A posteriori error estimates for generalized Schwarz method for HJB equation related to management of energy production with mixed boundary condition

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Abstract. In this paper, a posteriori error estimates for the generalized Schwarz method (GSM) for evolutionary Hamilton-Jacobi-Belmann equation with linear source terms related to management of energy production with mixed boundary condition (MBC) are established, using a theta scheme with a Galerkin spatial approximation and the techniques of the residual a posteriori error analysis.

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1 Introduction

We consider the following evolutionary inequalities: find $u\,(x,t)$ such that $u \in L^2\,(0,T;K(u))\,,$ $u_t \in L^2\left(0,T;L^2(\Omega)\right)$ and

$$\frac{\partial u^i}{\partial t} + \max_{i=1,\ldots,M}\left(-\Delta u^i + a_0^i u^i - f^i\right) = 0, \quad \text{in } K, \tag{1.1}$$

where K is an implicit convex set defined as follows:

$$K = \left\{\begin{array}{l} u^i \in L^2\left(0,T,H_0^1(\Omega)\right) \cap C^{\,2}\left(0,T,H^{-1}(\Omega)\right), \\ u^i\,(x) \le l + u^{i+1},\ u^i = 0 \text{ in } \Gamma,\quad u^i = 0 \text{ in } \Gamma/\Gamma_0, \\ u^i\,(x,0) = u_0^i \text{ in } \Omega, i = 1,...,M. \end{array}\right\},$$

Ω is a bounded smooth domain in $\mathbb{R}^d$, $d \ge 1$, Σ is a set in $\mathbb{R}\times\mathbb{R}^d$ defined as $\Sigma = [0,\ T]\times\Omega$ with $T < +\infty$, and $a_0^i \in L^2\,(0,T,L^\infty(\Omega)) \cap C^{\,0}\left(0,T,H^{-1}(\Omega)\right),\ i \in \overline{1,M}$ and the right hand side $f^i \in \left(L^2\left(0,T,L^2(\Omega)\right) \cap C^{\,1}\left(0,T,H^{-1}(\Omega)\right)\right)^M$.

The problem (1.1) can be approximated by the following system of the continuous parabolic inequalities: find $\left(u^1,u^2,...,u_h^M\right) \in \left(L^2\left(0,T,H_0^1(\Omega)\right)\right)^M$, solution to

$$\frac{\partial u^i}{\partial t} + A^i u^i \le f^{\;i} \text{ in } K,$$

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which is similar to that in [4] and [7] which investigated the evolutionary free boundary problems. The problem (1.1) can be transformed into the following system of evolutionary quasi variational inequalities: find $u^i \in L^2\left(0,T,H_0^1(\Omega)\right)$ solution of

$$(1.2)\qquad \begin{cases} \left(\frac{\partial u^i}{\partial t}, v-u^i\right)_\Omega + a^i\left(u^i, v-u^i\right) \geq \left(f^i, v-u^i\right)_\Omega + \left(\varphi^i, v-u^i\right)_{\Gamma_0}, \\ u^i = 0 \text{ in } \Gamma/\Gamma_0,\ u^i(x,0) = u_0^i \text{ in } \Omega, \\ \dfrac{\partial u^i}{\partial \eta} = \varphi^i \text{ in } \Gamma_0, i \in \overline{1,M}, \end{cases}$$

where $a^i(.,.)$ is the bilinear form defined as: for $u,v \in H_0^1(\Omega) : a^i\left(u^i,u^i\right) = \left(\nabla u^i, \nabla u^i\right) - \left(a_0^i u^i, u^i\right)$ and $a_0 \in L^2\left(0,T,L^\infty(\Omega)\right) \cap C^{\ 0}\left(0,T,H^{-1}(\Omega)\right)$ is sufficiently smooth functions and satisfy the following condition: $a_0(t,x) \geq \beta > 0,\ \beta$ is a constant.

M is an operator given by $Mu^i = k + \inf_{i\neq\mu} u^\mu$ where $k > 0$ and $\mu > 0$ and Γ_0 is the part of the boundary defined by: $\Gamma_0 = \left\{x \in \partial\Omega = \Gamma \text{ such that } \forall \xi > 0,\ x+\xi \notin \bar{\Omega}\right\}$ where $\overrightarrow{\eta}_i$ is the normal vector, the symbol $(.,.)_{\Gamma_0}$ stands for the inner product in $L^2(\Gamma_0)$ and in [11] M is satisfying the following proprieties: for all $u,v \in C(\Omega)$, we have

$$\begin{cases} M\left(\delta u + (1-\delta)v\right) \geq \delta M(u) + (1-\delta)M(v), \\ \text{For all } \eta \in \mathbb{R},\ \ M(u+\eta) = M(u) + \eta. \end{cases}$$

The system of evolutionary quasi-variational inequality (1.1) emerged from many scientific, engineering and economic problems, e.g., from the heat control problem, the Stefan problem, and the American option problem [1].

In this paper, we prove an a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains of evolutionary HJB equation with linear source terms using the theta time scheme combined with a finite element spatial approximation, similar to that in our published papers in [3, 4, 7, 8], which investigate the Laplace operator.

2 The discrete system of parabolic quasi-variational inequalities

In [4], the problem (1.2) can be reformulated as the following coercive discrete system of elliptic quasi-variational inequalities

$$(2.1)\qquad \begin{cases} b^i\left(u_h^{i,\theta,k}, v_h - u_h^{i,\theta,k}\right) \geq \left(f^{i,\ \theta,k} + \mu u_h^{i,k-1}, v_h - u_h^{i,\theta,k}\right)_\Omega \\ +\left(\varphi^{i,\theta,k}, \left(v_h - u_h^{i,\theta,k}\right)\right)_{\Gamma_0}, \\ v_h,\ u_h^{i,\theta,k} = \theta u_h^{i,k} + (1-\theta)u_h^{i,k-1} \in V_h^i,\ \theta \in [0,1], \\ f^{\ i,\theta,k} = \theta f^{i,k} + (1-\theta)f^{i,k-1},\ \varphi^{\ \theta,k} = \theta\varphi^k + (1-\theta)\varphi^{k-1}, \end{cases}$$

where

$$(2.2)\qquad \begin{cases} b^i\left(u_h^{i,\theta,k}, v_h - u_h^{i,\theta,k}\right) = \mu\left(u_h^{i,\theta,k}, v_h - u_h^{i,\theta,k}\right)_\Omega + a\left(u_h^{i,\theta,k}, v_h - u_h^{i,\theta,k}\right), \\ \mu = \dfrac{1}{\theta\Delta t} = \dfrac{p}{\theta T} \end{cases}.$$

and

$$V^{i,h}=\left\{\begin{array}{l} v^{i}\in\left(L^{2}\left(0,T,H_{0}^{1}(\Omega)\right)\cap C\left(0,T,H_{0}^{1}\left(\bar{\Omega}\right)\right)\right)^{M}, \\ \text{such that } v_{h}^{i}\mid_{K}=P_{1},\ k\in\tau_{h},\ v_{h}^{i}\leq r_{h}Mv_{h}^{i}, \\ v_{h}^{i}\left(.,0\right)=v_{^{i}h0}\text{ in }\Omega,\ \dfrac{\partial v_{h}^{i}}{\partial\eta}=\varphi^{i}\text{ in }\Gamma_{0},\ v_{h}^{i}=0\text{ in }\Gamma\backslash\Gamma_{0}, \end{array}\right\} \tag{2.3}$$

where P_1 Lagrangian polynomial of degree less than or equal to 1and r_h be the usual interpolation operator defined by

$$r_{h}v=\sum_{i=1}^{m(h)}v\left(M_{i}\right)\varphi_{i}\left(x\right).$$

3 The continuity space for the generalized Schwarz method

We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that $\Omega_1\cap\Omega_2=\Omega_{12},\ \ \partial\Omega_s\cap\Omega_t=\Gamma_s,\ s\neq t$ and $s,t=1,2$. We need the spaces

$$V_{s}=H^{1}(\Omega)\cap H^{1}(\Omega_{s})=\left\{v\in H^{1}(\Omega_{i})\mid v_{\partial\Omega_{i}\cap\partial\Omega}=0\right\},$$

and

$$W_{s}=H_{0}^{\frac{1}{2}}(\Gamma_{s})=\left\{v_{\Gamma_{s}}\mid v\in V_{s}\ \text{ and } v=0 \text{ on }\ \partial\Omega_{s}\backslash\Gamma_{s}\right\},$$

which is a subspace of

$$H^{\frac{1}{2}}(\Gamma_{s})=\left\{\psi\in L^{2}(\Gamma_{s})\mid\psi=\varphi_{\Gamma_{s}}\ \text{ for } \varphi\in V_{s},\ s=1,2\right\},$$

with its norm $\|\varphi\|_{W_s}=\inf\limits_{v\in V_s v=\varphi \text{ on } \Gamma_s}\|v\|_{1,\Omega}$.

We define the continuous counterparts of the continuous Schwarz sequences by $u_1^{i,k,m+1}\in\left(H_0^1(\Omega)\right)^M,\ m=0,1,2,...,i=1,...,M$ solution of

$$\left\{\begin{array}{l} c^{i}\left(u_{1}^{i,\theta,k,m+1},v-u_{1}^{i,\theta,k,m+1}\right)\geq \\ \left(F^{i,\theta}\left(u_{1}^{i,\theta,k-1,m+1}\right),v-u_{1}^{i,\theta,k,m+1}\right)_{\Omega_{1}}+\left(\varphi^{i},v-u^{i,\theta,k,m+1}\right)_{\Gamma_{0}}, \\ u_{1}^{i,\theta,k,m+1}=0,\quad\text{on }\partial\Omega_{1}\cap\partial\Omega=\partial\Omega_{1}-\Gamma_{1}, \\ \dfrac{\partial u_{1}^{i,\theta,k,m+1}}{\partial\eta_{1}}+\alpha_{1}u_{1}^{i,\theta,k,m+1}=\dfrac{\partial u_{2}^{i,\theta,k,m}}{\partial\eta_{1}}+\alpha_{1}u_{1}^{i,\theta,k,m}\text{ on }\Gamma_{1}, \end{array}\right. \tag{3.1}$$

where η_s is the exterior normal to Ω_s and α_s is a real parameter, $s\in\{1,2\}$.

In the next sections, our main interest is to obtain an a posteriori error estimate, we need for stopping the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in Laplace operator with the new boundary conditions of generalized Schwarz alternating method, we get

$$\left(-\Delta\ u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} = \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1}$$
$$- \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1}$$
$$= \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} - \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1}$$

and thus we can deduce

$$\left(-\Delta u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} = \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1}$$
$$- \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1}$$
$$= \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} -$$
$$\left(\frac{\partial u_2^{i,\theta,k,m+1}}{\partial \eta_2} + \alpha_1\ u_2^{i,\theta,k,m} - \alpha_1 u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1}$$
$$= \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} + \left(\alpha_1 u_1^{i,\theta,k,m+1}, v_1^i - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1}$$
$$= \left(\nabla u_1^{i,\theta,k,m+1}, \nabla\left(v_1 - u_1^{i,\theta,k,m+1}\right)\right)_{\Omega_1} + \left(\alpha_1 u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1}$$
$$- \left(\frac{\partial u_2^{i,\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{i,\theta,k,m}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1},$$

thus the problem (3.1) is equivalent to the problem of finding $u_1^{i,\theta,k,m+1} \in V_1$, such that

$$\begin{aligned}
&c(u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}) + \left(\alpha_1 u_1^{i,\theta,k,m}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\
&\geq \left(F^\theta(u_1^{i,\theta,k-1,m+1}), v_1 - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} + \left(\varphi^i, v - u_1^{i,\theta,k,m+1}\right)_{\Gamma_0} \\
&+ \left(\frac{\partial u_2^{i,\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{i,\theta,k,m}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1}, \forall v_1 \in V_1,
\end{aligned} \tag{3.2}$$

and we have $u_2^{i,\theta,k,m+1} \in V_2$,

$$\begin{aligned}
&c^i(u_2^{i,\theta,k,m+1}, v_2 - u_2^{i,\theta,k,m+1}) + \left(\alpha_2 u_2^{i,\theta,k,m+1}, v_2 - u_2^{i,\theta,k,m+1}\right)_{\Gamma_2} \\
&\geq \left(F^i(u_2^{i,\theta,k-1,m+1}), v_2 - u_2^{i,\theta,k,m+1}\right)_{\Omega_2} + \left(\varphi^i, v - u^{i,\theta,k,m+1}\right)_{\Gamma_0} \\
&\left(\frac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_2} + \alpha_2 u_1^{i,\theta,k,m}, v_2 - u_2^{i,\theta,k,m+1}\right)_{\Gamma_2}.
\end{aligned} \tag{3.3}$$

4 A posteriori error estimate

To define the auxiliary inequalities, we need to split the domain Ω into two sets of disjoint subdomains : (Ω_1, Ω_3) and (Ω_2, Ω_4) such that $\Omega = \Omega_1 \cup \Omega_3$, with $\Omega_1 \cap \Omega_3 = \varnothing$ $\Omega = \Omega_2 \cup \Omega_4$ and $\Omega_2 \cap \Omega_4 = \phi$.

Let $(u_1^{i,k,m}, u_2^{i,k,m})$ be the solution of problems (3.1), we define the couple $(u_1^{i,k,m}, u_3^{i,k,m})$ over (Ω_1, Ω_3) to be the solution of the following non-overlapping inequalities

(4.1)
$$\begin{cases} \dfrac{u_1^{i,k,m+1} - u_1^{i,k-1,m+1}}{\Delta t} - \Delta\ u_1^{i,\theta,k,m+1} + a_0^{i,k} u_1^{i,\theta,k,m+1} \geq F^{\ i,\theta}\ \left(u_1^{i,\theta,k-1,m+1}\right) \text{ in } \Omega_1, \\ u_1^{i,\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\ k = 1, ..., n, \\ \dfrac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_1} + \alpha u_1^{i,\theta,k,m} = \dfrac{\partial u_2^{i,\theta,k,m+1}}{\partial \eta_1} + \alpha_1 u_2^{i,\theta,k,m}, \text{ on } \Gamma_1, \end{cases}$$

and

(4.2)
$$\begin{cases} \dfrac{u_3^{i,k,m+1} - u_3^{i,k-1,m+1}}{\Delta t} - \Delta\ u_3^{i,\theta,k,m+1} + a_0^{i,k} u_3^{\theta,k,m+1} \geq F^{\theta}\left(u_3^{i,\theta,k-1,m+1}\right) \text{in } \Omega_3, \\ u_3^{i,\theta,k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_3^{i,\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_3^{i,\theta,k,m} \text{on } \Gamma_2 = \dfrac{\partial u_1^{i,\theta,k,m+1}}{\partial \eta_3} + \alpha_3 u_1^{i,\theta,k,m}, \text{ on } \Gamma_1. \end{cases}$$

We further take $\epsilon_1^{i,\theta,k,m} = u_2^{i,\theta,k,m+1} - u_3^{i,\theta,k,m+1}$ on Γ_1, the difference between the overlapping and the nonoverlapping solutions $u_2^{i,\theta,k,m+1}$ and $u_3^{i,\theta,k,m+1}$ of the problem (3.1) and (resp.,(4.1) and (4.2)) in Ω_3. Because both overlapping and the nonoverlapping problems converge see [10] that is, $u_2^{i,\theta,k,m+1}$ and $u_3^{i,\theta,k,m+1}$ tend to $u_3^{i,\theta,k}$ (resp. $u_3^{i,\theta,k}$), then $\epsilon_1^{i,\theta,k,m}$ should tend to naught when m tends to infinity in V_2.

By taking

$$\begin{aligned} \Lambda_3^{i,k,m} &= \frac{\partial u_2^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_2^{i,\theta,k,m},\ \Lambda_1^{i,k,m} = \frac{\partial u_1^{i,\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{i,\theta,k,m}, \\ \Lambda_3^{i,k,m} &= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \frac{\partial \epsilon_1^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,\theta,k,m}, \\ \Lambda_1^{i,k,m} &= \frac{\partial u_1^{i,\theta,k,m}}{\partial \eta_3} + \alpha_3 u_1^{i,\theta,k,m}, \end{aligned} \tag{4.3}$$

and by using Green formula, (4.1) and (4.2) can be reformulated as the following system of elliptic variational equations

$$\begin{aligned} & c(u_1^{i,\theta,k,m+1}, v_1 - u_1^{i,\theta,k,m+1}) + \left(\alpha_1 u_1^{i,\theta,k,m}, v_1 - u_1^{i,\theta,k,m+1}\right)_{\Gamma_1} \\ & \geq \left(F^{i,\theta}(u_1^{i,\theta,k-1,m+1}), v_1 - u_1^{i,\theta,k,m+1}\right)_{\Omega_1} + \left(\varphi^i, v - u_1^{i,\theta,k,m+1}\right)_{\Gamma_0} \\ & + \left(\Lambda_3^{k,m}, v_1 - u_1^{\theta,k,m+1}\right)_{\Gamma_1}, \forall v_1 \in V_1 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} & c(u_3^{i,\theta,k,m+1}, v_3 - u_3^{i,\theta,k,m+1}) + \left(\alpha_3 u_3^{i,\theta,k,m+1}, v_3 - u_3^{i,\theta,k,m+1}\right)_{\Gamma_1} \\ & \geq \left(F^{i,\theta}(u_3^{i,\theta,k-1,m+1}), v_3 - u_3^{i,\theta,k,m+1}\right)_{\Omega 3} + \left(\varphi^i, v - u_3^{i,\theta,k,m+1}\right)_{\Gamma_0} \\ & + \left(\Lambda_1^{i,k,m}, v_3 - u_3^{i,\theta,k,m+1}\right)_{\Gamma_1}, \forall v_3 \in V_3. \end{aligned} \tag{4.5}$$

On the other hand by taking

$$\theta_1^{i,k,m} = \frac{\partial \epsilon_1^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,\theta,k,m}, \tag{4.6}$$

we get

$$\begin{aligned} \Lambda_3^{i,\theta,k,m} &= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \frac{\partial (u_2^{i,\theta,k,m} - u_3^{i,\theta,k,m})}{\partial \eta_1} + \alpha_1 (u_2^{i,\theta,k,m} - u_3^{i,\theta,k,m}) \\ &= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \frac{\partial \epsilon_1^{i,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,k,m} \\ &= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \theta_1^{i,k,m}. \end{aligned} \tag{4.7}$$

By using (4.6), we have

(4.8)

$$\begin{aligned} \Lambda_3^{i,k,m+1} &= \frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,\theta,k,m} + \theta_1^{i,k,m+1} = -\frac{\partial u_3^{i,\theta,k,m}}{\partial \eta_3} + \alpha_1 u_3^{i,\theta,k,m} + \theta_1^{i,k,m+1} \\ &= \alpha_3 u_3^{i,\theta,k,m} - \frac{\partial u_1^{i,\theta,k,m}}{\partial \eta_3} - \alpha_3 u_1^{i,\theta,k,m} + \alpha_1 u_3^{i,\theta,k,m} + \theta_1^{i,k,m+1} \\ &= (\alpha_1 + \alpha_3) u_3^{i,\theta,k,m} - \Lambda_1^{i,k,m} + \theta_1^{i,k,m+1} \end{aligned}$$

and from the last equation in (4.8), we have

$$\begin{aligned} \Lambda_1^{i,k,m+1} &= -\frac{\partial u_1^{i,\theta,k,m}}{\partial \eta_1} + \alpha_3 u_1^{i,\theta,k,m} = \alpha_1 u_1^{i,\theta,k,m} - \frac{\partial u_2^{i,\theta,k,m}}{\partial \eta_1} - \alpha_1 u_2^{i,\theta,k,m} + \\ &\alpha_3 u_1^{i,\theta,k,m} + \alpha_3 u_1^{i,\theta,k,m} = (\alpha_1 + \alpha_3) u_1^{i,\theta,k,m} - \Lambda_3^{i,k,m} + \theta_3^{i,k,m+1}. \end{aligned} \tag{4.9}$$

Lemma 4.1. [11] *Let* $u_s^k = u_{\Omega s}^k$, $e_s^{\theta,k,m+1} = u_s^{\theta,k,m+1} - u_s^k$ *and* $\eta_s^{k,m+1} = \Lambda_s^{k,m+1} - \Lambda_s^k$. *Then for* $s, t = 1, 3, s \neq t$, *we have*

$$\begin{aligned} & c_s^i (e_s^{i.\theta,k,m+1}, v_s - e_s^{i,\theta,k,m+1}) + \left(\alpha_s e_s^{i,\theta,k,m+1}, v_s - e_s^{i,k,m+1}\right)_{\Gamma s} \\ &= \left(\eta_t^{i,k,m}, v_s - e_s^{i,k,m+1}\right)_{\Gamma_s}, \forall v_s \in V_s \end{aligned} \tag{4.10}$$

and

(4.11)

$$\left(\eta_s^{i,k,m+1}, \psi^i\right)_{\Gamma_s} = \left((\alpha_s + \alpha_t) e_s^{i,k,m+1}, v_s\right)_{\Gamma_s} - \left(\eta_t^{i,k,m}, \psi^i\right)_{\Gamma_s} + \left(\theta_t^{i,k,m+1}, \psi^i\right)_{\Gamma s}, \forall \psi \in V_s.$$

Theorem 4.2. [11] *The following hold true:*

$$\left\| u_{1,h}^{i,\theta,k,m+1} - u_{1,h}^{i,\theta,k} \right\|_{1,\Omega_1} + \left\| u_{3,h}^{i,\theta,k,m+1} - u_{3,h}^{i,\theta,k} \right\|_{1,\Omega_3} \leqslant C \left\| u_{1,h}^{i,\theta,k,m+1} - u_{3,h}^{i,\theta,k,m} \right\|_{W_1},$$

$$\left\| u_{2,h}^{i,\theta,k,m+1} - u_{2,h}^{i,\theta,k} \right\|_{1,\Omega_2} + \left\| u_{4,h}^{i,\theta,k,m+1} - u_{4,h}^{i,\theta,k} \right\|_{1,\Omega_4} \leqslant C \left\| u_{2,h}^{i,\theta,k,m+1} - u_{4,h}^{i,\theta,k,m} \right\|_{W_2},$$

and

$$\begin{aligned} & \left\| u_{1,h}^{i,\theta,k,m+1} - u_{1,h}^{i,\theta,k} \right\|_{1,\Omega_1} + \left\| u_{2,h}^{i,\theta,k,m} - u_{2,h}^{i,\theta,k} \right\|_{1,\Omega_2} \leqslant C \ (\left\| u_{1,h}^{i,\theta,k,m+1} - u_{2,h}^{i,\theta,k,m} \right\|_{W_1} \\ & + \left\| u_{2,h}^{i,\theta,k,m} - u_{1,h}^{i,\theta,k,m} \right\|_{W_2} + \left\| e_{1,h}^{i,k+1,m} \right\|_{W_1} + \left\| e_{2,h}^{i,k+1,m} \right\|_{W_2}). \end{aligned}$$

5 The main result

Theorem 5.1. *Let $u_s^{i,\theta,k} = u^{i,\theta,k} |_{\Omega_s}$ where u is the solution of problem (1.1), the sequences $\left(u_{1,h}^{i,\theta,k,m+1}, u_{2,h}^{i,\theta,k,m}\right)_{m\in\mathbb{N}}$ are solutions of the discrete problems (4.4) and (4.5). Then there exists a constant C independent of h such that*

$$\left\|u_{1,h}^{i,\theta,k,m+1} - u_1^{i,\theta,k}\right\|_{1,\Omega_1} + \left\|u_{2,h}^{i,\theta,k,m} - u_2^{i,\theta,k}\right\|_{1,\Omega_2} \leqslant C\left\{\sum_{i=1}^{2}\sum_{T\in\tau_h}\left(\eta_i^T\right) + \eta_{\Gamma_s}\right\},$$

where

$$\eta_{\Gamma_s} = \left\|u_{h,s}^{i,\theta,k,*} - u_{h,t}^{i,\theta,k,*-1}\right\|_{W_{h,s}} + \left\|\epsilon_{i,h}^{i,\theta,k,*}\right\|_{W_{h,s}}$$

and

$$\eta_s^T = h_T \left\| \begin{array}{c} F\left(u_{h,s}^{i,\theta,k-1,*}\right) + u_{h,s}^{i,\theta,k-1} + \\ \Delta\ u_{h,s}^{i,\theta,k,*} - \left(1+\lambda a_{h0}^k\right) u_{h,s}^{i,\theta,k} \end{array} \right\|_{0,T} + \sum_{E\in\varepsilon_h} h_E^{\frac{1}{2}} \left\| \left[\frac{\partial u_{h,s}^{i,\theta,k,*}}{\partial \eta_E}\right] \right\|_{0,E},$$

where C is a constant independent of h and k, and the symbol $*$ corresponds to $m+1$ when $s=1$, and to m when $s=2$.

Proof. We have by using the triangle inequality

$$\sum_{s=1}^{2}\left\|u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k,*}\right\|_{1,\Omega_s} \leqslant \sum_{s=1}^{2}\left\|u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}\right\|_{1,\Omega_s} + \sum_{s=1}^{2}\left\|u_{h,s}^{i,\theta,k} - u_{s,h}^{i,*}\right\|_{1,\Omega_s}. \tag{5.1}$$

The second term on the right-hand side of (5.1) is bounded by

$$\sum_{s=1}^{2}\sum_{i=1}^{2}\left\|u_{h,s}^{i,\theta,k} - u_{s,h}^{i,*}\right\|_{1,\Omega_s} \leqslant C\sum_{s=1}^{2}\eta_{\Gamma_s}^i.$$

To bound the first term on the right-hand side of (5.1) we use the residual equation and the technique of the residual a posteriori error estimation [10], to obtain for $v_h \in V^h$ the following relation

$$\left\{\begin{array}{l} c(u_s^{i,\theta,k} - u_{h,s}^{\theta,k}, v_s) = c(u_s^{i,\theta,k} - u_{h,s}^{i,\theta,k}, v_s - v_{h,s}) \\ \leq \sum\limits_{T\subset\Omega_s}\int\limits_T \left(\begin{array}{c} F^{i,\theta}\left(u_{h,s}^{i,\theta,k-1}\right) + u_{h,s}^{i,\theta,k-1} + \mu\Delta\ u_{h,s}^{i,\theta,k} - \\ \left(1+\mu a_{h0}^{i,k}\right) u_{h,s}^{i,\theta k} \end{array}\right)(v_s - v_{h,s})\, ds \\ -\sum\limits_{E\subset\Omega_s}\int\limits_E \left[\dfrac{\partial u_{h,s}^{i,\theta k}}{\partial \eta_E}\right](v_s - v_{h,s})\, ds - \sum\limits_{E\subset\Gamma_s}\int\limits_E \dfrac{\partial u_{h,s}^{i,\theta k}}{\partial \eta_E}(v_s - v_{h,s})\, ds' \\ +\sum\limits_{E\subset\Omega_s}\int\limits_T \left(F^\theta\left(u_s^{i,\theta,k}\right) - F^\theta\left(u_{h,s}^{i,\theta k}\right)\right)(v_s - v_{h,s})d\sigma + \left(\dfrac{\partial u_{h,s}^{i,\theta k}}{\partial \eta_s}, v_s - v_{h,s}\right)_{\Gamma_s}, \end{array}\right.$$

where $F^\theta\left(u_{h,s}^{i,\theta,k}\right)$ is any approximation of $F^\theta\left(u_s^{i,\theta,k}\right)$. Therefore, we have

(5.2)

$$\begin{aligned}
&\sum_{s=1}^{2} c(u_s^{i,\theta,k}-u_{h,s}^{i,\theta,k},v_s)\\
&\leq \sum_{s=1}^{2}\sum_{T\subset\Omega_s}\left\|\begin{array}{c} F^\theta\left(u_{h,s}^{i,\theta,k}\right)+u_{h,s}^{i,\theta,k-1}+\mu\Delta\ u_{h,s}^{i,\theta,k}\\ -\left(1+\mu a_{h0}^{i,k}\right)u_{h,s}^{i,\theta,k}\end{array}\right\|_{0,T}\|v_s-v_{h,s}\|_{0,T}\\
&+\sum_{s=1}^{2}\sum_{E\subset\Omega_s}\left\|\left[\frac{\partial u_{h,s}^{i,\theta,k}}{\partial\eta_E}\right]\right\|_{0,E}\|v_s-v_{h,s}\|_{0,E}+\sum_{s=1}^{2}\sum_{E\subset\Gamma_s}\left\|\frac{\partial u_{h,s}^{i,\theta,k}}{\partial\eta_E}\right\|_{0,E}\|v_s-v_{h,s}\|_{0,E}\\
&+\sum_{s=1}^{2}\sum_{T\subset\Omega_s}c\left\|u_s^{i,\theta,k}-u_{h,s}^{i,\theta,k}\right\|_{0,T}\|v_s-v_{h,s}\|_{0,T}+\sum_{s=1}^{2}\sum_{T\subset\Omega_s}\left\|\frac{\partial u_{h,s}^{i,\theta,k}}{\partial\eta_s}\right\|_{0,T}\|v_s-v_{h,s}\|_{0,T},
\end{aligned}$$

Then, using the inequality

$$\left\|u_s^{i,\theta,k}-u_{h,s}^{i,\theta,k}\right\|_{1,\Omega_s}\leqslant\sup_{v_s^i\in K}\frac{c(u_s^{i,\theta,k}-u_{h,s}^{i,\theta,k},v_s+ch_s^{i,T})}{\|v_s^i+ch_s^T\|_{1,\Omega_i}},$$

we readily infer

$$\sum_{s=1}^{2} c(u_s^{i,\theta,k}-u_{h,s}^{i,\theta,k},v_s+ch_s^{i,T})\leq\sum_{s=1}^{2}\left(\sum_{T\subset\Omega_s}\eta_s^{i,T}\right)\sum_{s=1}^{2}\|v_s\|_{1,\Omega_s}. \tag{5.3}$$

Finally, by combining (5.1) and (5.2), the claim follows. □

6 Conclusions

In this paper, a posteriori error estimates for the generalized Schwarz method (GSM) for the evolutionary Hamilton-Jacobi-Belmann equation with linear source terms related to management of energy production with mixed boundary condition (MBC), are established using a theta scheme with a Galerkin spatial approximation, using the techniques of the residual a posteriori error analysis. The convergence of the new scheme is established and the numerical example show that the new presented scheme is efficient.

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