^{d^2}} \leq | \Psi(t)Z(t) | \leq m, \forall t \geq 0.$$

Thus, $z(t) = \mathcal{V}ec(Z(t))$ is a $I_d \otimes \Psi-$ bounded solution on $\mathbb{R}_+$ of (2.1).
For the converse, suppose that $z(t)$ is a $I_d \otimes \Psi-$ bounded solution on $\mathbb{R}_+$ of (2.1). From Lemma 5, [6], Definition 2.1 and Lemma 6, [6], $Z(t) = \mathcal{V}ec^{-1}(z(t))$ satisfies

$$\frac{1}{d} \mid \Psi(t)Z(t) \mid \leq \| (I_d \otimes \Psi(t))\, z(t) \|_{\mathbb{R}^{d^2}} \leq m, \forall t \geq 0.$$

From Definition 2.4, $Z(t)$ is a $\Psi-$ bounded solution on $\mathbb{R}_+$ of (1.1).
b). Suppose that the solutions of (1.1) are $\Psi-$ uniformly bounded on $\mathbb{R}_+$ − see Definition 2.4. Let $z(t)$ be a solution on $\mathbb{R}_+$ of (2.1). From Lemma 5, [6], $Z(t) = \mathcal{V}ec^{-1}(z(t))$ is a solution on $\mathbb{R}_+$ of (1.1). For $\alpha > 0$, suppose that

$$\| (I_d \otimes \Psi(t_0))\, z(t_0) \|_{\mathbb{R}^{d^2}} < \frac{\alpha}{d}, \text{ for some } t_0 \geq 0.$$

From Lemma 6, [6], it follows that $\mid \Psi(t_0)Z(t_0) \mid < \alpha$. From Definition 2.4, there exists $H(\alpha) > 0$ such that $\mid \Psi(t)Z(t) \mid < H(\alpha)$, for all $t \geq t_0$. From Lemma 6, [6], again, $\| (I_d \otimes \Psi(t))\, z(t) \|_{\mathbb{R}^{d^2}} < H(\alpha)$, for all $t \geq t_0$. From Definition 2.3, it follows that the solutions of (2.1) are $I_d \otimes \Psi-$ uniformly bounded on $\mathbb{R}_+$.

For the converse, suppose that the solutions of (2.1) are $I_d \otimes \Psi-$ uniformly bounded on $\mathbb{R}_+-$ see Definition 2.3. Let $Z(t)$ be a solution on $\mathbb{R}_+$ of (1.1) such that

$$| \Psi(t_0)Z(t_0) |< \alpha, \text{ for some } t_0 \geq 0.$$

From Lemma 5, [6], $z(t) = \mathcal{V}ec(Z(t))$ is a solution on $\mathbb{R}_+$ of (2.1). From Lemma 6, [6], it follows that $\| (I_d \otimes \Psi(t_0))\, z(t_0) \|_{\mathbb{R}^{d^2}} < \alpha$. From Definition 2.3, there exists $H(\alpha) > 0$ such that $\| (I_d \otimes \Psi(t))\, z(t) \|_{\mathbb{R}^{d^2}} < H(\alpha)$, for all $t \geq t_0$. From Lemma 2.6 again, $| \Psi(t)Z(t) |< dH(\alpha)$, for all $t \geq t_0$. From Definition 2.4, the solutions of (1.1) are $\Psi-$ uniformly bounded on $\mathbb{R}_+$. □

# 3 $\Psi-$ boundedness of the solutions of a nonlinear Lyapunov matrix differential equations

The purpose of this section is to give sufficient conditions for $\Psi-$ (uniform) boundedness of the solutions of the Lyapunov matrix differential equations (1.1) and (1.2).

**Theorem 3.1.** *Suppose that the solutions of $Z' = A(t)Z$ are $\Psi-$ uniformly bounded on $R_+$ and let $F(t,Z)$ satisfy the inequality $|\Psi(t)F(t,Z)| \leq \gamma(t)\, |\Psi(t)Z|$ for $t \geq 0$ and $Z \in \mathbb{M}_{d\times d}$, where $\gamma : R_+ \to R_+$ is a continuous function and $\int_0^\infty \gamma(t)dt < +\infty$.*
*Then, the solutions of (1.2) are $\Psi-$ uniformly bounded on $R_+$.*
*If, in addition, the fundamental matrix $X(t)$ for $Z' = A(t)Z$ satisfies the condition $\lim_{t\to\infty} \Psi(t)X(t) = O_d$, then, for any solution $Z(t)$ of (1.2), $\lim_{t\to\infty} \Psi(t)Z(t) = O_d$.*

*Proof.* We first prove that there exists a positive constant $L$ such that if $t_0 \geq 0$, any solution $Z(t)$ of the equation (1.2) [that exists for $t \in [t_0, t_1)$] is defined and satisfies $|\Psi(t)Z(t)| \leq L\, |\Psi(t_0)Z(t_0)|$, for all $t \geq t_0$. If $Z(t)$ is a solution on $[t_0, t_1)$ of (1.2), then it is also a solution of the nonhomogeneous linear equation $Z' = A(t)Z + F(t, Z(t))$ on the same interval $[t_0, t_1)$. Therefore, by the Variation of constants formula,

$$Z(t) = X(t)X^{-1}(t_0)Z(t_0) + \int_{t_0}^{t} X(t)X^{-1}(s)F(s, Z(s))ds, \quad t \in [t_0, t_1).$$

(the proof is similar to the proof of well-known Variation of constants formula for the linear differential system $x' = A(t)x + f(t)$ – see [3], Chapter III, section 2(8)).

From Theorem 2, [10], it follows that for a fundamental matrix $X(t)$ for $Z' = A(t)Z$, there exists a positive constant $K$ such that $\left|\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right| \leq K$, for all $t \geq s \geq 0$. From hypotheses, it follows that

$$|\Psi(t)Z(t)| \leq K\, |\Psi(t_0)Z(t_0)| + K \int_{t_0}^{t} \gamma(s)\, |\Psi(s)Z(s)|\, ds,$$

for $t \in [t_0, t_1)$. Therefore, by Gronwall's inequality (see [3], Chapter I, Lemma 3)

$$|\Psi(t)Z(t)| \leq K\, |\Psi(t_0)Z(t_0)|\, e^{K \int_{t_0}^{t} \gamma(s)ds} \leq L\, |\Psi(t_0)Z(t_0)|\,,$$

for $t \in [t_0, t_1)$, where $L = Ke^{K \int_0^\infty \gamma(s)ds}$.

This shows that $t_1 = +\infty$ and hence, the solution $Z(t)$ is defined on $[t_0, \infty)$. Thus, we have $|\Psi(t)Z(t)| \leq L\, |\Psi(t_0)Z(t_0)|$, for $t \in [t_0, \infty)$. From this and Definition

2.4, the solutions of (1.2) are $\Psi-$ uniformly bounded on $\mathbb{R}_+$. Now, we prove that $|\Psi(t)Z(t)| \to 0$ as $t \to \infty$, if $\Psi(t)X(t) \to O_d$ as $t \to \infty$. From hypotheses, it follows that

$$|\Psi(t)Z(t)| \leq |\Psi(t)X(t)| \left|X^{-1}(t_0)\Psi^{-1}(t_0)\right| + \int_{t_0}^{t} \left|\Psi(t)X(t)X^{-1}(s)F(s,Z(s))\right| ds, t \geq t_0.$$

Given any $\varepsilon > 0$, we can choose $t_2 \geq t_0$ so large that

$$KL\,|\Psi(t_0)Z(t_0)| \int_{t2}^{\infty} \gamma(s)ds < \frac{\varepsilon}{2}.$$

From the above inequality, we thus obtain

$$|\Psi(t)Z(t)| \leq |\Psi(t)X(t)| \left[\left|X^{-1}(t_0)\Psi^{-1}(t_0)\right| + \int_{t_0}^{t_2} \left|X^{-1}(s)F(s,Z(s))\right| ds\right] + \int_{t2}^{t} \left|\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right| \gamma(s)\,|\Psi(s)Z(s)|\,ds,$$

and then

$$|\Psi(t)Z(t)| \leq |\Psi(t)X(t)| \left[\left|X^{-1}(t_0)\Psi^{-1}(t_0)\right| + \int_{t_0}^{t_2} \left|X^{-1}(s)F(s,Z(s))\right| ds\right] + +KL\,|\Psi(t_0)Z(t_0)| \int_{t2}^{\infty} \gamma(s)ds, \text{ for } t \geq t_2.$$

We can choose $t_3 \geq t_2$ so large that

$$|\Psi(t)X(t)| \leq \frac{\varepsilon}{2\left[|X^{-1}(t_0)\Psi^{-1}(t_0)| + \int_{t_0}^{t_2} |X^{-1}(s)F(s,Z(s))|\,ds\right]}, \text{ for } t \geq t_3.$$

Now, we have $|\Psi(t)Z(t)| \leq \varepsilon$, for $t \geq t_3$. This shows that $|\Psi(t)Z(t)| \to 0$ as $t \to \infty$. □

**Remark 3.1.** Similar results are Theorem 5, [6] (in connection with $\Psi-$ stability of a nonlinear Lyapunov matrix differential equation) and Theorem 5, [7] (in connection with $\Psi-$asymptotic stability of a nonlinear Lyapunov matrix differential equation. The proofs of these Theorems appeal to Kronecker product of matrices and to theory of systems of differential equations. The above Theorem 3.1 has a direct proof.

**Remark 3.2.** 1. The Theorem contains as a particular case a result concerning $\Psi-$uniform boundedness of solutions of the differential system $x' = A(t)x + f(t,x)$, situated in Theorem 6, [3], Chapter III, section 3. Indeed, consider in (1.2)

$$Z = \begin{pmatrix} x_1 & x_1 & \cdots & x_1 \\ x_1 & x_2 & \cdots & x_2 \\ \vdots & \vdots & \vdots & \vdots \\ x_d & x_d & \cdots & x_d \end{pmatrix} \text{ and } F(t,Z) = \begin{pmatrix} f_1(t,x) & f_1(t,x) & \cdots & f_1(t,x) \\ f_2(t,x) & f_2(t,x) & \cdots & f_2(t,x) \\ \vdots & \vdots & \vdots & \vdots \\ f_d(t,x) & f_d(t,x) & \cdots & f_d(t,x) \end{pmatrix}$$

where $z = (x_1, x_1, ...x_d)^T$ and $f(t,x) = (f_1(t,x), f_2(t,x), \cdots f_d(t,x))^T$.
Now, the definitions and conditions for $\Psi-$ (uniform) boundedness on $R_+$ of $x$ are the same for $\Psi-$ (uniform) boundedness on $R_+$ of $Z$ in (1.2).
2. In conditions of Theorem and a supplementary condition, the trivial solution of (1.2) is $\Psi-$ uniformly stable on $R_+$ (see [6]).
3. Theorem 3.1 generalizes Theorem 3.4, [11].

The Theorem from above is no longer true if we require that $\gamma(t) \to 0$ as $t \to \infty$ instead of the integrability of $\gamma(t)$ over the interval $[0, \infty)$. To see this, we give the next example, transformed after an example due to A. Wintner [16].

**Example 3.3.** Consider equation $Z' = A(t)Z$ with $A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
Then, $X(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ is a fundamental matrix for this equation.
Let $\Psi(t) = \frac{1}{\sqrt{t+1}} I_2$. We have

$$\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) = \sqrt{\frac{s+1}{t+1}} \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix},$$

and then $\left|\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right| \le \sqrt{2}$, $\forall t \ge s \ge 0$.

Thus, (Theorem 3.1, [11]) the solutions of $Z' = A(t)Z$ are $\Psi-$ uniformly bounded on $R_+$. If we take $F(t, Z) = \left(\begin{smallmatrix} 0 & 0 \\ -b(t) & 0 \end{smallmatrix}\right) Z$, where

$$b(t) = \frac{2\sin(t+1)\cos t + 6\cos(t+1)\sin t}{t+1} + \frac{2\cos t\cos(t+1) - 4\cos^2(t+1)\cos^2 t}{(t+1)^2},$$

we have $|\Psi(t)F(t, Z)| \le |b(t)|\,|\Psi(t)Z|$ and $\lim\limits_{t\to\infty} |b(t)| = 0$. The equation (1.2) becomes $Z' = \begin{pmatrix} 0 & 1 \\ -(1+b(t)) & 0 \end{pmatrix} Z$. Now, we take $x(t) = (\cos t)\, e^{2\int_0^t \frac{\cos(s+1)\cos s}{s+1} ds}$, $t \ge 0$.
It is easily verified by differentiation that $Z_0(t) = \begin{pmatrix} x(t) & 0 \\ x'(t) & 0 \end{pmatrix}$ is a solution of the equation (1.2). It follows that $\Psi(t)Z_0(t) = \begin{pmatrix} \frac{1}{\sqrt{t+1}} x(t) & 0 \\ \frac{1}{\sqrt{t+1}} x'(t) & 0 \end{pmatrix}$. Because

$$\frac{1}{\sqrt{t+1}} x(t) = \frac{\cos t}{\sqrt{t+1}} e^{2\int_0^t \frac{\cos(s+1)\cos s}{s+1} ds} = (\cos t)\, e^{2\int_0^t \frac{\cos(s+1)\cos s}{s+1} ds - \frac{1}{2}\ln(t+1)} =$$
$$(\cos t)\, e^{\int_0^t \frac{\cos(2s+1)+\cos 1}{s+1} ds - \frac{1}{2}\ln(t+1)} = (\cos t)\, e^{\int_0^t \frac{\cos(2s+1)}{s+1} ds + (\cos 1 - \frac{1}{2})\ln(t+1)},$$

the integral $\int_0^\infty \frac{\cos(2s+1)}{s+1} ds$ is divergent and $\cos 1 > \frac{1}{2}$, we have

$$\lim_{n\to\infty} \frac{x(2n\pi)}{\sqrt{2n\pi+1}} = +\infty.$$

Thus, $\Psi(t)Z_0(t)$ is not bounded on $R_+$. The affirmation of the Remark is correct.

**Theorem 3.2.** *Suppose that there exists a constant $K > 0$ such that the fundamental matrix $X(t)$ for the linear matrix differential equation $Z' = A(t)Z$ satisfies the condition $\int_0^t \left|\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right| ds \le K$, for all $t \ge 0$ and let $F(t, Z)$ satisfy the inequality $|\Psi(t)F(t, Z)| \le \gamma\,|\Psi(t)Z|$ for $t \ge 0$ and $Z \in \mathbb{M}_{d\times d}$, where $\gamma \in \left(0, \frac{1}{K}\right)$. Then, the solutions of (1.2) are $\Psi-$ bounded on $R_+$. In addition, for every solution $Z(t)$ of (1.2), $\Psi(t)Z(t) \to O_d$ as $t \to \infty$.*

*Proof.* We first prove that for any $t_0 \geq 0$, there exists a constant $L(t_0) > 0$ such that, any solution $Z(t)$ of the equation (1.2) (that exists on an interval $[t_0, t_1) \subset R_+$) is defined and satisfies $|\Psi(t)Z(t)| \leq L(t_0) |\Psi(t_0)Z(t_0)|$, for all $t \geq t_0$. From hypothesis and Lemma 1, [4], $\Psi(t)X(t) \to O_d$ as $t \to \infty$ and in particular, for any $t_0 \geq 0$, there exists a constant $M(t_0) > 0$ such that

$$|\Psi(t)X(t)| \leq M(t_0), \text{ for all } t \geq t_0.$$

For any solution $Z(t)$ of (1.2) we have, by Variation of constant formula,

$$Z(t) = X(t)X^{-1}(t_0)Z(t_0) + \int_{t_0}^{t} X(t)X^{-1}(s)F(s, Z(s))ds, \text{ for } t \in [t_0, t_1).$$

From hypotheses, it follows that

$$|\Psi(t)Z(t)| \leq M(t_0) \left|X^{-1}(t_0)\Psi^{-1}(t_0)\right| |\Psi(t_0)Z(t_0)| + \gamma K \sup_{t_0 \leq s \leq t} |\Psi(s)Z(s)|$$

and hence $\sup_{t_0 \leq s \leq t} |\Psi(s)Z(s)| \leq (1 - \gamma K)^{-1} M(t_0) \left|X^{-1}(t_0)\Psi^{-1}(t_0)\right| |\Psi(t_0)Z(t_0)|$.

Thus, $|\Psi(t)Z(t)| \leq (1 - \gamma K)^{-1} M(t_0) \left|X^{-1}(t_0)\Psi^{-1}(t_0)\right| |\Psi(t_0)Z(t_0)|$, for all $t \geq t_0$. This shows that $t_1 = +\infty$ and hence, the solution $Z(t)$ is defined on $[t_0, +\infty)$. Thus, we have $|\Psi(t)Z(t)| \leq L(t_0) |\Psi(t_0)Z(t_0)|$, for all $t \geq t_0$, where the constant L($t_0$) is $L(t_0) = (1 - \gamma K)^{-1} M(t_0) \left|X^{-1}(t_0)\Psi^{-1}(t_0)\right|$.

From this and Definition 2.2, the solutions $Z(t)$ of (1.2) are $\Psi-$ bounded on $\mathbb{R}_+$. Further on, let $\lambda = \overline{\lim_{t \to \infty}} |\Psi(t)Z(t)|$ and choose $\theta$ so that $\gamma K < \theta < 1$. If $\lambda > 0$, there exists $t_1 \geq t_0$ such that $|\Psi(t)Z(t)| \leq \theta^{-1}\lambda$, for all $t \geq t_1$. Then, for all $t \geq t_1$,

$$|\Psi(t)Z(t)| \leq |\Psi(t)X(t)| \left|X^{-1}(t_0)Z(t_0)\right| +$$
$$+ |\Psi(t)X(t)| \left|\int_{t_0}^{t_1} X^{-1}(s)F(s, Z(s))ds\right| + \gamma K \theta^{-1}\lambda.$$

Letting $t \to \infty$, we get $\lambda \leq \gamma K \theta^{-1}\lambda$, which is a contradiction. Therefore, $\lambda = 0$ and then, $\Psi(t)Z(t) \to O_d$ as $t \to \infty$. □

**Remark 3.4.** If $C(t)$ is a continuous matrix function defined on $R_+$ with $\Psi(t)C(t) \to O_d$ as $t \to \infty$, then by the same method we can prove that in conditions of Theorem, every solution $Z(t)$ of the equation $Z' = A(t)Z + F(t, Z) + C(t)$ is defined on $[t_0, +\infty)$ and $\Psi(t)Z(t) \to O_d$ as $t \to \infty$.

**Remark 3.5.** A similar results are in Theorems 7 and 8, [6] (in connection with $\Psi-$ *stability of a nonlinear Lyapunov matrix differential equation) and Theorems 7 and 8, [7] (*in connection with $\Psi-$ asymptotic stability of a nonlinear Lyapunov matrix differential equation*). The proofs of these Theorems appeal to Kronecker product of matrices and to theory of systems of differential equations. The above Theorem 3.2 has a direct proof.*

**Remark 3.6.** The Theorem 3.1 and Theorem 3.2 are no longer true if the solutions of $Z' = A(t)Z$ are only $\Psi-$ bounded on $R_+$. This is shown by the Example 3.5, [11], adapted to a linear matrix differential equation.

**Theorem 3.3.** *Suppose that:*
*a). the solutions of $Z' = A(t)Z + ZB(t)$ are $\Psi-$ uniformly bounded on $R_+$;*
*b). the matrix function $F(t, Z)$ satisfies the inequality $|\Psi(t)F(t, Z)| \leq \gamma(t) |\Psi(t)Z|$ for all $t \geq 0$ and $Z \in \mathbb{M}_{d \times d}$, where $\gamma : R_+ \to R_+$ is a continuous function such that $L = \int_0^{\infty} \gamma(t)dt < +\infty$.*
*Then, the solutions of (1.1) are $\Psi-$ uniformly bounded on $R_+$.*

*If, in addition, the solutions of $Z' = A(t)Z + ZB(t)$ tend to zero as $t \to \infty$, then the solutions of (1.1) are alike.*

*Proof.* We first prove there exists a positive constant $N$ such that if $t_0 \geq 0$, any solution $Z(t)$ of the equation (1.1) is defined and satisfies

$$|\Psi(t)Z(t)| \leq N\,|\Psi(t_0)Z(t_0)|\,, \text{ for all } t \geq t_0.$$

We will apply the Theorem 3.1, variant for differential systems (see Remark 3.2).
Let $Z(t)$ a solution of equation (1.1), with $Z(t_0) = Z_0$, $t_0 \geq 0$, that exists for $t \in [t_0, t_1)$. From Lemma 5, [6], we know that the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the differential system (2.1), on the same interval $[t_0, t_1)$. The hypothesis a) ensure, via Lemma 2.1 and Lemma 5, [6], that the solutions of the linear homogeneous differential system associated with (2.1) are $I_d \otimes \Psi-$ uniformly bounded on $R_+$. From hypothesis b) and Lemma 6, [6], it follows that

$$\| (I_d \otimes \Psi(t))\, f(t,z) \|_{R^{d^2}} = \| (I_d \otimes \Psi(t))\, \mathcal{V}ec(F(t,Z)) \|_{R^{d^2}} \leq$$
$$\leq |\Psi(t)F(t,Z)| \leq \gamma(t)\, |\Psi(t)Z| \leq d\gamma(t)\, \| (I_d \otimes \Psi(t))\, \mathcal{V}ec(Z) \|_{R^{d^2}} =$$
$$= d\gamma(t)\, \| (I_d \otimes \Psi(t))\, z \|_{R^{d^2}}, \text{ for all } t \in R_+ \text{ and } z \in \mathbb{R}^{d^2}.$$

We see from this that is ensured the second hypothesis of Theorem 3.1. From Theorem 3.1, Remark 3.2, Lemma 2.1, Lemma 5, [6], Theorem 2, [10] and Lemma 8, [6], there exists a constant $N > 0$ such that

$$\| (I_d \otimes \Psi(t))\, z(t) \|_{R^{d^2}} \leq N\, \| (I_d \otimes \Psi(t_0))\, z(t_0) \|_{R^{d^2}}, t \geq t_0.$$

From Lemma 5, [6], and Lemma 6, [6], the solution $Z(t)$ is defined for $t \geq t_0$ and $|\Psi(t)Z(t)| \leq dN\, |\Psi(t_0)Z(t_0)|\,, t \geq t_0$.
From this and Definition 2.4, the solutions of (1.1) are $\Psi-$ uniformly bounded on $\mathbb{R}_+$.
The last part of proof results from Lemma 7, [7], Theorem 3.1 and Lemma 6, [6]. □

**Remark 3.7.** 1. The Theorem contains as a particular case Theorem 3.1.
2. A similar results are in Theorem 5, [6], Theorem 5, [7], in connection with $\Psi-$ (asymptotic) stability of (1.1). The proof of Theorem is new, based on Theorem 3.1.
3. Theorem contains as a particular case a result concerning $\Psi-$ uniform boundedness of solutions of the differential system $x' = A(t)x + f(t,x)$, situated in Theorem 6, [3], Chapter III, section 3.

**Theorem 3.4.** *Suppose that there exists a constant $K > 0$ such that the fundamental matrices $X(t)$ and $Y(t)$ for the equations $Z' = A(t)Z$ and $Z' = ZB(t)$ respectively satisfy the condition*

$$\int_0^t \left|\left(Y^T(t)(Y^T)^{-1}(s)\right) \otimes \left(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\right)\right| ds \leq K$$

*for all $t \geq 0$ and let $F(t,Z)$ satisfy the inequality*

$$|\Psi(t)F(t,Z)| \leq \gamma\, |\Psi(t)Z|$$

*for $t \geq 0$ and $Z \in \mathbb{M}_{d\times d}$, where $\gamma \in \left(0, \frac{1}{dK}\right)$. Then, the solutions of (1.1) are $\Psi-$ bounded on $R_+$. In addition, for every solution $Z(t)$ of this equation,*

$$\Psi(t)Z(t) \to O_d \text{ as } t \to \infty.$$

*Proof.* We first prove that for any $t_0 \geq 0$, there exists a constant $N(t_0) > 0$ such that, any solution $Z(t)$ of the equation (1.1) (that exists on an interval $[t_0, t_1) \subset R_+$) is defined and satisfies

$$|\Psi(t)Z(t)| \leq N(t_0)\,|\Psi(t_0)Z(t_0)|\,,\text{ for all } t \geq t_0.$$

We will apply the Theorem 3.2, variant for differential systems (see Remark 3.2). Let $Z(t)$ a solution of equation (1.1), with $Z(t_0) = Z_0$, $t_0 \geq 0$, that exists for $t \in [t_0, t_1)$. From Lemma 5, [6], we know that the vector function $z(t) = \mathcal{V}ec(Z(t))$ is a solution of the differential system (2.1), on the same interval $[t_0, t_1)$. From Lemma 8, [6], we know that the matrix $U(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the linear homogeneous differential system associated with (2.1).
The first hypothesis ensures that the matrix $U(t)$ satisfies the first hypothesis of Theorem 3.2.
From hypothesis and Lemma 6, [6], it follows that

$$\begin{aligned} \| & \ (I_d \otimes \Psi(t))\, f(t,z)\ \|_{R^{d^2}} = \ \|\ (I_d \otimes \Psi(t))\, \mathcal{V}ec(F(t,Z))\ \|_{R^{d^2}} \leq \\ \leq & \ |\Psi(t)F(t,Z)| \leq \gamma\, |\Psi(t)Z| \leq d\gamma\ \|\ (I_d \otimes \Psi(t))\, \mathcal{V}ec(Z)\ \|_{R^{d^2}} = \\ = & \ d\gamma\ \|\ (I_d \otimes \Psi(t))\, z\ \|_{R^{d^2}}, \end{aligned}$$

for all $t \in R_+$ and $z \in \mathbb{R}^{d^2}$. We see from this that the function $f(t,z) = \mathcal{V}ec(F(t,Z))$, $z = \mathcal{V}ec(Z)$, ensures the second hypothesis of Theorem 3.2. Now, from this Theorem, it follows that for any $t_0 \geq 0$, there exists a constant $L(t_0) > 0$ such that the solution $z(t)$ of (2.1) is defined and satisfies

$$\|\ (I_d \otimes \Psi(t))\, z(t)\ \|_{R^{d^2}} \leq L(t_0)\ \|\ (I_d \otimes \Psi(t_0))\, z(t_0)\ \|_{R^{d^2}},\text{ for } t \geq t_0.$$

and

$$\lim_{t\to\infty}\ \|\ (I_d \otimes \Psi(t))\, z(t)\ \|_{R^{d^2}} = 0.$$

From Lemma 5, [6] and Lemma 6, [6], the solution $Z(t)$ is defined for $t \geq t_0$ and

$$|\Psi(t)Z(t)| \leq N(t_0)\,|\Psi(t_0)Z(t_0)|\,,\text{ for all } t \geq t_0,$$

where $N(t_0) = dL(t_0)$. From this and Definition 2.2, the solutions $Z(t)$ of (1.1) are $\Psi-$ bounded on $\mathbb{R}_+$. In addition, $\lim_{t\to\infty} |\Psi(t)Z(t)| = 0$. □

**Remark 3.8.** 1. The Theorem contains as a particular case Theorem 3.2.
2. If $C(t)$ is a continuous matrix function defined on $R_+$ with $\Psi(t)C(t) \to O_d$ as $t \to \infty$, then by the same method we can prove that in conditions of Theorem, every solution $Z(t)$ of the equation

$$Z' = A(t)Z + ZB(t) + F(t,Z) + C(t) \tag{3.1}$$

is defined on $[t_0, +\infty)$ and $\Psi(t)Z(t) \to O_d$ as $t \to \infty$.
3. A similar results are in Theorems 7 and 8, [7], in connection with $\Psi-$ *asymptotic stability of (1.1).*

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