Unification by structural discreteness

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Abstract. The aim of this note is to remedy what Albert Einstein has named "*lack of mathematical structures of discontinuum without calling upon continuum space-time as an aid*". The main idea is that similarly to continuity, which is studied in topological structures, discreteness has its specific structures, namely *horistologies*. Therefore we analyze several senses of the term "discreteness", including the structural one. Then, we suggest how to interpret the horistologies as an unifying framework of the Special Relativity, Quantum Physics, Cosmology (via Nottale's Scale Theory) and other topics where super-additivity plays a significant role.

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1 Introduction: structural pairs

In this note we support the idea that mathematical structures may decide about the continuous or discrete character of each particular result in the huge mathematical and physical literature. It is well known that the study of continuity is always related to topological structures. In counterpart, we will show how to study discreteness in horistological structures (introduced in [4], published in [5], [6], and later studied as in references).

A. Dual structures. Let us focus on the following **pairs of structures**:

Topology	Horistology
Uniform Topology	Uniform Horistology
Sub-additive Metric	Super-additive Metric
Sub-additive Norm	Super-additive Norm
(Semi-) Definite Inner Product	Indefinite Inner Product

In the left column we recognize the well known structures of continuum; we claim that the right column contains the structures of discreteness. For brevity we recall the (less-known) definitions of the right-column structures only.

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Pair # 1: Topology / Horistology. The peer of the topology takes the form of a function $\chi : W \to \mathcal{P}(\mathcal{P}(W))$, called *horistology* on W (from the Greek $\chi\omega\rho\iota\sigma\tau o\sigma$ = *separate*) which attaches to each $e \in W$ an ideal of *perspectives*, satisfying the conditions:

[HOR$_1$] $e \notin P$ for all $P \in \chi(e)$
[HOR$_2$] If $P \in \chi(e)$ and $Q \subseteq P$, then $Q \in \chi(e)$
[HOR$_3$] If $P, Q \in \chi(e)$, then $P \cup Q \in \chi(e)$
[HOR$_4$] $\forall\ P \in \chi(e)\ \exists\ T \in \chi(e)$ such that $[\ell \in P$ *and* $Q \in \chi(\ell)] \Longrightarrow [Q \subseteq T]$.

The (causal-like) *proper order* of the horistological world (W, χ) is $K^{=}(\chi) = K(\chi) \cup \delta$, where δ means *equality* and

$$K(\chi) = \{(e, \ell) \in W^2 :\ \{\ell\} \in \chi(e)\}.$$

Pair # 2: Uniform Topology / Uniform Horistology. A *uniform horistology* on W (briefly u.h.) is an ideal of relations $\mathcal{H} \subseteq \mathcal{P}(W^2)$, which satisfies the conditions:

[uh$_1$] $\pi \cap \delta = \varnothing$ for all $\pi \in \mathcal{H}$
[uh$_2$] If $\pi \in \mathcal{H}$ and $\lambda \subseteq \pi$, then $\lambda \in \mathcal{H}$
[uh$_3$] If $\lambda,\ \pi \in \mathcal{H}$, then $\lambda \cup \pi \in \mathcal{H}$
[uh$_4$] $\forall \pi \in \mathcal{H}\ \exists \theta \in \mathcal{H}$ such that $[\omega \in \mathcal{H}] \Longrightarrow [\theta \supseteq \pi \circ \omega$ and $\theta \supseteq \omega \circ \pi]$.

If $(W, \mathcal{H})$ is a u,h. world, then function $\chi_{\mathcal{H}} : W \to \mathcal{P}(\mathcal{P}(W))$ of values

$$\chi_{\mathcal{H}}(e) = \{P \in \mathcal{P}(W) : \exists \pi \in \mathcal{H}\ \text{ such that } P \subseteq \pi[e]\}$$

is a horistology on W. The *strict proper order* of $\mathcal{H}$ is

$$K(\mathcal{H}) \stackrel{def.}{=} \cup\{\lambda : \lambda \in \mathcal{H}\} = K(\chi_{\mathcal{H}}).$$

Pair # 3: Sub-additive / Super-additive metrics. Sub-additivity (briefly s.a.) is the classical property of a metric $\rho : W^2 \to \mathbb{R}_+$, meaning that

$$\rho(x, y) \leq \rho(x, z) + \rho(z, y),$$

while super-additivity (briefly S.a.) is the opposite inequality. Because super-additivity is not possible for each side of a triangle, we have to restrain the domain of such a metric to a strict part of $W \times W$, which usually is an order. More exactly, if $\Pi \subset W \times W$ is an order on W, then function $\rho : \Pi \to \mathbb{R}_+$ is a *super-additive metric* if it satisfies the conditions:

[S.a.m$_1$] $\rho(e_1, e_2) = 0 \Longleftrightarrow e_1 = e_2$
[S.a.m$_2$] $\rho(e_1, e_3) \geq \rho(e_1, e_2) + \rho(e_2, e_3)$ for all $(e_1, e_2), (e_2, e_3) \in \Pi$.

If only " $\Longleftarrow$ " holds in [S.a.m$_1$], we say that ρ is a *S.a. pseudo metric.* The triplet (W, Π, ρ) is called S.a. *metric* (respectively *pseudo metric*) *world.*

Further restrictions to $\Lambda \subset \Pi$ are always possible, but the converse process, that of prolongation, is also significant (see [5], etc.). Each S.a. metric $\rho : \Pi \to \mathbb{R}_+$ generates a u. horistology $\mathcal{H}_\rho$:on W, whose ideal base consists of hyperbolic prospects

$$\pi_r = \{(e_1, e_2) \in \Pi : \rho(e_1, e_2) > r\},$$

where $r > 0$, . In addition, $K^{=}(\mathcal{H}_\rho) = \Pi$.
Differently from the usual (s.a.) ones, the super-additive metrics allow quantizations.

Using a real constant $\hbar > 0$, to each (p-) metric $\rho : \Pi \to \mathbb{R}_+$, we may attach the S.a. pseudo metric $\rho_\hbar : \Pi \to \mathbb{R}_+$, of values

$$\rho_\hbar(e_1, e_2) = \begin{cases} 0 & if \;\; \rho(e_1, e_2) \le \hbar \\ \rho(e_1, e_2) & if \;\; \rho(e_1, e_2) > \hbar \end{cases}$$

called *$\hbar$-quantization* of ρ.

Pair # 4: Sub-additive / Super-additive norms. If W is a real linear space and $\Pi \subset W \times W$ is an order on W, then $\rfloor \cdot \lfloor : \Pi[0] \to \mathbb{R}_+$ is a *super-additive* (S.a.) *norm* if it satisfies the conditions:

[S.a.n$_1$] $\rfloor e \lfloor = 0 \iff e = 0$;
[S.a.n$_2$] $\rfloor \lambda e \lfloor = \lambda \rfloor e \lfloor$ for all $e \in \Pi[0]$ and $\lambda \ge 0$;
[S.a.n$_3$] $\rfloor e_1 + e_2 \lfloor \ge \rfloor e_1 \lfloor + \rfloor e_2 \lfloor$ for all $e_1, e_2 \in \Pi[0]$.

The triplet $(W, \Pi, \rfloor \cdot \lfloor)$ defines a *S.a. normed world* (*space*). It is a particular case of a S.a. metric world, in the sense that $\rho : \Pi \to \mathbb{R}_+$ of values $\rho(e_1, e_2) = \rfloor e_2 - e_1 \lfloor$ is a S.a. (pseudo) metric.

Pair # 5: Semi-definite / Indefinite inner products. Let W be a linear space over Γ, which is either $\mathbb{R}$ or $\mathbb{C}$. Function $\langle . | . \rangle : W \times W \to \Gamma$ is an *inner product* on W (see [3], etc.) if it satisfies the conditions:

[I$_1$] $\langle \alpha e_1 + \beta e_2 \mid e_3 \rangle = \alpha \langle e_1 | \, e_3 \rangle + \beta \langle e_2 | \, e_3 \rangle$
[I$_2$] $\langle e_1 | \, e_2 \rangle = \overline{\langle e_2 | \, e_1 \rangle}$.

If there exist $e_1, e_2 \in W$ such that $\langle e_1 | \, e_1 \rangle > 0$ and $\langle e_2 | \, e_2 \rangle < 0$, the inner product is *indefinite*. In indefinite (real) inner product spaces, we obtain S.a. norms by restricting the inner product to subspaces of Π_1 Pontrjagin type (as in [7]). These subspaces look like relativist worlds of events $W = \mathbb{R} \times H$, where $(H, (\cdot|\cdot))$ is a scalar product space. Then the inner product of the events $e_1 = (t_1, s_1)$ and $e_2 = (t_2, s_2)$ takes the form $\langle e_1 | \, e_2 \rangle = c^2 t_1 t_2 - (s_1 | s_2)$, where $c > 0$ usually stands for the speed of light. Function $\rfloor \cdot \lfloor_t : K[0] \to \mathbb{R}_+$, of values $\rfloor e \lfloor_t = \sqrt{c^2 t^2 - s^2}$, where $s^2 = (s \mid s)$, is a S.a. norm.

B. Dual morphisms. The morphisms of the horistological structures are *discrete functions*. More exactly, if (W, χ) and (V, ψ) are horistological worlds, then function $f : W \to V$ has *germ* $g \in V$ at e_0, and we write $g = \underset{e_0 \to e}{germ} \, f(e)$, if for every $P \in \chi(e_0)$ we have $f(P) \in \psi(g)$. We may note the set of all germs of f at e_0 by $Germ(f, e_0)$. If $f(e_0) \in Germ(f, e_0)$, we say that f is *discrete* at e_0. Briefly, this means $f(\chi(e_0)) \subseteq \psi(f(e_0))$.

Discrete functions preserve the proper orders of the horistologies.

If $(W, \mathcal{H})$ and $(V, \mathcal{U})$ are uniform horistological worlds, we similarly define the *uniform discreteness* of f, namely $f_{II}(\mathcal{H}) \overset{def}{=} \{ f_{II}(\pi) : \pi \in \mathcal{H} \} \subseteq \mathcal{U}$. The uniform discreteness implies the point-wise discreteness of f on W.

Similarly to convergence, emergence represents discreteness of the function that defines the net, relative to the intrinsic horistology of a directed set (see [6], [14], etc.).

C. Bornologies. *Bornology* (see [10], etc.), is similar to horistology in many aspects. A family of parts $\mathcal{B} \subseteq \mathcal{P}(S)$ defines a bornology on the non-void set S if the following conditions hold:

[b$_1$] $\cup\{B : B \in \mathcal{B}\} = S$;
[b$_2$] $[(B \in \mathcal{B})\ \&\ (C \subseteq B)] \Longrightarrow (C \in \mathcal{B})$;
[b$_3$] $(B, C \in \mathcal{B}) \Longrightarrow (B \cup C \in \mathcal{B})$.

The pair $(S, \mathcal{B})$ is named *bornological space*, and the elements of $\mathcal{B}$ are called *bounded sets*. Each bornological space carries an intrinsic horistology.

2 Physical arguments

Usually, discreteness reduces to finiteness. Consequently, a lot of physical arguments refer to discrete sets outside any topology. To exemplify such approaches of discreteness, we mention the following three directions:

(i) Reducing the relativist space-time to a cubic lattice. This variant avoids infinities, but it is hardly accepted because of the Lorentz invariance failure.

(ii) The process of sprinkling reduces discreteness to local finiteness by renouncing the regularity of a cubic lattice.

(iii) In quantum physics, we represent the physical quantities (including space and time) by Hermitian operators whose spectra consist of possible results of the measurements. Consequently, discreteness may refer to these spectra, but difficulties concerning the Lorentz invariance still persist.

Besides the reduction of discreteness to (eventually local) finiteness, many scientists claim that the topologies (especially the Euclidean ones) are not adequate to study topics like the relativist and quantum physics, and suggest to look for other structures. In this respect we mention:

(I) Einstein was initially convinced that our universe is a four-dimensional continuum. However, the development of the general relativity and quantum physics have led him to doubt about the continuum character of the space-time. In a letter to Walter Dallenbach (1916) we find the opinion that "The problem seems to me how one can formulate statements about a discontinuum without calling upon continuum space-time as an aid; the latter should be banned from the theory as a supplementary construction not justified by the essence of the problem, which corresponds to nothing "real". But we still lack the mathematical structure unfortunately".

(II) A fundamental assumption in the quantum physics is the existence of quanta of energy and other physical quantities, including space and time. However, the usual (s.a.) metrics do not allow quantizations.

In addition, the Heisenberg's uncertainty principle $|\Delta x| \cdot |\Delta p| \geq \hbar$ hides a particular type of horistological spaces since the functional of values $\sqrt{|\Delta x| \cdot |\Delta p|}$ is a super-additive norm.

(III) Based on his remarkable theorem "Causality implies the Lorentz group", Zeeman has severely criticized (see [15]) the use of the Euclidean topology on the Minkowskian space-times. This topology is locally homogeneous (while the space-time is not), and the group of all homeomorphisms is too large and has no physical significance (in comparison to the Lorentz group).

3 Discrete sets

It is well known what "discreteness" means in topology. However, except the trivial case of the discrete topology, the topologically discrete sets do not respect some generally expected features of discreteness. In this respect we mention the following objections:

[Ob$_1$] Finite sets may be not topologically discrete.

[Ob$_2$] It is impossible to impose lower limits for the size of neighborhoods, i.e. there is no quantization.

[Ob$_3$] Continuous functions do not preserve discreteness.

The horistological discreteness expresses the same idea of "separating" the events of a set. Let Λ be a strict order on the horistological world (W, χ), such that $\Lambda \subseteq K(\chi)$, and let M be a subset of W. We say that an event $e \in M$ is Λ-*detachable* from M (alternatively, M is Λ-*discrete* at e, etc.) if $M \cap \Lambda[e] \in \chi(e)$. The set of all Λ-detachable points of M forms the Λ-*discrete part* of M, noted $\partial_\Lambda(M)$. If each point of M is Λ-detachable, i.e. $\partial_\Lambda(M) = M$, then M is Λ-*discrete*. Function $\partial_\Lambda : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, which extracts the Λ-discrete part $\partial_\Lambda(M)$ of each subset $M \in \mathcal{P}(W)$, is called *operator of* Λ-*discreteness.*

Considering discrete sets in horistological structures makes the above objections [Ob$_1$], [Ob$_2$] and [Ob$_3$] disappear. Discreteness has the following properties (established in [4]):

[d_1] $card\ M \in \mathbb{N} \Longrightarrow \partial_\Lambda(M) = M$ (unlike [Ob$_1$]);

[d_2] $L \subseteq M \Longrightarrow L \cap \partial_\Lambda(M) \subseteq \partial_\Lambda(L;)$

[d_3] $\partial_\Lambda(M) \cap \partial_\Lambda(L) \subseteq \partial_\Lambda(M \cup L)$;

[d_4] $e \in \partial_\Lambda(M) \Longleftrightarrow e \in M \cap \partial_\Lambda(\{e\} \cup \Lambda[M \cap \Lambda[e]])$;

Conversely, we may recover the horistological structure from.an *abstract operator of discreteness*, which satisfies $[d_1] - [d_4]$.

Condition [d_1] annihilates [Ob$_1$]. Relative to [Ob$_2$], it is easy to see that In horistological structures we may impose lower bounds to the perspectives, hence [Ob$_2$] disappears. To invalidate objection [Ob$_3$], let (W_1, χ_1) and (W_2, χ_2) be horistological spaces, and let function $f : W_1 \to W_2$ be $1 : 1$ and strictly monotonic relative to the orders $K(\chi_1)$ and $K(\chi_2)$. If $f_{II}(K(\chi_1)) = K(\chi_2)$, then ($f$ is discrete on W_1) $\Longrightarrow$ (f preserves detachability).

4 Unifying by horistologies

Considering horistological structures obviously is a process of generalization. As usually, the utility of a generalization consists in giving unified vision on particular fields previously considered independent. In the sequel we put forward some common horistological features in relativity, quantum physics, cosmology and other topics. Consequently, this unification by horistological structures offers a common mathematical language to these fields.

1.**Special Relativity**. Nowadays, the structure of a real indefinite inner product, which preserves the entire physical significance in universes of events, is increasingly replacing the Minkowskian "complexified" structure .

A. Intrinsic horistology of the universe of events. The intrinsic inner product of the special relativity is

$$\langle e_1 |\ e_2 \rangle = c^2 t_1 t_2 - x_1\ y_1 - x_2\ y_2 - x_3\ y_3,$$

where $e_1 = (t_1, x_1, y_1, z_1)$ and $e_2 = (t_2, x_2, y_2, z_2)$ are events in $W = \mathbb{R} \times \mathbb{R}^3$. The corresponding *quadratic form*

$$Q(e) = \langle e \mid e \rangle = c^2 t^2 - x^2 - y^2 - z^2,$$

where $e = (t, x, y, z)$, allows the construction of the relation of *causality*,

$$K = \{(e_1, e_2) : Q(e_1 - e_2) > 0, t_2 > t_1\}.$$

This inner product generates the *temporal norm* $\rfloor \cdot \lfloor_t : K^{=}[0] \to \mathbb{R}_+$, of values

$$\rfloor e \lfloor_t = \sqrt{c^2 t^2 - x^2 - y^2 - z^2},$$

and the *temporal metric* $\rho : K^{=} \longrightarrow \mathbb{R}_+$, defined by

$$\rho(e_1, e_2) = \sqrt{c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}\ ,$$

which measures *proper time*. The *Aczél's Inequality* (see [1], [12], etc.)

$$|\ \langle e_1 |\ e_2 \rangle\ | \geq \langle e_1 |\ e_1 \rangle\ \langle e_2 \mid e_2 \rangle\ ,$$

assures the super-additivity of the temporal norm and metric. Finally, using the *hyperbolic perspectives* of vertex e and radius r,

$$H(e, r) = \{\ell \in K[e] : \rho(e, \ell) > r\},$$

we obtain the intrinsic horistology $\chi : W \to \mathcal{P}(\mathcal{P}(W))$, of values

$$\chi(e) = \{P \in \mathcal{P}(W) : \exists r > 0 \ \textit{such that}\ P \subseteq H(e, r)\}.$$

B. Quantized time. Because the physical meaning of $\rho(e_1, e_2) = \rfloor e_2 - e_1 \lfloor_t$ is proper time, quantizing ρ represents a quantization of time corresponding to a generic constant $\hbar$. First, in the extended causality

$$\overline{K} = \{(e_1, e_2) : Q(e_1 - e_2) \geq 0, t_2 \geq t_1\},$$

we identify the strict order

$$K_\hbar = \{(e_1, e_2) : Q(e_1 - e_2) > \hbar^2, t_2 > t_1\},$$

then we define $\rho_\hbar : \overline{K}_\hbar \longrightarrow \mathbb{R}_+$ by

$$\rho_\hbar(e_1, e_2) = \begin{cases} \rho(e_1, e_2)\ \textit{if}\ \ (e_1, e_2) \in K_\hbar \\ 0 \qquad\qquad \textit{if}\ \ (e_1, e_2) \in \overline{K} \setminus K_\hbar\ . \end{cases}$$

C. Discrete functions. The property of a function $f : W \to W$ of being discrete at an event $e \in W$ takes the form

$$\forall \delta > 0 \ \exists \varepsilon > 0 \ \textit{such that} \ [\rho(e, \ell) > \delta \Longrightarrow \rho(f(e), f(\ell)) > \varepsilon].$$

In particular, the Lorentz transformations are discrete functions since they are isometries (hyperbolic rotations).

The discreteness of a set $M \subset W$ expresses a separation in proper time between each event $e \in M$ and its causal consequences in M. Discrete functions, including Lorentz transformations, preserve the discreteness of the sets of events.

To conclude, the Einsteinean universe of events is structurally discrete in the sense that its intrinsic structure is horistological.

2.Quantum Physics. The definitive characteristic of the quantum physics is the presence of quanta for physical quantities. The simplest case concerns scalar quantities, for which the result of a measurement is a number $x \in \mathbb{R}$.

A. Scalar quanta. Almost unanimously, $\mathbb{R}$ is considered a pattern of continuum, especially if we endow it with its Euclidean topology, but according to [Ob_2], the topological structures do not allow quanta. It is significant to specify that we may construct $\mathbb{R}$ on a purely horistological way, hence it equally is a pattern of structural discreteness. In addition, we may endow $\mathbb{R}$ with a "quantized" horistology $\chi_\hbar : \mathbb{R} \to \mathcal{P}(\mathcal{P}(\mathbb{R}))$, which corresponds to the $\hbar$-quantified metric $\rho_\hbar$, where $\rho(x, y) = y - x$ is defined on the usual order of $\mathbb{R}$. More exactly,

$$P \in \chi_\hbar(x) \Longleftrightarrow [\exists r > \hbar \ \textit{such that} \ P \subseteq H(x, r)].$$

Because $\rho_\hbar(x, y)$ is either 0 or greater than $\hbar$, we may use it to measure scalar quantized physical quantities.

B. Heisenberg horistologies. The Heisenberg relations of uncertainty deal with super-additive functionals. For example, relation $|\Delta x| \cdot |\Delta p| \geq \hbar$ involves a quadratic functional that measures area. In addition, if Π is the product order on $S = \mathbb{R} \times \mathbb{R}$, for which $\Pi[0] = \mathbb{R}_+ \times \mathbb{R}_+$ represents the positive cone, then functional $\rfloor \cdot \lfloor : \Pi[0] \to \mathbb{R}_+$, of values $\rfloor(\Delta x, \Delta p)\lfloor = \sqrt{\Delta x \Delta p}$, is a S-a norm on $\mathbb{R}^2$. Consequently, the Heisenberg's relation of uncertainty takes the form $\rfloor(|\Delta x|, |\Delta p|)\lfloor \geq \sqrt{\hbar}$, i.e. at each measurement of x and p, point $(|\Delta x|, |\Delta p|)$ necessarily places in a sufficiently large hyperbolic $\rfloor \cdot \lfloor$-perspective of the origin. The $\hbar$-quantized resulting metric $\rho_\hbar$ generates the so called Heisenberg horistology, noted $\chi_\hbar$.

3.Cosmology. In relativity, the Lorentz transformations are discrete functions, hence they preserve discreteness. In quantum physics, the discrete functions preserve quantization.

Most differences between quantum physics, relativity and cosmology are due to the diversity of scales at which these theories operate. According to [13], we may connect these theories by adequate *scale transformations*.

Let us transform the field φ into φ' by the change of scale of ratio $q = \Delta x / \Delta x'$ according to the *power law* $\varphi' = \varphi \cdot q^{\mathfrak{d}}$. In terms of renormalization group, $\mathfrak{d}$ represents the *anomalous dimension* of the field φ, while in fractal interpretation we have $\mathfrak{d} = D - D_T$, where D is the fractal dimension and D_T is the topological dimension. This power law allows the form

$$\ln(\varphi' / \varphi_0) = \ln(\varphi \ / \ \varphi_0) + \mathfrak{d} \ \ln(\Delta x / \Delta x'),$$

which is strongly resembling the Galilean transformation $(x,t) \to (x^/, t^/)$,

$$\begin{cases} x^/ = x + vt \\ t^/ = t \ . \end{cases}$$

A critical analysis of these transformations has led Nottale to adopt a Lorentz-type transformation of a field under a change of scale. More exactly, the transformation $(\ln\varphi, \mathfrak{d}) \to (\ln\varphi^/, \mathfrak{d}^/)$, caused by a change of scale of characteristic $\log_k \sigma$, has the form

$$\begin{cases} \log_k(\varphi^/ / \varphi_0) = \frac{\log_k(\varphi/\varphi_0) + \mathfrak{d}\ \log_k \sigma}{\sqrt{1-\log_k^2 \sigma}} \\ \mathfrak{d}^/ = \frac{\mathfrak{d}\ + (\log_k \sigma) \log_k(\varphi/\varphi_0)}{\sqrt{1-\log_k^2 \sigma}} \ . \end{cases}$$

To stress on the analogy with the relativity of motion, we may remark the correspondence

$$x \longleftrightarrow \log_k(\varphi/\varphi_0), \ \ ct \longleftrightarrow \mathfrak{d}, \text{ and } \frac{v}{c} \longleftrightarrow \log_k \sigma.$$

An immediate consequence of this similarity is the intrinsic structure of the plane of coordinates $\log_k(\varphi/\varphi_0)$ and $\mathfrak{d}$. The specific condition $\log_k \sigma \in (-1, 1)$ shows that the base k of the logarithms should be great enough, to assure $\sigma \in (\frac{1}{k}, k)$ for all physically accepted σ. For each fixed k, the fundamental invariant of the Nottale transformations, which is $\mathfrak{d}^2 - \log_k^2(\varphi/\varphi_0)$, furnishes a "causal" order Λ, and a S.a. norm $\dashv \cdot \vdash : \Lambda[(0,0)] \to \mathbb{R}_+$, of values

$$\dashv (\log_k(\varphi/\varphi_0), \mathfrak{d}) \vdash \ \ = \sqrt{\mathfrak{d}^2 - \log_k^2(\varphi/\varphi_0)}$$

Finally, the plane $\mathbb{R} \times \mathbb{R}$, of the variables $\log_k(\varphi \ / \ \varphi_0)$ and $\mathfrak{d}$, becomes a horistological space via the S.a. metric generated by $\dashv \cdot \vdash$. In addition, the Nottale's change of scale is a discrete function on $\mathbb{R} \times \mathbb{R}$ relative to the horistology generated by $\dashv \cdot \vdash$ because the isometries are always discrete functions.

5 Unifying other topics by discreteness

Without going into details, we mention several topics where S.a. and horistology play an important role, hence discreteness realizes their unification.

1. Theory of proof in formal logic. The general form of a theorem is "If H (hypothesis), then C (conclusion)", briefly $H \Longrightarrow C$. The proof consists in finding a deductive sequence, which consists of at least one intermediate fact, say I, such that $H \Longrightarrow I \Longrightarrow C$. If ∂ denotes the difficulty of deduction, then obviously $\partial(H, C) > \partial(H, I) + \partial(I, C)$, which means super-additivity. In other terms, S.a. is the essential justification of every proof. Thus, logic becomes a horistological world.

2. Indefinite inner products. The Aczél's inequality, S.a. of the norms and metrics, hence the presence of a horistology, are possible only in indefinite inner product spaces (see [7], etc.).

3. Duality theory of L^p spaces. It is well known that for $p \geq 1$, the space of linear and continuous functionals (called dual) on L^p is L^q, where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, L^p is a Banach space with the norm $\|f\|_p = \left(\int |f|^p\right)^{\frac{1}{p}}$. If $p < 1$, we face serious difficulties: $\|\cdot\|_p$ is a S.a. norm, the only linear and continuous functional on L^p is identically null, etc. In [9] we see how to obtain results similar to the classical ones by using this S.a. norm, horistology and discreteness.

4. Stability / instability. Most frequently, we express the property of stability of a particular evolution of a dynamical system by the continuity of the function "*initial state* $\to$ *evolution*". If the space of initial states and that of evolutions allow horistological structures (see [8]), then the discreteness of this function defines the *discrete instability*

5. Concave gauge optimization. There are practical problems asking optimization of a concave gauge functions (see [2], etc.). These functions act on convex parts of a cone, which usually are perspectives of S.a. norms.

6. The invariant description of the movement. Using S.a. metrics, indefinite inner products and horistology we may express all relativist topics in real variables. The same tools allow relativist interpretation of many results in hyperbolic geometry. For example, a relativist Frenet referential gives an invariant description of the movement of a particle.

7. Relativistic dynamical systems. The relativistic character of a dynamical system results by the description of its evolution in terms of events. The main advantage of this approach is the direct relieve of the intrinsic properties, which are invariant under changes of observers (see [11]. etc.).

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