Global existence and boundedness of solutions for a type of nonlinear integro-differential equations of third order

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Abstract. In this article, we consider a particular nonlinear integro-differential equation of the third order and discuss the continuability and boundedness of solutions of this equation. Lyapunov's second method is used in the proof of the main theorem by building an appropriate Lyapunov function. The result obtained in this paper contains and improves some well known results on the third order nonlinear integro-differential equations in literature. We also provide an example to illustrate the method.

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Key words: Lyapunov function; global existence; boundedness; integro-differential equation.

1 Introduction

In this paper we discuss the continuability and boundedness of solutions of the third order nonlinear integro-differential equation of the form

$$\left(\sigma(x)x^{'}\right)^{''}+u\left(t\right)\Gamma\left(x,x^{'}\right)x^{''}+v\left(t\right)\Psi\left(x^{'}\right)+w\left(t\right)\Omega\left(x\right)=\int_0^t\Delta\left(t,s\right)x^{'}\left(s\right)ds, \tag{1.1}$$

where $t\in\Re^+$, $\Re^+=[0,\infty)$; σ is positive and continuously differentiable functions on $\Re^+$; $u,v,w\in C^1(\Re^+,(0,\infty))$; $\Gamma\in C(\Re\times\Re,\Re^+)$; $\Psi\in C(\Re,\Re^+)$; $\Omega\in C^1(\Re,\Re)$ and $\Delta\left(t,s\right)$ is countinuous for $0\leq t\leq s<\infty$. Also $x^{'}, x^{''}$ and $x^{'''}$ denote the first, second and third derivatives of the function $x\left(t\right)$ with respect to t.

It is well known that the studies on the qualitative behaviors of solutions, continuability, stability, boundedness, convergence, instability and so forth, are significant in the theory and applications of integro-differential equations. Therefore, nonlinear integro-differential equations have been widely studied in many scientific areas such as, control theory, chemistry, biology, atomic energy, economy, engineering, physics, information theory, mechanics, medicine and so on. It should be explained that the

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studies concerning the qualitative behaviors of solutions of integro-differential equations are very few in comparison to that on the nonlinear differential equations of third order. The qualitative properties of differential and integro-differential equations of third order have been studied by many authors in the literature. There are many interesting works on the global existence, boundedness, asymptotic behavior, periodicity and stability of solutions for nonlinear differential equations of third order. In particular, for some works on the stability and boundedness of solutions of nonlinear differential equations of third order, the readers can refer to the papers or books of Ademola and Arawomo [1,2], Ahmad and Rama Mohana Rao [3], Burton [5,6], Ezeilo [9], Hara [11], Mehri [13], Miller and Michel [14], Omeike [16-17], Oudjedi et al. [18], Qian [19], Reissig et al. [20], Remili and Oudjedi [21], Remili and Beldjerd [22], Swick [23], Tunç [26-27], Tunç and Ayhan [28], Yoshizawa [30], Zhang and Yu [31]. In this regards, It should be noted that, in 1960, Ezeilo [9] investigated the stability of solutions of the following nonlinear third order differential equation:

$$x^{'''} + \psi\left(x, x^{'}\right) x^{''} + \phi\left(x^{'}\right) + g\left(x\right) = p(t, x, x^{'}, x'')$$

Also, in 2010, Omeike [16] took into consideration the following ordinary differential equation of third order:

$$x^{'''} + \psi\left(x, x^{'}\right) x^{''} + f\left(x, x^{'}\right) = p\left(t\right).$$

The author established sufficient conditions for asymptotic behaviour of solutions of the equation by building a suitable Lyapunov function.

Recently, in 2014, Remili and Beldjerd [22] obtained sufficient conditions which guarantee the uniformly asymptotically stable and boundedness of solutions of the following third order non-autonomous differential equation with delay $r > 0$:

$$\left(\Psi(x)x^{'}\right)^{''} + a\left(t\right) x^{''} + b\left(t\right) \Phi(x)x^{'} + c\left(t\right) f\left(x\left(t-r\right)\right) = e(t).$$

We should note that the continuability and boundedness of solutions of equation (1.1) have not been discussed in the literature until now. In addition to the results established in the above mentioned papers, the motivation for our work comes chiefly from the articles of Baxley [4], Changian et al. [7], Constantin [8], Graef and Tunç [10], Napoles Valdes [15], Tidke and Dhakne [24], Tidke [25] and Tunç and Ayhan [29]. In all these studies, it is pointed out that the Lyapunov's second method is used as a fundamental tool to reach the results there.

It should be noticed that the problem of defining appropriate Lyapunov functions remains as an open problem in the literature until now. The main goal of this paper is to give sufficient conditions which ensure the continuability and boundedness of solutions of equation (1.1) by defining an appropriate Lyapunov function. On the other hand, this paper may be beneficial for researchers working on the qualitative behaviors of solutions of third order integro-differential equations.The result obtained in this investigation improves the existing results on the third order nonlinear integro-differential equations in the literature.

We presume that there are positive constants $\sigma_0, \sigma_1, u_0, u_1, n, N, \delta_0, \delta_1, \delta_2, \delta_3, \delta_4$ and δ_5 such that the undermentioned conditions hold:

$(A1)$ $\sigma_0 \leq \sigma(x) \leq \sigma_1,\ \int_{-\infty}^{+\infty} |\sigma'(u)|\, du < \infty,$

$(A2)$ $u_0 \leq u\,(t) \leq u_1,\ u^{'}\,(t) \leq 0,$

$(A3)$ $n \leq w(t) \leq v(t) \leq N,\ v^{'}\,(t) \leq w^{'}\,(t) \leq 0,$

$(A4)$ $\Omega(0) = 0,\ \frac{\Omega(x)}{x} \geq \delta_0,\ (\ x \neq 0),\ |\Omega'(x)| \leq \delta_1,$

$(A5)$ $\Psi(0) = 0$ and $0 < \delta_2 \leq \frac{\Psi(y)}{y} \leq \delta_3,\ (y \neq 0),$

$(A6)$ $\delta_4 \leq \Gamma\,(x, y) \leq \delta_5,\ y\Gamma_x\,(x, y) \leq 0,$

$(A7)$ $\max\left(\frac{\alpha\sigma_0}{2} \int_0^t |\Delta\,(t, s)|\, ds + \int_t^{\infty} |\Delta\,(u, t)|\, du\right) \leq R.$

2 Preliminaries

Firstly, we give a well known preliminary result which will be used in the proof of our main result.

Consider the non-autonomous differential system

$$\frac{dx}{dt} = F\,(t, x) \tag{2.1}$$

where x is an $n-$ vector, $t \in [0, \infty)$. Suppose that $F(t, x)$ is continuous in (t, x) on D, where D is a connected open set in $\Re \times \Re^n$. Now, we give the following theorem.

Theorem 2.1. *Let $F \in C\,(D)$ and $|F| \leq M$ on D. Suppose that φ is a solution of (2.1) on the interval $j = (\alpha, \beta)$ such that the following conditions hold:*

(i) The two limits $\lim_{t\to\alpha^+} \varphi\,(t) = \varphi\,(\alpha^+)$ and $\lim_{t\to\beta^-} \varphi\,(t) = \varphi\,(\beta^-)$ exist,

(ii) $(\alpha, \varphi\,(\alpha^+))$ (respectively, $(\beta, \varphi\,(\beta^-))$) is in D.

Then the solution φ can be continued to the left pass the point $t = \alpha$ (respectively, to the right pass the point $t = \beta$).

Proof. See Hsu [12]. □

3 The main result

Now we give the following theorem as the main result.

Theorem 3.1. *Presume that conditions $(A1) - (A7)$ hold. If*

$$\frac{N\delta_1}{\sigma_0} + \frac{R}{\sigma_0^2} \leq \frac{\alpha n \delta_2}{\sigma_1^2},\ \ \alpha + \frac{\sigma_1 R}{\alpha\sigma_0} \leq u_0\delta_4,$$

$$\delta_1 < \alpha \leq \sigma_0 \text{ and } \sigma_1 \leq \delta_2.$$

Then all the solutions of (1.1) are continuable and bounded.

Proof. We use the undermentioned differential system which is equivalent to (1.1),

$$
\begin{aligned}
x^{'} &= \frac{y}{\sigma(x)}, \\
y^{'} &= z, \\
z^{'} &= \int_0^t \Delta\left(t,s\right)\frac{y(s)}{\sigma(x(s))}ds - u\left(t\right)\Gamma\left(x, \frac{y}{\sigma(x)}\right)\frac{z}{\sigma(x)} \\
&\quad + \frac{u\left(t\right)\sigma^{'}\left(x\right)}{\sigma^{3}(x)}\Gamma\left(x, \frac{y}{\sigma(x)}\right)y^{2} - v\left(t\right)\Psi\left(\frac{y}{\sigma(x)}\right) - w\left(t\right)\Omega\left(x\right).
\end{aligned}
\tag{3.1}
$$

To prove the Theorem, we define a continuously differantiable Lyapunov function $V\left(t\right) = V\left(t, x\left(t\right), y\left(t\right), z\left(t\right)\right)$ by

$$
V\left(t\right) = e^{-\frac{\rho(t)}{\varepsilon}}U\left(t, x\left(t\right), y\left(t\right), z\left(t\right)\right), \tag{3.2}
$$

where ε is a positive constant which will be determine later,

$$
\begin{aligned}
\rho(t) &= \int_0^t \left|\varphi(s)\right| ds = \int_0^t \left|\frac{x'(s)\sigma'(x(s))}{\sigma^2(x(s)}\right| ds \\
&= \int_{\alpha_1(t)}^{\alpha_2(t)} \left|\frac{\sigma'(u)}{\sigma^2(u)}\right| du \le \frac{1}{\sigma_0^2}\int_{-\infty}^{+\infty} \left|\sigma'(u)\right| du < \infty,
\end{aligned}
$$

for $\varphi(t) = \frac{x'(t)\sigma'(x(t))}{\sigma^2(x(t)}$, $\alpha_1(t) = \min\left\{x(0), x(t)\right\}$, $\alpha_2(t) = \max\left\{x(0), x(t)\right\}$ and

$$
\begin{aligned}
U\left(t\right) &= U\left(t, x\left(t\right), y\left(t\right), z\left(t\right)\right) = \alpha w\left(t\right)\int_0^x \Omega\left(u\right)du + w\left(t\right)\Omega\left(x\right)y \\
&\quad + v(t)\sigma(x)\int_0^{\frac{y}{\sigma(x)}} \Psi(\tau)d\tau + \frac{1}{2}z^2 + \frac{\alpha}{\sigma(x)}yz \\
&\quad + \alpha u(t)\int_0^{\frac{y}{\sigma(x)}} \Gamma(x,\tau)\tau d\tau + \int_0^t\int_t^\infty \left|\Delta\left(u,s\right)\right| \frac{y^2(s)}{\sigma^2\left(x(s)\right)}duds.
\end{aligned}
\tag{3.3}
$$

Further, since $\int_0^t\int_t^\infty \left|\Delta\left(u,s\right)\right| \frac{y^2(s)}{\sigma^2(x(s))}duds$ is non negative, from the definition of $U\left(t\right)$ in (3.3), we get

$$
\begin{aligned}
U\left(t\right) &\ge w\left(t\right)\left(\alpha\int_0^x \Omega\left(u\right)du + \Omega\left(x\right)y + \frac{v(t)}{w\left(t\right)}\sigma(x)\int_0^{\frac{y}{\sigma(x)}} \Psi(\tau)d\tau\right) \\
&\quad + \frac{1}{2}z^2 + \frac{\alpha}{\sigma(x)}yz + \alpha u(t)\int_0^{\frac{y}{\sigma(x)}} \Gamma(x,\tau)\tau d\tau.
\end{aligned}
$$

From the assumptions $(A1)-(A6)$ we obtain

$$\begin{aligned}
U(t) &\geq w(t)\begin{pmatrix} \int_0^x \left(\alpha-\Omega'(u)\right)\Omega(u)\,du+\frac{1}{2}\left(\Omega(x)+y\right)^2 \\ +\sigma(x)\int_0^{\frac{y}{\sigma(x)}}\left(\frac{\Psi(\tau)}{\tau}-\sigma(x)\right)\tau d\tau \end{pmatrix} \\
&\quad +\frac{1}{2}\left(z+\frac{\alpha}{\sigma(x)}y\right)^2+\alpha u(t)\int_0^{\frac{y}{\sigma(x)}}\left(\Gamma(x,\tau)-\frac{\alpha}{u(t)}\right)\tau d\tau \\
&\geq w(t)\left(\int_0^x \left(\alpha-\delta_1\right)\frac{\Omega(u)}{u}u du+\sigma(x)\int_0^{\frac{y}{\sigma(x)}}\left(\delta_2-\sigma_1\right)\tau d\tau\right) \\
&\quad +\frac{1}{2}\left(z+\frac{\alpha}{\sigma(x)}y\right)^2+\alpha u(t)\int_0^{\frac{y}{\sigma(x)}}\left(\delta_4-\frac{\alpha}{u_0}\right)\tau d\tau.
\end{aligned}$$

Thus, if $\delta_1<\alpha<u_0\delta_4$ and $\sigma_1\leq\delta_2$, we have

$$U(t)\geq\frac{n\left(\alpha-\delta_1\right)\delta_0}{2}x^2+\frac{1}{2}\left(z+\frac{\alpha}{\sigma(x)}y\right)^2+\frac{\alpha\left(u_0\delta_4-\alpha\right)}{2\sigma_1^2}y^2.$$

From the terms included in the last inequality, it is easy to see that there exists a sufficiently small positive δ_6 constant such that

$$U(t)\geq\delta_6\left(x^2+y^2+z^2\right), \tag{3.4}$$

which implies $V(t)\geq 0$. Therefore, the function $V(t)$ defined by the expression (3.2) is positive definite.

Let $(x(t),y(t),z(t))$ be any solution of (1.1). Calculating the time derivative of the function $U(t)$, along the trajectories of system (3.1), we obtain

$$\begin{aligned}
U'(t) &= \alpha w'(t)\int_0^x \Omega(u)\,du+w'(t)\,\Omega(x)\,y+v'(t)\sigma(x)\int_0^{\frac{y}{\sigma(x)}}\Psi(\tau)d\tau \\
&\quad +\frac{w(t)}{\sigma(x)}\Omega'(x)\,y^2+\varphi(t)v(t)\sigma^2(x)\int_0^{\frac{y}{\sigma(x)}}\Psi(\tau)d\tau \\
&\quad -\varphi(t)v(t)\sigma(x)\Psi(\frac{y}{\sigma(x)})y+z\int_0^t \Delta(t,s)\,\frac{y(s)}{\sigma(x(s))}ds \\
&\quad -\frac{u(t)}{\sigma(x)}\Gamma\left(x,\frac{y}{\sigma(x)}\right)z^2+u(t)\,\varphi(t)\Gamma\left(x,\frac{y}{\sigma(x)}\right)zy \\
&\quad -\alpha\varphi(t)yz+\frac{\alpha}{\sigma(x)}z^2+\frac{\alpha y}{\sigma(x)}\int_0^t \Delta(t,s)\,\frac{y(s)}{\sigma(x(s))}ds \\
&\quad -\frac{\alpha v(t)}{\sigma(x)}\Psi\left(\frac{y}{\sigma(x)}\right)y+\alpha u'(t)\int_0^{\frac{y}{\sigma(x)}}\Gamma(x,\tau)\tau d\tau \\
&\quad +\frac{\alpha u(t)}{\sigma(x)}y\int_0^{\frac{y}{\sigma(x)}}\Gamma_x(x,\tau)\tau d\tau+\frac{y^2(t)}{\sigma^2(x(t))}\int_t^\infty \left|\Delta(u,t)\right|du \\
&\quad -\int_0^t \left|\Delta(t,s)\right|\frac{y^2(s)}{\sigma^2(x(s))}ds.
\end{aligned}$$

It is obvious that

$$\begin{aligned}
U^{'}(t) &= w^{'}(t)\left(\alpha\int_0^x \Omega(u)\,du+\Omega(x)\,y+\frac{v^{'}(t)}{w^{'}(t)}\sigma(x)\int_0^{\frac{y}{\sigma(x)}}\Psi(\tau)d\tau\right)\\
&+\left(\frac{w(t)}{\sigma(x)}\Omega^{'}(x)-\frac{\alpha v(t)}{\sigma(x)}\frac{\Psi\left(\frac{y}{\sigma(x)}\right)}{y}\right)y^2\\
&+\varphi(t)\left(v(t)\sigma^2(x)\int_0^{\frac{y}{\sigma(x)}}\Psi(\tau)d\tau-v(t)\sigma(x)\Psi(\frac{y}{\sigma(x)})y\right)\\
&+\varphi(t)\left(u(t)\,\Gamma\left(x,\frac{y}{\sigma(x)}\right)-\alpha\right)zy+\frac{1}{\sigma(x)}\left(\alpha-u(t)\,\Gamma\left(x,\frac{y}{\sigma(x)}\right)\right)z^2\\
&+z\int_0^t\Delta(t,s)\,\frac{y(s)}{\sigma(x(s))}ds+\frac{\alpha y}{\sigma(x)}\int_0^t\Delta(t,s)\,\frac{y(s)}{\sigma(x(s))}ds\\
&+\alpha u^{'}(t)\int_0^{\frac{y}{\sigma(x)}}\Gamma(x,\tau)\tau d\tau+\frac{\alpha u(t)}{\sigma(x)}y\int_0^{\frac{y}{\sigma(x)}}\Gamma_x(x,\tau)\tau d\tau\\
&+\frac{y^2(t)}{\sigma^2(x(t))}\int_t^\infty|\Delta(u,t)|\,du-\int_0^t|\Delta(t,s)|\,\frac{y^2(s)}{\sigma^2(x(s))}ds.
\end{aligned}$$

Thus, from the assumptions $(A1)-(A7)$ and the inequality $|ab|\leq\frac{1}{2}\left(a^2+b^2\right)$, we obtain the following inequality

$$\begin{aligned}
U^{'}(t) &\leq w^{'}(t)\begin{pmatrix}\int_0^x\left(\alpha-\Omega^{'}(u)\right)\Omega(u)\,du+\frac{1}{2}\left(\Omega(x)+y\right)^2\\ +\sigma(x)\int_0^{\frac{y}{\sigma(x)}}\left(\frac{\Psi(\tau)}{\tau}-\sigma(x)\right)\tau d\tau\end{pmatrix}\\
&+\left(\frac{N\delta_1}{\sigma_0}-\frac{\alpha n\delta_2}{\sigma_1^2}\right)y^2+|\varphi(t)|\left(N\delta_3-n\delta_2\right)y^2\\
&+|\varphi(t)|\left(u_1\delta_5-\alpha\right)|yz|+\left(\frac{\alpha-u_0\delta_4}{\sigma_1}+\frac{1}{2}\int_0^t|\Delta(t,s)|\,ds\right)z^2\\
&+\frac{1}{\sigma_0^2}\left(\frac{\alpha\sigma_0}{2}\int_0^t|\Delta(t,s)|\,ds+\int_t^\infty|\Delta(u,t)|\,du\right)y^2\\
&+\left(\frac{\alpha}{2\sigma_0}-\frac{1}{2}\right)\int_0^t|\Delta(t,s)|\,\frac{y^2(s)}{\sigma^2(x(s))}ds\\
&\leq w^{'}(t)\left(\int_0^x\left(\alpha-\delta_1\right)\frac{\Omega(u)}{u}udu+\sigma(x)\int_0^{\frac{y}{\sigma(x)}}\left(\delta_2-\sigma_1\right)\tau d\tau\right)\\
&+\left(\frac{N\delta_1}{\sigma_0}-\frac{\alpha n\delta_2}{\sigma_1^2}+\frac{R}{\sigma_0^2}\right)y^2+\left(\frac{\alpha-u_0\delta_4}{\sigma_1}+\frac{R}{\alpha\sigma_0}\right)z^2\\
&+|\varphi(t)|\left(\left(N\delta_3-n\delta_2+u_1\delta_5-\alpha\right)y^2+\left(u_1\delta_5-\alpha\right)z^2\right)\\
&+\left(\frac{\alpha}{2\sigma_0}-\frac{1}{2}\right)\int_0^t|\Delta(t,s)|\,\frac{y^2(s)}{\sigma^2(x(s))}ds.
\end{aligned}$$

If

$$\frac{N\delta_1}{\sigma_0}+\frac{R}{\sigma_0^2}\leq\frac{\alpha n\delta_2}{\sigma_1^2},\ \alpha+\frac{\sigma_1 R}{\alpha\sigma_0}\leq u_0\delta_4$$

$$\delta_1 < \alpha \le \sigma_0 \text{ and } \sigma_1 \le \delta_2,$$

then we have

$$U^{'}(t) \le |\varphi(t)|\,\delta_7\left(y^2 + z^2\right), \tag{3.5}$$

where $\delta_7 = N\delta_3 - n\delta_2 + u_1\delta_5 - \alpha$.

It is now evident that the time derivative of the function $V(t)$ defined by (3.2) throughout any solution of system (3.1) is as follows

$$V^{'}(t) = e^{-\frac{\rho(t)}{\varepsilon}}\left(-\frac{\varphi(t)}{\varepsilon}U(t, x(t), y(t), z(t)) + U^{'}(t, x(t), y(t), z(t))\right).$$

Thus, using (3.4), (3.5) and taking $\varepsilon = \frac{\delta_6}{\delta_7}$, the last equality becomes

$$V^{'}(t) \le e^{-\frac{\rho(t)}{\varepsilon}}\left(-|\varphi(t)|\,\delta_7(y^2 + z^2) + |\varphi(t)|\,\delta_7\left(y^2 + z^2\right)\right) = 0,$$

which implies $V^{'}(t) \le 0$.

Since all the functions in equation (1.1) are continuous, by the Cauchy-Peano existence theorem, then it is clear that there exists at least one solution of equation (1.1) defined on $[t_0, t_0 + a)$ for some $a > 0$. We want to demonstrate that the solution can be expanded to the entire interval $[t_0, \infty)$. We suppose, on the contrary, that there is a first time $T < \infty$ such that the solution exists on $[t_0, T)$ and

$$\lim_{t \to T^-}(|x(t)| + |y(t)| + |z(t)|) = \infty.$$

Let $(x(t), y(t), z(t))$ be such a solution of system (3.1) under initial condition (x_0, y_0, z_0). Since $V(t)$ is positive definite and decreasing function on the trajectories of system (3.1), we can say that $V(t)$ is bounded on $[t_0, T)$, that is,

$$V(x(T), y(T), z(T)) \le V(t_0, x_0, y_0, z_0) = V_0.$$

Hence, it follows from (3.2) and (3.4) that

$$x^2(T) + y^2(T) + z^2(T) \le \frac{V_0}{K},$$

where $K = \delta_6 e^{-\frac{\rho(T)}{\varepsilon}}$. The last inequality implies that $|x(t)|, |y(t)|$ and $|z(t)|$ are bounded as $t \to T^-$. Therefore, we conclude that $T < \infty$ is not possible, we must have $T = \infty$. This finishes the proof of the theorem. □

Example. We take into consideration the undermentioned nonlinear integro-differential equation of third order

$$\left(\left(5 + \frac{1}{1 + x^2}\right)x^{'}\right)^{''} + \left(4 + \frac{1}{1 + t}\right)\left(2 + \frac{1}{1 + y^2}\right)x^{''}$$
$$+ \left(8 + \frac{1}{1 + t}\right)\left(10y + \frac{y}{1 + y^2}\right) + \left(8 + \frac{1}{2 + t}\right)\left(2x + \frac{x}{1 + x^2}\right)$$

$$= \int_0^t \frac{t}{(50t^2+1)^2} x'(s)\,ds. \tag{3.6}$$

When we compare (3.6) with (1.1), the following expressions are obtained.

$\sigma_0 = 5 \le \sigma(x) = 5 + \frac{1}{1+x^2} \le 6 = \sigma_1,$

$\int_{-\infty}^{+\infty} |\sigma'(u)|\,du \le \int_{-\infty}^{+\infty} \left|\frac{2u}{(1+u^2)^2}\right| du = 0 < \infty,$

$u_0 = 4 \le u(t) = 4 + \frac{1}{1+t} \le 5 = u_1,\ u^{'}(t) = -\frac{1}{(1+t)^2} \le 0,$

$v(t) = 8 + \frac{1}{1+t},\ v^{'}(t) = -\frac{1}{(1+t)^2},\ w(t) = 8 + \frac{1}{2+t},\ w^{'}(t) = -\frac{1}{(2+t)^2},$

$n = 8 \le w(t) \le v(t) \le 9 = N,\ v^{'}(t) \le w^{'}(t) \le 0,$

$\delta_4 = 2 \le \Gamma\left(x, x^{'}\right) = 2 + \frac{1}{1+y^2} \le 3 = \delta_5,\ \ y\Gamma_x\left(x, x^{'}\right) = 0,$

$\Psi(0) = 0,\ \delta_2 = 10 \le \frac{\Psi(y)}{y} = \left(10 + \frac{1}{1+y^2}\right) \le 11 = \delta_3,$

$h(0) = 0,\ \frac{h(x)}{x} = 2 + \frac{1}{1+x^2} \ge 2 = \delta_0,\ (x \ne 0),\ |h'(x)| \le 3 = \delta_1,$

$$\frac{\alpha\sigma_0}{2}\int_0^t |\Delta(t,s)|\,ds + \int_t^\infty |\Delta(u,t)|\,du \le \frac{25}{2}\int_0^t \frac{t}{(50t^2+1)^2}ds + \int_t^\infty \frac{u}{(50u^2+1)^2}du \le \frac{1}{100} = R$$

Thus, all assumptions of Theorem 3.1 hold. Therefore, we can finalize that all solutions of equation (3.6) are continuable and bounded.

Also the trajectories of solutions of equation (3.6) are shown in Figure 1.

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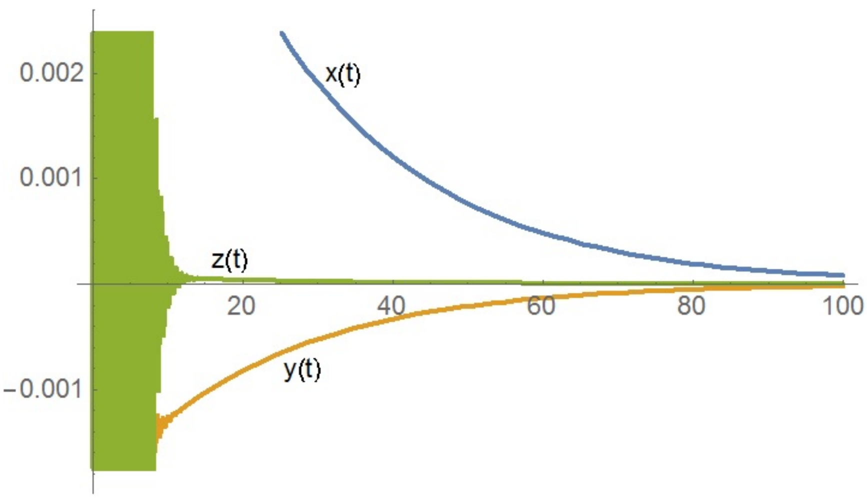

Figure 1: Time evolution of the states $x(t)$, $y(t)$ and $z(t)$ of equation (3.6).

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