The Dirac equation in parabolic cylindric coordinates and possible effects of the spinor structures in quantum physics

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Abstract. The Physics of spinor Lorentz group significantly differs from the one based on the orthogonal Lorentz group $L_{+-}^{\uparrow\downarrow}$, and only experiments may decide on this problem. In this context, the study of the fermion parity problem based on investigating possible single-valued representations of spinor coverings of the extended Lorentz group shows that P-parity and T-parity do not exist as separate concepts. Instead of this, only some unified concept of (PT)-parity can be determined in group-theoretical terms.

The extension procedure which describes a space with spinor structure is performed by relying on cylindrical parabolic coordinates. This is done by expanding the region $G(t,u,v,z) \rightsquigarrow \tilde{G}(t,u,v,z)$, so that instead of the half-plane $(u, v > 0)$ the entire plane (u, v) is used, while considering new identification rules for the boundary points. In the Cartesian picture, this procedure corresponds to taking the two-sheet surface $(x', y') \oplus (x'', y'')$ instead of the one-sheet surface (x, y).

The solutions of the Klein–Fock–Gordon equation classify into four types: Ψ_{++}, Ψ_{--}, Ψ_{+-}, Ψ_{-+}. The first two ones, Ψ_{++} and Ψ_{--}, provide single-valued functions of the vector space points, whereas the last two, Ψ_{+-}, Ψ_{-+}, have discontinuities in the frame of vector spaces, and therefore they are be discarded in this model. However, all the four types of functions are continuous ones, while regarded in the spinor space. It is established that all the solutions Ψ_{++}, Ψ_{--}, Ψ_{+-} and Ψ_{-+}, are orthogonal to each other, provided that integration is done over the extended region of integration which covers the spinor space. Similar results are obtained for the Dirac equation. The solutions of the type $(--), (++)$ are single-valued in the space with vector structure, whereas the solutions of the types $(-+), (+-)$ are not single-valued in the space with vector structure, so the solutions of types $(-+)$ and $(+-)$ must be discarded. However, they are valid solutions in the space with spinor structure.

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Introduction

In the literature [1–38], there exist three different approaches for describing the spinor Lorentz group, whose intrinsic essence is similar: a space-time spinor structure (Penrose, Rindler et al.); the Hopf bundle and the Kustaanheimo-Stiefel bundles.

In Hopf's technique, one uses only complex 2-spinors (ξ) and their conjugates (ξ^*), instead of real-valued 4-vector (tensor) objects. In the Kustaanheimo-Stiefel approach, there are used four real-valued coordinates, the real and imaginary parts of 2-spinor components.

The formalism developed in the present work exploits as well the possibilities given by spinors to construct 3-vectors; however, the emphasis is put on doubling the set of spacial points, so that we get an extended space model. In such a space, instead of the 2π-rotation, the 4π-rotation is considered - which transfers the space into itself. The procedure of extending the set of manifold points is achieved much easier by using curvilinear coordinates.

Within the framework of applications of spinor theory to Quantum Mechanics, we discuss two actual issues: exact linear representations for spinor coverings of the full Lorentz group and internal space-time parity of a relativistic fermion; the role of spinor space structure in classifying solutions of scalar and spinor wave equations specified for cylindric parabolic coordinates.

1 The concept of relativistic fermion parity

To study the problem of fermion parity, we will use 4-spinors instead of 2-spinors. An additional motivation for this approach is that among 4-spinors there exist the real-valued ones – the so-called Majorana 4-spinors; moreover, in this way we shall be able to describe discrete symmetries by means of linear transformations[1].

The obtained results will provide the grounds for a new discussion of the old fermion parity problem of investigating possible linear single-valued representations of spinor coverings of the extended Lorentz group. It is shown that, in the frame of this theory, P-parity and T-parity for a fermion do not exist as separate concepts; instead of these, only some unified concept of (PT)-parity can be described in a group-theoretical language.

1.1 The spin covering for the full Lorentz group $L_{+-}^{\uparrow\downarrow}$

We attach to the proper orthochronous Lorentz matrices $L(k,k^*) = L(-k,-k^*)$:

$$L_a{}^b(k,k^*) = \bar{\delta}_a^b(-\delta_c^b k^n k_n^* + k_c k^{b*} + k_c^* k^b + i\epsilon_c{}^{bmn} k_m k_n^*),$$

the following two linear operations

$$P: \; L_a^{(P)b} = +\bar{\delta}_a^b \; ; \qquad T: \; L_a^{(T)b} = -\bar{\delta}_a^b,$$

[1] We shall mainly consider only the problem of the accurate description of the single-valued representations of four different spinor groups, each of them covering the full Lorentz group $L_{+-}^{\uparrow\downarrow}$, including the P and T-reflections.

where $\bar{\delta}_a^b = \text{diag}(+1,-1,-1,-1)$, of which one readily produces the full Lorentz group $L_{+-}^{\uparrow\downarrow}$. The commutation rules between $L_a{}^b(k,k^*)$ and the discrete elements P,T are

$$\bar{\delta}_a^b \; L_b{}^c(k,k^*) = L_a{}^b(\bar{k}^* \;, \bar{k}) \; \bar{\delta}_b^c \; .$$

The group $L_{+-}^{\uparrow\downarrow}$ has four types of vector representations:

$$T_a{}^b(L) = f(L) \; L_a{}^b \; , \quad L \in L_{+-}^{\uparrow\downarrow} \; ,$$

namely

$$f_1(L) = 1 \; , \qquad f_2(L) = \det(L) \; ,$$
$$f_3(L) = \text{sgn} \; (L_0{}^0) \; , \quad f_4(L) = \det(L) \; \text{sgn} \; (L_0^0) \; .$$

We emphasize that the above-described extension of the group $L_a{}^b(k,k^*)$ by adding the two discrete operations P and T is not an extension of the spinor group $SL(2,\mathbb{C})$: actually this is just an expansion of the orthogonal group $L_+^{\uparrow}$. From the spinor point of view, the operations P and T are transformations which act on the space of 2-rank spinors, and *not* on the space of 1-rank spinors. Evidently, a more comprehensive study of the P,T-symmetry can be done in the framework of first-rank spinors, when one extends the covering group $SL(2,\mathbb{C})$ by adding spinor discrete operations.

Now we can start solving the following task. A covering group for the total Lorentz group can be constructed by adding two specific 4×4-matrices to the known set of 4-spinor transformations of the group $SL(2,\mathbb{C})$,

$$S(k,\bar{k}^*) = \left(\begin{array}{cc} B(k) & 0 \\ 0 & B(\bar{k}^*) \end{array} \right) \in \widetilde{SL}(2,\mathbb{C}).$$

These two new matrices are taken from the set

$$M = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix} \right), \quad M' = iM, \qquad N = \left(\begin{smallmatrix} 0 & -iI \\ +iI & 0 \end{smallmatrix} \right), \quad {}'N = iN.$$

Having added any two elements of the four ones, we provide the full extension of the group $\widetilde{SL}(2,\mathbb{C})$, by means of two new operations only. Also, we note that since the group $L(2,\mathbb{C})$ contains $-I$, the extension of the group by any two elements of $\{-M,-M',-N,-{}'N\}$, leads to the same result. However, if one takes any other phase factor, different from $+1,-1,+i,-i$ for $M,M',N,{}'N$, then this will result in substantially new extended groups.

The multiplication table for these four discrete elements is

	M	M'	N	$'N$
M	$\left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} iI & 0 \\ 0 & iI \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} +iI & 0 \\ 0 & -iI \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} -I & 0 \\ 0 & +I \end{smallmatrix}\right)$
M'	$\left(\begin{smallmatrix} +iI & 0 \\ 0 & +iI \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} -I & 0 \\ 0 & -I \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} -I & 0 \\ 0 & +I \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} -iI & 0 \\ 0 & +iI \end{smallmatrix}\right)$
N	$\left(\begin{smallmatrix} -iI & 0 \\ 0 & +iI \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} +I & 0 \\ 0 & -I \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} +I & 0 \\ 0 & +I \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} +iI & 0 \\ 0 & +iI \end{smallmatrix}\right)$
$'N$	$\left(\begin{smallmatrix} +I & 0 \\ 0 & -I \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} +iI & 0 \\ 0 & -iI \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} +iI & 0 \\ 0 & +iI \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} -I & 0 \\ 0 & -I \end{smallmatrix}\right)$

Hence we obtain six covering groups,

$$
\begin{aligned}
G_M &= \{\, S(k,\bar{k}^*) \uplus M \uplus M' \,\} , & G_N &= \{\, S(k,\bar{k}^*) \uplus N \uplus {}'N \,\} ,\\
G' &= \{\, S(k,\bar{k}^*) \uplus M' \uplus N \,\} , & {}'G &= \{\, S(k,\bar{k}^*) \uplus' N \uplus M \,\} ,\\
G &= \{\, S(k,\bar{k}^*) \uplus M \uplus N \,\} , & {}'G' &= \{\, S(k,\bar{k}^*) \uplus M' \uplus' N \,\} ,
\end{aligned}
$$

with the corresponding multiplication tables

$$
\begin{array}{llll}
G_M : & M^2 = +I \; , & {M'}^2 = -I \; , & MM' = (M')M \; ;\\
G_N : & N^2 = +I \; , & {}'N^2 = -I \; , & N({}'N) = ({}'N)N \; ;\\
G' : & {M'}^2 = -I \; , & N^2 = +I \; , & (M')N = -N(M') \; ;\\
{}'G : & ({}'N)^2 = -I \; , & M^2 = +I \; , & ({}'N)M = -M({}'N) \; ;\\
G : & M^2 = +I \; , & N^2 = +I \; , & MN = -NM \; ;\\
{}'G' : & (M')^2 = -I \; , & {N'}^2 = -I \; , & (M')({}'N) = -({}'N)(M') \; ,
\end{array}
$$

and

$$
F \; S(k,\bar{k}^*) = S(\bar{k}^*,k) \; F \; , \quad F \in \{M, M', N, {}'N\} \; .
$$

One can notice that the multiplication laws for the groups G_M and G_N coincide; the same happens for G' and $'G$. This implies that the groups G_M and G_N (and respectively G' and $'G$) represent the same abstract group. Indeed, it is readily verified that G_M and G_N (and, also, G' and $'G$), can be mutually transformed by similarity transformations:

$$
\begin{array}{lll}
G_N = A \; G_M \; A^{-1}, & A \; S(k,\bar{k}^*) = S(\bar{k}^*,k) \; A, & \\
A \; M \; A^{-1} = +N, & A \; M' \; A^{-1} = +{}'N, & A = \text{const} \cdot \begin{pmatrix} -iI & 0 \\ 0 & +I \end{pmatrix} \; ,\\
{}'G = A \; G' \; A^{-1}, & A \; S(k,\bar{k}^*) = S(\bar{k}^*,k) \; A \; , & \\
A \; M' \; A^{-1} = +{}'N, & A \; N \; A^{-1} = -M, & A = \text{const} \cdot \begin{pmatrix} -iI & 0 \\ 0 & +I \end{pmatrix} \; .
\end{array}
$$

In other words, we define here only four different covering groups. Since in literature all the six variants are discussed, we shall accordingly consider them all.

1.2 Representations of the extended spinor groups

We shall construct now the exact linear representations of the groups G_M, G_N, G', $'G$, G, $'G'$. It suffices to consider in detail only one group; for convenience, let this be G_M. Its multiplication table is

$$
\begin{gathered}
M^2 = -I \; , \; {M'}^2 = -I \; , \; M \; M' = M \; M \; ,\\
F \; S(k,\bar{k}^*) = S(\bar{k}^*,k) \; F \; , \quad (\;\; F = M \; , M' \;\;) \; ,\\
(k_1,\bar{k}_1^*)(k_2,\bar{k}_2^*) = (\langle k_1,k_2\rangle, \; \langle \bar{k}_1^*,\bar{k}_2^*\rangle) \; .
\end{gathered}
\tag{1.1}
$$

where the symbol $\langle \; , \; \rangle$ stands for the known multiplication rule in the group $SL(2,\mathbb{C})$:

$$
\langle k_1,k_2\rangle = (k_1^0 k_2^0 + \vec{k}_1\vec{k}_2; \; \vec{k}_1 k_2^0 + k_2^0\vec{k}_1 + i[\vec{k}_1\vec{k}_2]) \; .
$$

Let us look for the solution of the problem of constructing the simplest irreducible representations of the spinor groups as mappings of the form

$$T(g) = f(g)\, g \;, \qquad g \in G_M \;, \qquad f(g_1)\cdot f(g_2) = f(g_1\cdot g_2) \tag{1.2}$$

where $f(g)$ is a numerical function on the group G_M. Substitution of (1.2) into (1.1) yields

$$[f(M)]^2 = f(I)\;, \qquad [f(M')]^2 = f(-I)\;, \qquad f(S(k,\bar{k}^*)) = f(S(\bar{k}^*,k))\;,$$
$$f(S(k_1,\vec{k}_1^*))\; f(S(k_2),\vec{k}_2^*)) = f(S(\langle k_1,k_2\rangle\;,\;\langle \vec{k}_1^*\;,\;\vec{k}_2^*\rangle))\;.$$

There exist four different such functions f_i, described by:

G_M	$f_1(g)$	$f_2(g)$	$f_3(g)$	$f_4(g)$
$S(k,\bar{k}^*)$	$+1$	$+1$	$+1$	$+1$
M	$+1$	-1	$+1$	-1
M'	$+1$	-1	-1	$+1$,

which provide four representations $T_i(g)$ of the group G_M.

In the same manner, one can construct the analogous representation $T_i(g)$ of the remaining five groups. All these are described by the following table

	g	$T_1(g)$	$T_2(g)$	$T_3(g)$	$T_4(g)$
	$S(k,\bar{k}^*)$	$S(k,\bar{k}^*)$	$S(k,\bar{k}^*)$	$S(k,\bar{k}^*)$	$S(k,\bar{k}^*)$
G_M	M	$+M$	$-M$	$+M$	$-M$
	M'	$+M'$	$-M'$	$-M'$	$+M'$
G_N	N	$+N$	$-N$	$+N$	$-N$
	$'N$	$+'N$	$-'N$	$-'N$	$+'N$
G'	M'	$+M'$	$-M'$	$+M'$	$-M'$
	N	$+N$	$-N$	$-N$	$+N$
$'G$	$'N$	$+'N$	$-'N$	$+'N$	$-'N$
	M	$+M$	$-M$	$-M$	$+M$
G	M	$+M$	$-M$	$+M$	$-M$
	N	$+N$	$-N$	$+N$	$-N$
$'G'$	M'	$+M'$	$-M'$	$+M'$	$-M'$
	$'N$	$+'N$	$-'N$	$-'N$	$+'N$.

For each of these groups, one can ask whether the four representations $T_i(g)$ are equivalent, or not. With the help of the relations

$$F = const\begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix}, \quad F\; S(k,\bar{k}^*)\; F^{-1} = S(k,\bar{k}^*)\;,$$
$$F\; M\; F^{-1} = -M, \qquad F\; M'\; F^{-1} = -M'\; F\;,$$
$$N\; F^{-1} = -N, \qquad F\; 'N\; F^{-1} = -'N,$$

it is easily follows that the type $T_2(g)$ is equivalent to the type $T_1(g)$, as well, $T_4(g)$ is equivalent to $T_3(g)$:

$$T_2(g) = F\ T_1(g)\ F^{-1}\ , \quad T_4(g) = F\ T_3(g)\ F^{-1}\ .$$

Summarizing, for each of the six groups, only two non-equivalent representations $g \to T(g) = f(g)\ g$ are possible:

$$T_1(g) \sim T_2(g)\ , \quad T_3(g) \sim T_4(g)\ .$$

Evidently, this result does not depend on the explicit realization of the discrete spinor transformations.

The above study of the exact linear representations of the extended spinor groups leads to a new concept of a space-time intrinsic parity of a fermion. In group-theoretical terms, P-parity and T-parity do not have any sense; instead, only their joint characteristic, which might be called (PT)-parity, can be defined in the group-theoretic framework.

1.3 Representations of the coverings for partly extended groups $L^{\uparrow}_{+-}$ and $L^{\uparrow\downarrow}_{+}$

Now we are going to consider the problem of linear representations of the spinor groups that cover the partly extended Lorentz groups $L^{\uparrow}_{+-}$ and $L^{\uparrow\downarrow}_{+}$ (improper orthochronous and proper non-orthochronous, respectively). Such groups can be constructed by adding any matrix from $\{M, M', N, ' N\}$.

The case of the orthogonal group $L^{\uparrow}_{+-}$ leads to

$$T_1 = T_3\ ; \quad L \Longrightarrow L = (\text{sgn}\ {L_0}^0)\ L, \tag{1.3}$$

$$T_2 = T_4\ ; \quad L \Longrightarrow L = (\det L) L = (\det L)(\text{sgn}\ {L_0}^0)\ L\ , \tag{1.4}$$

and the case of the group $L^{\uparrow\downarrow}_{+}$ is characterized by

$$T_1 = T_4\ ; \quad L \Longrightarrow L = (\det L)(\text{sgn}\ {L_0}^0)\ L\ , \tag{1.5}$$

$$T_2 = T_3\ \ ; \quad L \Longrightarrow\ L = (\det L)\ L = (sgn\ {L_0}^0)\ L\ . \tag{1.6}$$

With the use of one additional discrete operation, one can determine four extended spinor groups:

$$\widetilde{SL}(2, \mathbb{C})_M = \{\ S(k, \bar{k}^*)\ \oplus\ M\}\quad \text{and so on}\ . \tag{1.7}$$

We conclude that the extended groups $\widetilde{SL}(2, \mathbb{C})_M$, $\widetilde{SL}(2, \mathbb{C})_N$ turn out to be isomorphic. Analogously, $\widetilde{SL}(2, \mathbb{C})_{M'}$ is isomorphic to $\widetilde{SL}(2, \mathbb{C})_{'N}$. Each of them covers both $L^{\uparrow\downarrow}_{+}$ and $L^{\uparrow}_{+-}$,

$$\widetilde{SL}(2, \mathbb{C})_M\ \sim\ \widetilde{SL}(2, \mathbb{C})_N\ \ ,\ \ \widetilde{SL}(2, \mathbb{C})_{M'}\ \sim\ \widetilde{SL}(2, \mathbb{C})_{'N}\ . \tag{1.8}$$

Now, we shall list the simplest representations of these groups. The obtained result is as follows: all the representations $T_i(g)$ from above, while confining them to the subgroups $SL(2, \mathbb{C})_{M(N)}$ and $SL(2, \mathbb{C})_{M',('N)}$, lead to representations which mutually change by a similarity transformation. In other words, in fact there exists only one representation of these partly extended spinor groups. This may be understood as the impossibility to determine any group-theoretical parity concept (P or T) within the limits of partly extended spinor groups.

1.4 On reducing spinor groups to a real form

We have considered above all the spinor groups $G_M \sim G_N$, $G' \sim' G$, G , $'G'$ as possible group covering candidates for the full Lorentz group $L^{\uparrow\downarrow}_{+-}$. It is desirable to formulate some extra arguments in order to choose only *one* spinor group as a natural (physical) covering.

We note that in the bispinor space, a special basis can be found by using the bispinor wave function

$$\Phi_M(x) = \varphi(x) + i\xi(x) ,$$

which transforms under the action of the group $SL(2,\mathbb{C})$ by means of real (4×4)-matrices. Therefore, the real 4-spinors $\varphi(x)$ and $\xi(x)$, two constituents of the complex-valued spinor $\Phi_M(x)$, transform as independent irreducible 4-dimensional real-valued spinor representations. In the physical context, this reads as a group-theoretical permission to exist real Majorana fermions.

But these arguments rely only on continuous $SL(2,\mathbb{C})$-transformations, while the idea is to extend them to discrete operations too. So we must find the answer to the question of which of the extended spinor groups of matrices can be reduced to real-valued forms. To this goal, we write down the bispinor matrix a the form that does not depend on the randomly chosen basis[2]:

$$\begin{aligned} S = \; & \tfrac{1}{2}(k_0+k_0^*) + \tfrac{1}{2}(k_0-k_0^*)\gamma^5 + (k_1+k_1^*)\sigma^{01} + (k_1-k_1^*)i\sigma^{23} \\ & +(k_2+k_2^*)\sigma^{02} + (k_2-k_2^*)i\sigma^{31} + (k_3+k_3^*)\sigma^{03} + (k_3-k_3^*)i\sigma^{13}. \end{aligned}$$

Any Majorana basis satisfies the relations

$$(\gamma^a_M)^* = -\gamma^a_M \; , \;\; (\gamma^5_M)^* = -\gamma^5_M \; , \;\; (\sigma^{ab}_M)^* = \sigma^{ab}_M \Longrightarrow S^* = S \; .$$

It remains to write down all the used discrete (matrix) operations $M, M'N,'N$ in terms of Dirac matrices:

$$M = +\gamma^0 \; , \qquad M' = +i\,\gamma^0 \; , \qquad N = +i\,\gamma^5\,\gamma^0 \; , \qquad 'N = -\gamma^5\,\gamma^0 \; .$$

In Majorana frames, the (continuous and discrete) group operations obey the following properties

$$S^* = S \; , \; M^* = -M \; , \; (M')^* = +M' \; , \; N^* = -N \; , \; ('N)^* = +'N.$$

Thus, the six spinor groups behave under complex conjugation as indicated below

G_M	G_N	G'	$'G$	G	$'G'$
$S^* = S$	$S^* = S$	$S^* = S$	$S^* = S$	$S^* = S$	$S^* = S$
$M^* = -M$	$N^* = -N$	$M'^* = +M'$	$'N^* = +'N$	$M^* = -M$	$M'^* = +M$
$(M')^* = M'$	$'N^* = +'N$	$N^* = -N$	$M^* = -M$	$N^* = -N$	$'M^* =' M.$

Only the group $'G'$ can be reduced to a real-valued form, and only this group allows real-valued spinor representations, namely the Majorana fermions[3].

[2] We employed above the Weyl basis.
[3] This variant coincides with the known in the literature *Racah group*.

2 Space with spinor structure and solutions of the Klein–Fock–Gordon equation

2.1 Cylindric parabolic coordinates

Let us start with the parabolic cylindrical coordinates

$$x = (u^2 - v^2)/2 \, , \qquad y = u\, v \, , \qquad z = z.$$

In order to cover the vector space (x, y, z), it suffices to make a choice out of the following four possibilities:

$$\begin{array}{ll} v = +\sqrt{-x + \sqrt{x^2 + y^2}} \, , & u = \pm\sqrt{+x + \sqrt{x^2 + y^2}} \, , \\ v = -\sqrt{-x + \sqrt{x^2 + y^2}} \, , & u = \pm\sqrt{+x + \sqrt{x^2 + y^2}} \, , \\ v = \pm\sqrt{-x + \sqrt{x^2 + y^2}} \, , & u = +\sqrt{+x + \sqrt{x^2 + y^2}} \, , \\ v = \pm\sqrt{-x + \sqrt{x^2 + y^2}} \, , & u = -\sqrt{+x + \sqrt{x^2 + y^2}} \, . \end{array}$$

For definiteness, let us use the first variant from the above:

$$v = +\sqrt{-x + \sqrt{x^2 + y^2}} \, , \qquad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}} \, .$$

which is illustrated in Fig. 1.

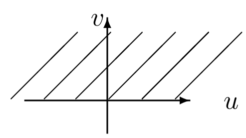

Fig. 1. The region $G(u, v)$, used to parameterize the vector model.

The correspondence between the points (x, y) and (u, v) can be illustrated by the following formulas and Fig. 2:

$$\begin{array}{lll} u = k\ \cos\phi, & v = k\ \sin\phi, & \phi \in [\ 0,\ \pi\ ]\ ; \\ x = (k^2/2)\ \cos 2\phi, & y = (k^2/2)\ \sin 2\phi, & 2\phi \in [0,\ 2\pi]. \end{array}$$

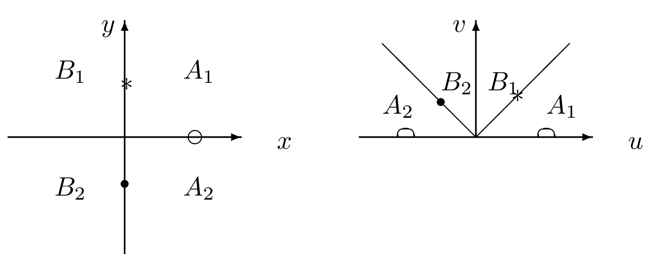

Fig. 2. The mapping $G(x, y) \Longrightarrow G(u, v)$; identification rules

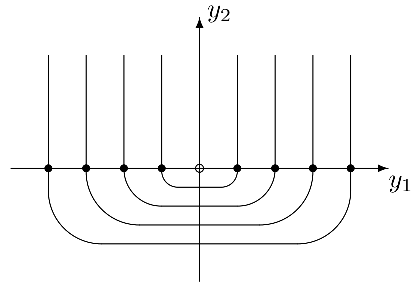

Fig. 3. Parabolic cylindrical coordinates

In Fig. 3, the identified points on the boundary are connected by lines, and the domain $G(y_1, y_2)^{y_3}$ (at arbitrary y_3) ranging in the half-plane (y_1, y_2) covers the whole vector plane $(x_1, x_2)^{x_3}$.

When turning to the case of spinor space, we shall see the complete symmetry between the coordinates u and v; they relate to the Cartesian coordinates of the extended model $(x, y, z) \oplus (x', y', z')$ through the formulas (see Fig. 4.):

$$v = \pm\sqrt{-x + \sqrt{x^2 + y^2}}\,, \qquad u = \pm\sqrt{+x + \sqrt{x^2 + y^2}}$$

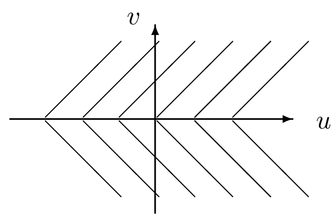

Fig. 4. $\tilde{G}(u, v)$ covering the spinor space

The metric of space-time in parabolic cylindrical coordinates has the form

$$dS^2 = c^2 dt^2 - (u^2 + v^2)(du^2 + dv^2) - dz^2 \,.$$

2.2 Solutions of the Klein–Fock–Gordon equation and spinors

Let us consider the KFG equation

$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{u^2 + v^2} \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) - \frac{m^2 c^2}{\hbar^2} \right] \Psi = 0. \tag{2.1}$$

After separating the variables by the substitution

$$\Psi(t, u, v, \phi) = e^{-i\epsilon t/\hbar}\, e^{ipz/\hbar}\, U(u)\, V(v),$$

one deduces

$$\left[\frac{1}{U}\frac{d^2 U}{du^2} + \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2} \right) u^2 \right] + \left[\frac{1}{V}\frac{d^2 V}{dv^2} + \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2} \right) v^2 \right] = 0 \,. \tag{2.2}$$

In the following, we shall use the notation

$$\lambda^2 = \left(\frac{\epsilon^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2} - \frac{p^2}{\hbar^2} \right) , \qquad [\lambda] = \frac{1}{\text{meter}} \; . \tag{2.3}$$

By introducing two separation constants, a and b $(a + b = 0)$, we can derive from (2.2) two distinct equations:

$$\frac{d^2 U}{du^2} + (\, \lambda^2 \, u^2 - a \,) \, U = 0 \; , \qquad \frac{d^2 V}{dv^2} + (\, \lambda^2 \, v^2 - b \,) \, V = 0 \; . \tag{2.4}$$

The transition in equations (2.4) to the canonical form is obtained by using dimensionless variables:

$$\sqrt{2\lambda}\, u \;\; \rightarrow \;\; u \; , \; \frac{a}{2\lambda} \;\; \rightarrow \;\; a \; , \; \sqrt{2\lambda}\, v \;\; \rightarrow \;\; v \; , \; \frac{b}{2\lambda} \;\; \rightarrow \;\; b \; . \tag{2.5}$$

The equations (2.4) will take the form:

$$\frac{d^2 U}{du^2} \; + \; \left(\, \frac{u^2}{4} - a \right) \; U = 0 \; , \qquad \frac{d^2 V}{dv^2} \; + \; \left(\, \frac{v^2}{4} + a \right) \; V = 0 \; . \tag{2.6}$$

The solutions of these similar equations can be found in series form:

$$F(\xi) = c_0 + c_1 \xi + \; c_2 \xi^2 + \sum_{k=1,2,...} c_{2k+1} \xi^{2k+1} + \sum_{k=1,2,...} c_{2k+2} \xi^{2k+2}. \tag{2.7}$$

We note that in (2.7) the terms of even and odd powers of ξ are separated.

After tedious calculation, one derives the following two independent groups of recurrent relations:

for even powers

$$\begin{array}{rl} \xi^0 : & 2 \, c_2 - \alpha \, c_0 = 0 \; , \\ \xi^2 : & c_4 \; 4 \times 3 + \frac{c_0}{4} - \alpha \, c_2 = 0 \; , \\ \xi^4 : & c_6 \; 6 \times 5 + \frac{c_2}{4} - \alpha \, c_4 = 0 \; , \\ n = 3, 4, ..., \;\; \xi^{2n} : & c_{2n+2} (2n+2)(2n+1) + \frac{1}{4} \, c_{2n-2} - \alpha \, c_{2n} \; = 0 \; ; \end{array} \tag{2.8}$$

for odd powers

$$\begin{array}{rl} \xi^1 : & c_3 \; 3 \times 2 - \alpha \, c_1 = 0 \; , \\ \xi^3 : & c_5 \; 5 \times 4 + \frac{c_1}{4} - \alpha \, c_3 = 0 \; , \\ n = 3, 4, ..., \;\; \xi^{2n-1} : & c_{2n+1} (2n+1)(2n) + \frac{1}{4} \, c_{2n-3} - \alpha \, c_{2n-1} = 0 \; . \end{array} \tag{2.9}$$

So one can construct the following two linearly independent solutions

even

$$F_1(\xi^2) = 1 + a_2 \frac{\xi^2}{2!} + a_4 \frac{\xi^4}{4!} + ...,$$

$$a_2 = \alpha \; , \qquad a_4 = \alpha^2 - \frac{1}{2} \; , \qquad c_6 = \alpha^3 - \frac{7}{2} \alpha \; ,$$

$$n = 3, 4, ... : \qquad a_{2n+2} = \alpha \; a_{2n} - \frac{(2n)(2n-1)}{4} \; a_{2n-2} \; ; \tag{2.10}$$

odd

$$F_2(\xi) = \xi + a_3 \, \frac{\xi^3}{3!} + a_5 \, \frac{\xi^5}{5!} + ... \, ,$$

$$a_3 = \alpha \, , \qquad a_5 = \alpha^2 - \frac{3}{2} \, ,$$

(2.11) $$n = 3, 4, ... : \qquad a_{2n+1} = \alpha \, a_{2n-1} \; - \; \frac{(2n-1)(2n-2)}{4} \, a_{2n-3} \, .$$

2.3 Manifestation of vector and spinor space structures

Having combined the two previous solutions F_1 and F_2, we can obtain four types of wave functions[4]

(2.12) $$\begin{array}{ll} (\text{even} \otimes \text{even}) : & \Phi_{++} = E(a,u^2) \; E(-a,v^2), \\ (\text{odd} \otimes \text{odd}) : & \Phi_{--} = O(a,u) \; O(-a,v \,) \;\; , \\ (\text{even} \otimes \text{odd}) : & \Phi_{+-} = E(a,u^2) \; O(-a,v \,) \; , \\ (\text{odd} \otimes \text{even}) : & \Phi_{-+} = O(a,u) \; E(-a,v^2) \; . \end{array}$$

We note the behavior of the constructed wave functions:

(2.13) $$\begin{array}{ll} \Phi_{++}(x=0, y=0) \neq 0 \, , & \Phi_{--}(x=0, y=0) = 0 \, , \\ \Phi_{+-}(x>0, y=0) = 0 \, , & \Phi_{-+}(x<0, y=0) = 0 \, . \end{array}$$

Now let us find which restrictions for the wave functions Ψ follow from the requirement of single-valuedness. To this aim, two properties of the parametrization are essential:

$$\underline{v=0} \; : \; x = +\frac{u^2}{2} \geq 0, \; y = 0; \qquad \underline{u=0} \; : \; x = -\frac{v^2}{2} \leq 0, \; y = 0 \, .$$

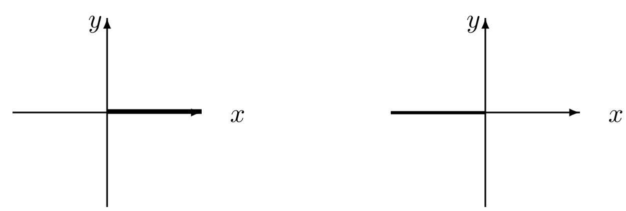

Fig. 5. The peculiarities of the parametrization

The four solutions behave in special regions, as follows:

(2.14) $$\begin{array}{l} \Phi_{++}(a; u=0, v) = + \, \Phi_{++}(a; u=0, -v), \\ \Phi_{++}(a; +u, v=0) = + \, \Phi_{++}a; -u, v=0), \\ \Phi_{--}(a; u=0, +v) = + \, \Phi_{--}(a; u=0, -v) = 0, \\ \Phi_{--}(a; u, v=0) = + \, \Phi_{--}(a; -u, v=0) = 0, \end{array}$$

(2.15) $$\begin{array}{l} \Phi_{+-}(a; u=0, +v) = - \, \Phi_{+-}(a; u=0, -v), \\ \Phi_{+-}(a; u, v=0) = \Phi_{+-}(a; -u, v=0) = 0, \\ \Phi_{-+}(a; u=0, +v) = \Phi_{-+}(a; u=0, -v) = 0, \\ \Phi_{-+}(a; +u, v=0) = - \, \Phi_{-+}(a; -u, v=0) \; . \end{array}$$

[4]We will change the notation: $F_1 \Longrightarrow E$; $F_2 \Longrightarrow O$.

The boundary properties of the constructed wave functions can be illustrated by the schemes described in Fig. 6.

So we conclude that the solutions Ψ of the types $(++)$ and $(--)$ are single-valued in the space with vector structure, whereas the solutions of the types $(+-)$ and $(-+)$ are not single-valued in such a space, so these latter types $(+-)$ and $(-+)$ must be discarded. However, these solutions ($(+-)$ and $(-+)$) must be retained in the space with *spinor* structure.

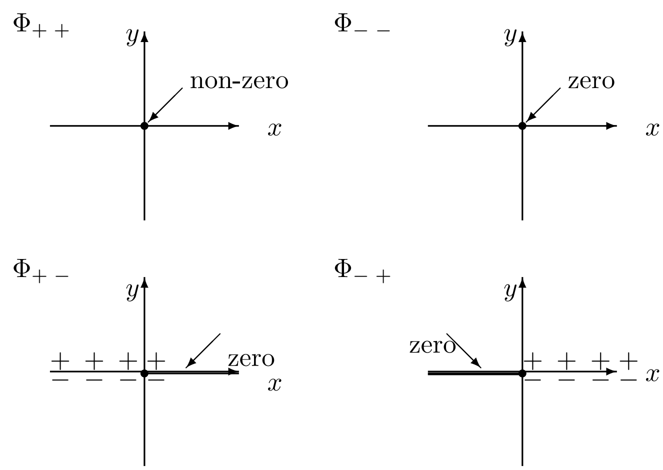

Fig 6. Boundary behavior of the wave functions in the (x, y)-plane

When using the spinor space model, the two sets (u, v) and $(-u, -v)$ represent different geometrical points in the spinor space, so the requirement of single valuedness, like applied in the case of the spinor space does not assume that the values of the wave functions must be equal at the points (u, v) and $(-u, -v)$:

$$\Phi(u, v) = \Phi(x, y) \neq \Phi(-u, -v) = \Phi(x', y') \ .$$

The splitting of the basis wave functions into two subsets may be mathematically formalized with the help of the special discrete operator acting in the spinor space:

$$\hat{\delta} = \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right), \qquad \hat{\delta} \left(\begin{array}{c} u \\ v \end{array} \right) = \left(\begin{array}{c} -u \\ -v \end{array} \right) \ . \tag{2.16}$$

It is easily verified that the solutions which are single-valued in the vector space model are eigenfunctions of δ for the eigenvalue $\delta = +1$:

$$\hat{\delta} \ \Phi_{++}(a; u, v) = + \ \Phi_{++}(a; u, v) \ , \quad \hat{\delta} \ \Phi_{--}(a; u, v) = + \ \Phi_{--}(a; u, v) \ ,$$

and the additional ones - which are acceptable only in the spinor space model - are eigenfunctions for the eigenvalue $\delta = -1$:

$$\hat{\delta} \ \Phi_{+-}(a; u, v) = - \ \Phi_{+-}(a; u, v) \ , \quad \hat{\delta} \ \Phi_{-+}(a; u, v) = - \ \Phi_{-+}(a; u, v) \ .$$

2.4 Orthogonality and completeness of the bases for vector and spinor spaces

Now let us consider the scalar multiplication

$$\int \Psi^*_{\mu'}\ \Psi_\mu\ \sqrt{-g}\ dtdzdudv\ .$$

of the basic constructed wave functions:

$$\Psi_{++}(\epsilon,p,a) = e^{i\epsilon t}\ e^{ipz}\ \Phi_{++}(a;u,v)\ , \quad \Psi_{--}(\epsilon,p,a) = e^{i\epsilon t}\ e^{ipz}\ \Phi_{--}(a;u,v)\ ,$$
$$\Psi_{+-}(\epsilon,p,a) = e^{i\epsilon t}\ e^{ipz}\ \Phi_{+-}(a;u,v)\ , \quad \Psi_{-+}(\epsilon,p,a) = e^{i\epsilon t}\ e^{ipz}\ \Phi_{-+}(a;u,v)\ . \tag{2.17}$$

where μ and μ' stand for generalized quantum numbers.

First of all, we note some interesting integrals[5]:

in the vector space

$$I_0 = \int_0^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{++}\ \Phi_{--}\ (u^2+v^2),$$

in the spinor space

$$I_1 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{++}\ \Phi_{--}\ (u^2+v^2)\ ,$$
$$I_2 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{+-}\ \Phi_{-+}\ (u^2+v^2)\ ,$$
$$I_3 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{++}\ \Phi_{+-}\ (u^2+v^2)\ ,$$
$$I_4 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{++}\ \Phi_{-+}\ (u^2+v^2)\ ,$$
$$I_5 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{--}\ \Phi_{+-}\ (u^2+v^2)\ ,$$
$$I_6 = \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} du\ \Phi^*_{--}\ \Phi_{-+}\ (u^2+v^2)\ .$$

All these seven integrals $I_0, I_1...I_6$ are equal to zero, which means that the constructed functions provide us with an orthogonal basis for the Hilbert space $\Psi(t,z,u,v)$, where (u,v,z) belong to the extended (spinor) space model.

2.5 The Schrödinger equation

The study of the analytical properties of the Klein-Fock-Gordon wave solutions in vector and spinor space models is still applicable, with slight changes, to the non-relativistic Schrödinger equation as well:

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2m}\ \left[\ \frac{\partial^2}{\partial z^2} + \frac{1}{u^2+v^2}\ \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\ \right]\ \Psi\ ,$$

where the substitution for the wave functions is the same

$$\Psi(t,u,v,z) = e^{-i\epsilon t/\hbar}\ e^{ipz/\hbar}\ U(u)V(v)\ ,$$

and then, the equation for $U(u)V(v)$ is

$$\left[\ \frac{\hbar^2}{2m}\ \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\ +\ \left(\epsilon - \frac{p^2}{2m}\right)(u^2+v^2))\ \right] U(u)V(v) = 0\ .$$

[5]The arguments $(a;u,v)$ are omitted here.

3 The Dirac particle and the space with spinor structure

3.1 The separation of variables

We shall apply a tetrad based form of the Dirac equation to construct solutions of the Dirac equation in parabolic cylindric coordinates. The last ones are defined by the formulas

$$x_1 = \frac{u^2 - v^2}{2} , \qquad x_2 = uv , \qquad x_3 = z .$$

To parameterize the space with vector structure (x, y, z), it is enough to make use of the following formulas

$$v = +\sqrt{-x_1 + \sqrt{x_1^2 + x_1^2}} , \qquad u = \pm\sqrt{+x_1 + \sqrt{x_1^2 + x_2^2}} ,$$

However, to parameterize the space $(x, y, z) \oplus (x, y, z)$ with spinor structure, one must use for u and v more symmetrical, substantially different formulas:

$$v = \pm\sqrt{-x_1 + \sqrt{x_1^2 + x_2^2}} , \quad u = \pm\sqrt{+x_1 + \sqrt{x_1^2 + x_2^2}} .$$

We will further denote the parabolic cylindric coordinates by $(u, v, z) = (y^1, y^2, y^3)$, and use the explicit formulas: $dx^i = \frac{\partial x^i}{\partial y^j} dy^j$, and

$$(S^i{}_j) = \left(\frac{\partial x^i}{\partial y^j}\right) = \begin{pmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad (S^{-1})^j{}_k = \left(\frac{\partial y^j}{\partial x^k}\right) = \frac{1}{u^2 + v^2} \begin{pmatrix} u & v & 0 \\ -v & u & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The metric of the Minkowski 4-space in cylindric parabolic coordinates is given by

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(u^2 + v^2) & 0 & 0 \\ 0 & 0 & -(u^2 + v^2) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} ,$$

and the most simple tetrad to use is

$$e_{(k)}^{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (u^2 + v^2)^{-1/2} & 0 & 0 \\ 0 & 0 & (u^2 + v^2)^{-1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

To specify the Dirac equation in this tetrad, it is convenient to start with the general form of the equation in any orthogonal coordinate system [37]

$$\left[i\gamma^k \left(e^{\alpha}_{(k)}(y) \frac{\partial}{\partial y^{\alpha}} + \frac{1}{2} e^{\alpha}_{(k);\alpha}(y) \right) - M \right] \Psi(y) = 0 . \tag{3.1}$$

By using the known formula

$$A^{\alpha}_{;\alpha} = \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g} A^{\alpha}}{\partial y^{\alpha}}, \qquad g = \det(g_{\alpha\beta}) ,$$

we get

$$e^{\alpha}_{(0);\alpha} = 0 \; , \quad e^{\alpha}_{(3);\alpha} = 0 \; , \quad e^{\alpha}_{(1);\alpha}(y) = \frac{u}{(u^2+v^2)^{3/2}} \; , \quad e^{\alpha}_{(2);\alpha}(y) = \frac{v}{(u^2+v^2)^{3/2}} \; .$$

Therefore, (3.1) admits the equivalent form:

$$\left\{ i\gamma^0 \frac{\partial}{\partial t} + i\gamma^3 \frac{\partial}{\partial z} - M + \frac{i}{\sqrt{u^2+v^2}} \right.$$
$$\left. \times \left[\gamma^1 \left(\frac{\partial}{\partial u} + \frac{u}{2(u^2+v^2)} \right) + \gamma^2 \left(\frac{\partial}{\partial v} + \frac{v}{2(u^2+v^2)} \right) \right] \right\} \Psi(y) = 0 \; .$$

In spinor basis, this is written in 2-block form as follows:

$$\left\{ i\frac{\partial}{\partial t} + i\sigma^3 \frac{\partial}{\partial z} + \frac{i}{\sqrt{u^2+v^2}} \left[\sigma^1 \left(\frac{\partial}{\partial u} + \frac{u}{2(u^2+v^2)} \right) + \sigma^2 \left(\frac{\partial}{\partial v} + \frac{v}{2(u^2+v^2)} \right) \right] \right\} E = MH \; ,$$
$$\left\{ i\frac{\partial}{\partial t} - i\sigma^3 \frac{\partial}{\partial z} - \frac{i}{\sqrt{u^2+v^2}} \left[\sigma^1 \left(\frac{\partial}{\partial u} + \frac{u}{2(u^2+v^2)} \right) + \sigma^2 \left(\frac{\partial}{\partial v} + \frac{v}{2(u^2+v^2)} \right) \right] \right\} H = ME \; .$$

By using the substitutions

$$E(t,z,u,v) = e^{-i\epsilon t} e^{ikz} E(u,v), \qquad H(t,z,u,v) = e^{-i\epsilon t} e^{ikz} H(u,v) \; ,$$

we further obtain

$$\left\{ \epsilon - k\sigma^3 + \frac{i}{\sqrt{u^2+v^2}} \left[\sigma^1 \left(\frac{\partial}{\partial u} + \frac{u}{2(u^2+v^2)} \right) + \sigma^2 \left(\frac{\partial}{\partial v} + \frac{v}{2(u^2+v^2)} \right) \right] \right\} E = MH \; ,$$
$$\left\{ \epsilon + k\sigma^3 - \frac{i}{\sqrt{u^2+v^2}} \left[\sigma^1 \left(\frac{\partial}{\partial u} + \frac{u}{2(u^2+v^2)} \right) + \sigma^2 \left(\frac{\partial}{\partial v} + \frac{v}{2(u^2+v^2)} \right) \right] \right\} H = ME \; .$$

It is convenient to make the following substitutions for the following 2-component entities

$$E(u,v) = (u^2+v^2)^{-1/4} e(u,v) \; , \qquad H(u,v) = (u^2+v^2)^{-1/4} h(u,v),$$

which yields

$$\left[(\epsilon - k\sigma^3) + \frac{i}{\sqrt{u^2+v^2}} \left(\sigma^1 \frac{\partial}{\partial u} + \sigma^2 \frac{\partial}{\partial v} \right) \right] e(y) = Mh(y) \; ,$$

$$\left[(\epsilon + k\sigma^3) - \frac{i}{\sqrt{u^2+v^2}} \left(\sigma^1 \frac{\partial}{\partial u} + \sigma^2 \frac{\partial}{\partial v} \right) \right] h(y) = Me(y) \; . \tag{3.2}$$

The equations (3.2) are rather complicated, and to proceed with them we shall diagonalize the helicity operator $(\mathbf{sp})_0$. We shall translate it to cylindric coordinates and then translate it (see [37]) to the cylindric parabolic tetrad $(\mathbf{sp}) = S(\mathbf{sp})_0 S^{-1}$, where:

$$S = \left(\begin{array}{cc} B(y) & 0 \\ 0 & B(y) \end{array} \right), \; B(y) = \frac{1}{(u^2+v^2)^{1/4}} \left(\begin{array}{cc} \sqrt{u+iv} & 0 \\ 0 & \sqrt{u-iv} \end{array} \right).$$

In this way, we get

$$(\,\vec{\sigma}\vec{p}\,) = B(\,\vec{\sigma}\vec{p}\,)_0 B^{-1}$$

$$= -i \left\{ \sigma^3 \frac{\partial}{x^3} + \frac{1}{\sqrt{u^2+v^2}} \left[\sigma^1 \left(\frac{\partial}{\partial u} + \frac{u}{2(u^2+v^2)} \right) + \sigma^2 \left(\frac{\partial}{\partial v} + \frac{v}{2(u^2+v^2)} \right) \right] \right\} .$$

The explicit form of the eigenvalue equation for the 4-spinor wave function can be simplified by the following substitution

$$(3.3) \qquad \begin{aligned} (\,\vec{\sigma}\vec{p}\,)\; E(y) &= \lambda E(y) , \qquad & E(y) &= (u^2+v^2)^{-1/4} e(y) , \\ (\,\vec{\sigma}\vec{p}\,)\; H(y) &= \lambda H(y) , \qquad & H(y) &= (u^2+v^2)^{-1/4} h(y) ; \end{aligned}$$

so we obtain

$$(3.4) \qquad \begin{aligned} \left[\sigma^3 k - \frac{i}{\sqrt{u^2+v^2}} \left(\sigma^1 \frac{\partial}{\partial u} + \sigma^2 \frac{\partial}{\partial v} \right) \right] e(u,v) &= \lambda\, e(u,v) , \\ \left[\sigma^3 k - \frac{i}{\sqrt{u^2+v^2}} \left(\sigma^1 \frac{\partial}{\partial u} + \sigma^2 \frac{\partial}{\partial v} \right) \right] h(u,v) &= \lambda\, h(u,v) . \end{aligned}$$

Let us turn back to eqs. (3.2). With the help of (3.4), from (3.2) we infer the algebraic system

$$(\epsilon - \lambda) e(u,v) = M h(u,v) , \quad (\epsilon + \lambda) h(u,v) = M e(u,v) ,$$

which leads to the following solutions

$$(3.5) \qquad \begin{aligned} &\lambda = \pm \sqrt{\epsilon^2 - M^2} , \quad \mu = \frac{(\epsilon - \lambda)}{M} = \frac{M}{(\epsilon + \lambda)} , \\ &h_j = \mu\, e_j , \qquad \Psi = \frac{e^{-i\epsilon t} e^{ikz}}{\sqrt{u^2+v^2}} \begin{pmatrix} e_1(u,v) \\ e_2(u,v) \\ \mu e_1(u,v) \\ \mu e_2(u,v) \end{pmatrix} . \end{aligned}$$

Thus, we have only two independent variables, $e_1(u,v)$ and $e_2(u,v)$, which obey the following equations

$$(3.6) \qquad \begin{aligned} (\lambda - k)\; e_1 + \frac{i}{\sqrt{u^2+v^2}} (\partial_u - i\partial_v)\; e_2 = 0 , \\ (\lambda + k)\; e_2 + \frac{i}{\sqrt{u^2+v^2}} (\partial_u + i\partial_v)\; e_1 = 0 . \end{aligned}$$

Using the substitutions $e_1 = \sqrt{u+iv}\; F$, $e_2 = \sqrt{u-iv}\; G$, and taking into account the identities

$$\begin{aligned} (\partial_u + i\partial_v)\, \sqrt{u+iv}\; F = \sqrt{u+iv}\; (\partial_u + i\partial_v)\; F , \\ (\partial_u - i\partial_v)\, \sqrt{u-iv}\; G = \sqrt{u-iv}\; (\partial_u - i\partial_v)\; G , \end{aligned}$$

we simplify (3.6) to the simpler form

$$(3.7) \qquad \begin{aligned} (\lambda - k)\; (u+iv)\; F + i\; (\partial_u - i\partial_v)\; G = 0 , \\ (\lambda + k)\; (u-iv)\; G + i\; (\partial_u + i\partial_v)\; F = 0 . \end{aligned}$$

These equations are associated with the wave functions of the form

$$(3.8) \qquad \Psi = \frac{e^{-i\epsilon t} e^{ikz}}{\sqrt{u^2+v^2}} \begin{pmatrix} \sqrt{u+iv}\; F(u,v) \\ \sqrt{u-iv}\; G(u,v) \\ \mu\, \sqrt{u+iv}\; F(u,v) \\ \mu\, \sqrt{u-iv}\; G(u,v) \end{pmatrix} ;$$

the transition of (3.8) to Cartesian tetrads is made according to the rule

$$\Psi_0(x) = \frac{1}{(u^2+v^2)^{1/4}} \begin{pmatrix} \sqrt{u-iv} & 0 & 0 & 0 \\ 0 & \sqrt{u+iv} & 0 & 0 \\ 0 & 0 & \sqrt{u-iv} & 0 \\ 0 & 0 & 0 & \sqrt{u+iv} \end{pmatrix} \Psi(x) \; ,$$

whence it follows

$$\Psi_0(x) = \frac{e^{-i\epsilon t} e^{ikz}}{(u^2+v^2)^{1/4}} \begin{pmatrix} F(u,v) \\ G(u,v) \\ \mu\, F(u,v) \\ \mu\, G(u,v) \end{pmatrix} .$$

Note that after introducing the two complex variables

$$u + iv = x, \; u - iv = y \; , \quad (\partial_u + i\partial_v) = 2\partial_y, \; (\partial_u - i\partial_v) = 2\partial_x \; ,$$

the system (3.7) takes the form

$$(\lambda - k)\; x\; F + 2i\partial_x\; G = 0 \; , \quad (\lambda + k)\; y\; G + 2i\partial_y\; F = 0.$$

By eliminating the function G, we get

$$G = -\frac{2i}{(\lambda+k)}\; \frac{1}{y}\partial_y\; F \; , \quad (\lambda - k)\; x\; F - 2i\partial_x \frac{2i}{(\lambda+k)}\; \frac{1}{y}\partial_y\; F \; . = 0 \; .$$

Searching for solutions $F(x,y)$ of the form $F = X(x)Y(y)$, we infer:

$$-\frac{\lambda^2 - k^2}{4} = \frac{X'}{X}\frac{1}{x}\; \frac{Y'}{Y}\frac{1}{y} \quad \Longrightarrow \quad \frac{X'}{X}\frac{1}{x} = \alpha, \quad \frac{Y'}{Y}\frac{1}{y} = \beta, \quad \alpha\beta = -\frac{\lambda^2-k^2}{4} \; .$$

The separated equations can be readily integrated,

$$X = C_1 e^{\alpha x^2/2}, \qquad Y = C_2 e^{\beta y^2/2} \; ;$$

to which there correspond a quite definite function $G(x,y)$:

$$G = -\frac{2i}{\lambda+k}\frac{1}{y}\frac{\partial}{\partial y} XY = -\frac{2i}{\lambda+k} C_1 e^{\alpha x^2/2} \beta C_2 e^{\beta y^2/2} \; .$$

In fact, the constructed solutions represent the well known *plane waves*. Indeed, by assuming that $x_1 = (u^2 - v^2)/2, \; x_2 = uv$, we get:

$$X(x)Y(y) \sim e^{\alpha(u^2-v^2+2iub)/2} e^{\beta(u^2-v^2-2iub)/2} = e^{(\alpha+\beta)x_1} e^{(\alpha-\beta)x_2} = e^{ik_1x_1} e^{ik_2x_2} \; ,$$

whence

$$\alpha + \beta = ik_1, \quad \alpha - \beta = k_2 \quad \Longrightarrow$$

$$\alpha = \frac{ik_1 + k_2}{2}, \quad \beta = \frac{ik_1 - k_2}{2}, \quad \alpha\beta = -\frac{k_1^2 + k_2^2}{4} = -\frac{\lambda^2 - k_3^2}{4} \; ;$$

so we get the claimed identity $\epsilon^2 = M^2 + k_1^2 + k_2^2 + k_3^2$.

Let us turn again to the system (3.7). First, by eliminating the function G from (3.7), we derive

$$G = -\frac{i}{\lambda+k}\; \frac{1}{u-iv} \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right) F \; ,$$

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + (\lambda^2 - k^2)(u^2+v^2) \right] F(u,v) = 0 \; . \tag{3.9}$$

Alternatively, by eliminating the function F from (3.7), we obtain

$$F = -\frac{i}{\lambda - k}\ \frac{1}{u+iv}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) G\ ,$$

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + (\lambda^2 - k^2)(u^2+v^2)\right] G((u,v) = 0\ . \tag{3.10}$$

The second order differential equation is the same for both cases, (3.9) and (3.10). For definiteness, let us examine the case (3.9), starting with the equation

$$\left[\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} + \sigma^2(u^2+v^2)\right] F(u,v) = 0\ , \quad \sigma^2 = (\lambda^2 - k^2) = \epsilon^2 - M^2 - k^2 > 0\ . \tag{3.11}$$

Let $F(u,v) = U(u)V(v)$; then, instead of (3.11), we obtain

$$\left(\frac{1}{U}\ \frac{d^2U}{du^2} + \sigma^2 u^2\right) + \left(\frac{1}{V}\ \frac{d^2V}{dv^2} + \sigma^2 v^2\right) = 0\ .$$

By introducing a separating constant Λ, we get two equations in variables u and v, respectively:

$$\frac{d^2U}{du^2} + (\ \sigma^2\ u^2 + \Lambda\ )\ U = 0\ , \qquad \frac{d^2V}{dv^2} + (\ \sigma^2\ v^2 - \Lambda\ )\ V = 0\ .$$

For functions U, V we have similar equations, which differ only by the sign of the separation constant Λ. For definiteness, let us consider the function $U(u)$ given by

$$\frac{d^2U}{du^2} + (\ \sigma^2\ u^2 + \Lambda)\ U = 0\ .$$

Further, rewriting in terms of the variable $bu^2 = z$, this takes the form:

$$\left(z\frac{d^2}{dz^2} + \frac{1}{2}\frac{d}{dz} + \sigma^2\frac{z}{4b^2} + \frac{\Lambda}{4b}\right) U = 0\ .$$

Separating a simple factor, $U(z) = e^{-z/2} f(z)$, we derive

$$zf'' - zf' + \frac{1}{4}zf + \frac{1}{2}\left(f' - \frac{1}{2}f\right) + \sigma^2\frac{z}{4b^2}f + \frac{\Lambda}{4b}f = 0\ .$$

Let $b^2 = -\sigma^2$ and $b = +i\sigma,\ \sigma > 0$; then get

$$z\frac{d^2 f}{dz^2} + \left(\frac{1}{2} - z\right)\frac{df}{dz} - \frac{1 + i\Lambda/\sigma}{4} f = 0\ ,$$

which is in fact a confluent hypergeometric equation

$$z\frac{d^2 f}{dz^2} + (c - z)\frac{df}{dz} - af = 0\ , \quad c = \frac{1}{2}\ , \quad a = \frac{1 + i\Lambda/\sigma}{4}\ .$$

We further use two independent solutions

$$U_- = e^{-z/2}\Phi\left(a, \frac{1}{2}; z\right)\ , \quad U_+ = e^{-z/2}\sqrt{z}\ \Phi\left(a + \frac{1}{2}, \frac{3}{2}; z\right)\ .$$

Analogous results for (3.10) are obtained by the formal change $\Lambda \rightsquigarrow -\Lambda$:

$$V(v) = e^{-y/2} g(y), \quad y = +i\sigma v^2 \,,$$
$$\frac{d^2 g}{dy^2} + \left(\frac{1}{2} - y\right) \frac{dg}{dy} - \frac{1 - i\Lambda/\sigma}{4} \, g = 0 \,,$$
$$c' = \frac{1}{2} = c \,, \quad a' = \frac{1 - i\Lambda/\sigma}{4} = \frac{1}{2} - a \,; \tag{3.12}$$

the two independent solutions are written as follows

$$V_-(y) = e^{-y/2} \Phi\left(\frac{1}{2} - a, \frac{1}{2}; y\right) \,, \quad V_+(y) = e^{-y/2} \sqrt{y}\, \Phi\left(1 - a, \frac{3}{2}; y\right) \,.$$

Thus, the total functions $F(u,v)$ and $G(u,v)$ will be constructed in terms of the following solutions

$$U_-(u) = e^{-i\sigma u^2/2} \Phi\left(a, \tfrac{1}{2}; i\sigma u^2\right) \,, \qquad U_+(u) = e^{-i\sigma u^2/2} u \Phi\left(a + \tfrac{1}{2}, \tfrac{3}{2}; i\sigma u^2\right) \,,$$
$$V_-(v) = e^{-i\sigma v^2/2} \Phi\left(\tfrac{1}{2} - a, \tfrac{1}{2}; i\sigma v^2\right) \,, \quad V_+(v) = e^{-i\sigma v^2/2} v \Phi\left(1 - a, \tfrac{3}{2}; i\sigma v^2\right) \,;$$

$$U'_-(u) = e^{-i\sigma u^2/2} \Phi\left(a', \tfrac{1}{2}; i\sigma u^2\right) \,, \qquad U'_+(u) = e^{-i\sigma u^2/2} u \Phi\left(a' + \tfrac{1}{2}, \tfrac{3}{2}; i\sigma u^2\right) \,,$$
$$V'_-(v) = e^{-i\sigma v^2/2} \Phi\left(\tfrac{1}{2} - a', \tfrac{1}{2}; i\sigma v^2\right) \,, \quad V'_+(v) = e^{-i\sigma v^2/2} v \Phi\left(1 - a', \tfrac{3}{2}; i\sigma v^2\right) \,.$$

Now, we should turn to the first order equation in (3.9)

$$i(\lambda + k) G = \frac{1}{u - iv} \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v}\right) F;$$

by using it and starting with any known $F(u,v)$, we can find an explicit form for the corresponding function $G(u,v)$.

3.2 Continuity properties of the Dirac solutions and spinor space structure

Let us discuss in more detail some subtleties of the properties of the solutions and of their relations to the stricture of spatial, vector and spinorial models.

While considering the relation between parabolic and Cartesian coordinates:

$$\left\{ \begin{array}{l} x_1 = \frac{u^2 - v^2}{2}, \\ x_2 = u\, v \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} \underline{v = 0\ :} \; x_1 = +\frac{u^2}{2} \geq 0, \; x_2 = 0 \,, \\ \underline{u = 0\ :} \; x = -\frac{v^2}{2} \leq 0, \; y = 0, \end{array} \right.$$

we noted existence of two special regions (see. Fig. 5).

Now let us consider which restrictions for the Dirac wave functions Ψ follow from the requirement of single-valuedness.

Let us consider the variant $(--)$:

$$F_{--}(u,v) = U_-(u) V_-(v) = e^{-i\sigma u^2/2} \Theta_1 e^{-i\sigma v^2/2} \Theta_2,$$

$$
\begin{aligned}
i(\lambda+k)G &= \tfrac{1}{(u-iv)}\left(\tfrac{\partial}{\partial u}+i\tfrac{\partial}{\partial v}\right)U_-(u)V_-(v) = \\
&= \tfrac{1}{(u-iv)}\left(\tfrac{\partial}{\partial u}+i\tfrac{\partial}{\partial v}\right)e^{-i\sigma u^2/2}\Theta_1 e^{-i\sigma v^2/2}\Theta_2 =
\end{aligned}
$$

$$
= \frac{1}{(u-iv)}e^{-i\sigma u^2/2}e^{-i\sigma v^2/2}\left[\left(-i\sigma\ u\ \Theta_1 + 2i\sigma\ u\ \frac{d}{d(i\sigma u^2)}\ \Theta_1\right)\Theta_2 + \right.
$$

$$
\left. +i\Theta_1\left(-i\sigma\ v\ \Theta_2 + 2i\sigma\ v\ \frac{d}{d(i\sigma v^2)}\Theta_2\right)\right].
$$

where $\Theta_1 = \Phi\left(a, \frac{1}{2}; i\sigma u^2\right)$ and $\Theta_2 = \Phi\left(\frac{1}{2}-a, \frac{1}{2}; i\sigma v^2\right)$.
We note that related pair of functions, $F_{--} \rightsquigarrow G_{--}(u,v)$ are single-valued in the special region:

$$
\begin{aligned}
&F_{--}(u=0,v) = +F_{--}(u=0,-v) \neq 0\ , \quad F_{--}(u,v=0) = +G_{--}(-u,v=0) \neq 0\ ; \\
&G_{--}(u=0,v) = +G_{--}(u=0,-v) \neq 0\ , \quad G_{--}(u,v=0) = +G_{--}(-u,v=0) \neq 0\ .
\end{aligned}
$$

Let us consider the variant (++):

$$
F_{++}(u,v) = U_+(u)V_+(v) = e^{-i\sigma u^2/2}\ u\ \Theta_1 e^{-i\sigma v^2/2}\ v\ \Theta_2\ ,
$$

$$
\begin{aligned}
i(\lambda+k)G &= \tfrac{1}{(u-iv)}\left(\tfrac{\partial}{\partial u}+i\tfrac{\partial}{\partial v}\right)U_+(u)V_+(v) = \\
&= \tfrac{1}{(u-iv)}\left(\tfrac{\partial}{\partial u}+i\tfrac{\partial}{\partial v}\right)e^{-i\sigma u^2/2}\ u\ \Theta_1 e^{-i\sigma v^2/2}\ v\ \Theta_2 =
\end{aligned}
$$

$$
= e^{-i\sigma u^2/2}e^{-i\sigma v^2/2}\frac{1}{(u-iv)}\left[\left((-i\sigma\ u^2+1)\ \Theta_1 + 2i\sigma\ u^2\ \frac{d}{d(i\sigma u^2)}\ \Theta_1\right)\ v\ \Theta_2 + \right.
$$

$$
\left. +iu\Theta_1\left((-i\sigma\ v^2+1)\ \Theta_2 + 2i\sigma\ v^2\ \frac{d}{d(i\sigma v^2)}\Phi\left(1-a, \frac{3}{2}; i\sigma v^2\right)\right)\right],
$$

where $\Theta_1 = \Phi\left(a+\frac{1}{2}, \frac{3}{2}; i\sigma u^2\right)$ and $\Theta_2 = \Phi\left(1-a, \frac{3}{2}; i\sigma v^2\right)$.
The related functions, $F_{++} \rightsquigarrow G_{++}(u,v)$ are single-valued in the spacial region:

$$
\begin{aligned}
&F_{++}(u=0,+v) = +F_{++}(u=0,-v) = 0\ , \quad F_{++}(u,v=0) = +F_{++}(-u,v=0) = 0\ ; \\
&G_{++}(u=0,+v) = +G_{++}(u=0,-v) \neq 0\ , \quad G_{++}(u,v=0) = +G_{++}(-u,v=0) \neq 0\ .
\end{aligned}
$$

Let us consider the variant (−+):

$$
F_{-+}(u,v) = U_-(u)V_+(v) = e^{-i\sigma u^2/2}\Theta_1 e^{-i\sigma v^2/2}\ v\ \Phi\left(1-a, \frac{3}{2}; i\sigma v^2\right)\ ,
$$

$$
\begin{aligned}
i(\lambda+k)G &= \tfrac{1}{(u-iv)}\left(\tfrac{\partial}{\partial u}+i\tfrac{\partial}{\partial v}\right)U_-(u)V_+(v) = \\
&= \tfrac{1}{(u-iv)}\left(\tfrac{\partial}{\partial u}+i\tfrac{\partial}{\partial v}\right)e^{-i\sigma u^2/2}\Theta_1 e^{-i\sigma v^2/2}\ v\ \Theta_2 =
\end{aligned}
$$

$$
= e^{-i\sigma u^2/2}e^{-i\sigma v^2/2}\frac{1}{(u-iv)}\left[\left(-i\sigma\ u\ \Theta_1 + 2i\sigma\ u\ \frac{d}{d(i\sigma u^2)}\ \Theta_1\right) v\ \Theta_2 + \right.
$$

$$
\left. +i\Theta_1\left((-i\sigma\ v^2+1)\ \Theta_2 + 2i\sigma\ v^2\ \frac{d}{d(i\sigma v^2)}\Theta_2\right)\right].
$$

where $\Theta_1 = \Phi\left(a, \frac{1}{2}; i\sigma u^2\right)$ and $\Theta_2 = \Phi\left(1-a, \frac{3}{2}; i\sigma v^2\right)$.
The related functions, $F_{-+} \rightsquigarrow G_{-+}(u,v)$, are double-valued in the spacial region:

$$
\begin{aligned}
&F_{-+}(u=0,+v) = -F_{-+}(u=0,-v) \neq 0\ , \quad F_{-+}(u,v=0) = -F_{-+}(-u,v=0) = 0\ , \\
&G_{-+}(u=0,+v) = -G_{-+}(u=0,-v) \neq 0\ , \quad G_{-+}(u,v=0) = -G_{-+}(-u,v=0) \neq 0\ .
\end{aligned}
$$

Let us consider the variant $(+-)$:

$$F_{+-}(u,v) = U_+(u)V_-(v) = e^{-i\sigma u^2/2} u\Theta_1 e^{-i\sigma v^2/2}\Phi\left(\frac{1}{2}-a,\frac{1}{2};i\sigma v^2\right) ,$$

$$\begin{aligned} i(\lambda+k)G &= \tfrac{1}{(u-iv)}\left(\tfrac{\partial}{\partial u}+i\tfrac{\partial}{\partial v}\right)U_+(u)V_-(v) = \\ &= \tfrac{1}{(u-iv)}\left(\tfrac{\partial}{\partial u}+i\tfrac{\partial}{\partial v}\right)e^{-i\sigma u^2/2}u\Theta_1 e^{-i\sigma v^2/2}\Theta_2 = \end{aligned}$$

$$= e^{-i\sigma u^2/2}e^{-i\sigma v^2/2}\frac{1}{(u-iv)}\left[\left((-i\sigma\, u^2+1)\,\Theta_1 + 2i\sigma\, u^2\,\frac{d}{d(i\sigma u^2)}\,\Theta_1\right)\Theta_2 \right.$$

$$\left. + iu\Theta_1\left(-i\sigma\, v\,\Theta_2 + 2i\sigma\, v\,\frac{d}{d(i\sigma v^2)}\Theta_2\right)\right] .$$

where $\Theta_1 = \Phi\left(a+\frac{1}{2},\frac{3}{2};i\sigma u^2\right)$ and $\Theta_2 = \Phi\left(\frac{1}{2}-a,\frac{1}{2};i\sigma v^2\right)$.

The related functions, $F_{+-} \rightsquigarrow G_{+-}(u,v)$, are double-valued in the spacial region:

$$F_{+-}(u=0,+v) = -F_{+-}(u=0,-v) = 0 , \quad F_{+-}(+u,v=0) = -F_{+-}(-u,v=0) \neq 0 ;$$

$$G_{+-}(u=0,+v) = -G_{+-}(u=0,-v) \neq 0 , \quad G_{+-}(+u,v=0) = -G_{+-}(-u,v=0) \neq 0 .$$

When using the spinor space model, the two sets of couples (u,v) and $(-u,v)$ (or, similarly, the sets of couples (u,v) and $(u,-v)$) represent different geometrical points, so the requirement of single valuedness in the case of a spinor space does not imply that the values of the wave functions must be equal at the points (u,v) and $(-u,v)$:

$$\Psi(u,v) = \Psi((x_1,x_2)^{(1)}) \neq \Psi(-u,v) = \Psi((x_1,x_2)^{(2)}) .$$

Therefore, we conclude that the solutions $F(u,v), G(u,v)$ of the types $(--)$ and $(++)$ are single-valued in the spaces with vector structure, whereas the solutions $F(u,v), G(u,v)$ of the types $(-+)$ and $(+-)$ are not single-valued in spaces with vector structure, so the solutions of these two types $(-+)$ and $(+-)$ must be discarded. However, the types of solutions $(-+)$ and $(+-)$ are valid in the space with spinor structure.

4 Conclusions

The study of the fermion parity problem by means of investigating possible single-valued representations of spinor coverings of the extended Lorentz group shows that P-parity and T-parity for a fermion do not exist as separate concepts. Instead of this, only some unified concept of (PT)-parity can be determined in group-theoretical terms.

The extension procedure which describes a space with spinor structure is performed by relying on cylindrical parabolic coordinates. This is done through the expansion of the region, $G(t,u,v,z) \rightsquigarrow \tilde{G}(t,u,v,z)$, so that instead of the half plane $(u, v > 0)$ now the entire plane (u,v) should be used, accompanied with new identification rules for the boundary points. In the Cartesian picture, this procedure corresponds to taking the two-sheet surface $(x',y') \oplus (x'',y'')$ instead of the one-sheet surface (x,y).

The solutions of the Klein–Fock–Gordon and Schrödinger equations are constructed in terms of parabolic cylindric functions. Four types of solutions are possible: Ψ_{++}, Ψ_{--};

Ψ_{+-}, Ψ_{-+}. The first two ones, Ψ_{++} and Ψ_{--}, provide us with single-valued functions of the vector space points, whereas the last two, Ψ_{+-} and Ψ_{-+}, have discontinuities in the framework of vector spaces, and therefore they must be discarded in this model. All the four types of functions are continuous while regarded in the spinor space. It is established that all solutions Ψ_{++}, Ψ_{--}, Ψ_{+-} and Ψ_{-+}, are orthogonal to each other, provided that integration is done over the extended region of integration which covers (corresponds to) the spinor space.

Similar results are obtained for the Dirac equation. The solutions of the type $(--), (++)$ are single-valued in the space with vector structure, whereas the solutions of the types $(-+), (+-)$ are not single-valued in the space with vector structure, so the solutions of types $(-+)$ and $(+-)$ must be discarded. However, they must are valid solutions in the space with spinor structure.

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