On L^1 convergence of barycentric sequences of empirical measures

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Abstract. In a previous article we proved a pointwise ergodic theorem for or sequences of barycentres of empirical measures which are defined from the action of Fuchsian groups and for L^2 maps valuated in $CAT(0)$ spaces. In this note we extend this result to the L^1-setting for maps valuated in nonpositively curved spaces.

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1 Introduction

The extension of the ergodic theorem to action groups as dynamics and to maps valuated in more general spaces, like $CAT(0)$ spaces or nonpositively curved spaces, is an issue of interest in the area of Ergodic Theory and Dynamical Systems.

The usual ergodic averages for measure spaces (X, ν) and maps $T : X \to X$, $\varphi : X \to \mathbb{R}$ are defined as

$$\mathcal{S}_{N,\varphi}(x) := \frac{1}{N}\sum_{n=0}^{N-1} \varphi\left(T^n(x)\right).$$

For maps valuated in more general spaces a notion of ergodic average can be introduced. Let $\varphi : X \to Y$, with Y not necessarily $\mathbb{R}$ and let us consider the following empirical measures

$$\mathcal{E}_{N,\varphi}(x) := \frac{1}{N}\sum_{n=0}^{N-1} \delta_{\varphi(T^n(x))}, \tag{1.1}$$

where δ is the point mass Dirac measure. For action groups as dynamics, a more general class of empirical measures can be defined as we describe below.

In [1] Austin considered maps defined on a probability space (X, ν) and valuated on a $CAT(0)$ space Y. If μ is a measure on Y with second finite moment , i.e.,

$$\int_Y d(y,z)^2 d\mu(z) < \infty \quad \text{ for any } y \in Y,$$

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then a *barycenter* $bar\,(\mu)$ can be assigned to μ in the following way: for any $\mu \in \mathcal{M}_2\,(Y)$, there is an unique $y \in Y$ which minimizes $\int_Y d(y,z)^2 d\mu(z)$[4], thus is defined $bar\,(\mu) = y$. The function $\mu \to bar\,(\mu)$ is called the *barycenter map.* The sequence of the barycenter of empirical measures can be considered as the analogue of the usual ergodic averages for real valuated maps. Austin established the convergence of the barycenter of the sequences of empirical measures from the Lindenstrauss ergodic theorem for real valuated maps[6]. Those empirical measures were defined from amenable action groups and maps $\varphi : X \to Y$ satisfying $\int_X d(\varphi(x), y)^2 d\nu(x) < \infty$, for any $y \in Y$. The space of such a maps is denoted by $L^2(X, Y, \nu)$.

The barycentric convergence of sequences of empirical measures from amenable action groups in L^1 spaces was studied by Navas[8]. He extended the result by Austin to L^1 maps valuated in nonpositively curved spaces and with a new definition of barycenter adapted to the more general setting.

In [7] we have considered functions $\varphi \in L^2(X, Y, \nu)$, with (X, ν) a probability space and Y a complete, separable, $CAT(0)$ space, and Fuchsian groups Γ acting on the hyperbolic disc H^2. For a Fuchsian group Γ acting on X by measure preserving actions $T_\gamma(x) = \gamma x$, $\gamma \in \Gamma$, we have defined the empirical measures

$$\mathcal{E}_{N,}\varphi(x) := \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{cardS(n)} \sum_{\gamma \in S(n)} \delta_{\varphi(T_\gamma(x))}, \tag{1.2}$$

where $S(n)$ denotes the sphere of radius n in the word metric in Γ and δ is the point mass measure.

In the setting of [7] we have not needed, unlike in [1] or [8], a topological structure on the group and the existence of Følner sequences. We have proved in [7] the pointwise convergence of the barycenter of the empirical measures $\mathcal{E}_{N,}\varphi(x)$ in $L^2(X, Y, \nu)$. We based on an ergodic theorem by Bufetov and Series for real valuated maps [3]. We call the real case when $Y = \mathbf{R}$.

One of the objectives of this note is to extend the result of [7] to the L^1 setting. More precisely let (Y, d) be a complete, separable metric space with nonpositive curvature in the sense of Busemann. Let (X, ν) be a probability space and let $\varphi : X \to Y$ be a map satisfying $\int_X d(\varphi(x), y) d\nu(x) < \infty$, for any $y \in Y$. The space of maps with this property is denoted by $L^1(X, Y, \nu)$. Thus, we want to prove that the sequence $\{bar(\mathcal{E}_{N,}\varphi(x))\}$ converges in $L^1(X, Y, \nu)$ for any $\varphi \in L^1(X, Y, \nu)$. Like in our previous article [7], we will not need herein the existence of Følner sequences and conditions of amenability in the action group. Although the L^2 setting is itself interesting, the L^1 spaces constitute the most important framework for the ergodic theorems.

2 Preliminaries

A space (Y, d) is nonpositively curved in the sense of Busemann if for any pair of points $x_1, y_1 \in Y$ and $x_2, y_2 \in Y$ the corresponding, unique, midpoints m_1, m_2 satisfy

$$d\,(m_1, m_2) \le \frac{d\,(x_1, x_2)}{2} + \frac{d\,(y_1, y_2)}{2}. \tag{2.1}$$

This is equivalent to say that the distance function along geodesics is convex.

Let us recall that by $L^1(X,Y,\nu)$ is denoted the space of maps $\varphi : X \to Y$ with the property $\int_X d(\varphi(x), y) d\nu(x) < \infty$, for any $y \in Y$. The space $L^1(X,Y,\nu)$ is endowed with the metric

$$d_1(\varphi,\psi) := \sqrt{\int_X d(\varphi(x), \psi(x)) \, d\nu(x)}.$$

A measure μ on Y has *finite first moment* if

$$\int_Y d(y,z) d\mu(z) < \infty, \text{ for any } y \in Y.$$

Let us denote by $\mathcal{M}_1(Y)$ denotes the set of measures μ in Y with first finite moment.

A *coupling* of two measures $\mu_1, \mu_2 \in \mathcal{M}(Y)$ is a measure $m \in \mathcal{M}(Y \times Y)$ which on the first and second factor projects on μ_1, μ_2 respectively. The $1-$*Wasserstein metric* in $\mathcal{M}(Y)$ is defined as $W_1(\mu_1,\mu_2) = \inf_{\substack{m \text{ coupling of} \\ \mu_1,\mu_2 \in \mathcal{M}(Y)}} \sqrt{\int_{Y\times Y} d(y,z)\, dm(y,z)}$.

Now we display the definition of barycenter map given in [8], for more details about the constructions see that article. When Y is a Banach space the barycenter of the measure $\frac{1}{N}\sum_{n=0}^{N-1} \delta_{x_i}$ is the Dirac measure concentrated in $\frac{1}{N}\sum_{n=0}^{N-1} x_i$. For a measure $\mu \in \mathcal{M}_1(Y)$ the barycenter of μ is defined as $bar(\mu) = \int y d\mu(y)$.

For a nonpositively curved spaces the concept is constructed as follows. The first step is to define the barycenter of any finite family of points $(y_1, y_2, ..., y_n)$, for $n = 1$ is set $bar_1(x) = x$. For $n = 2$ is defined $bar_2(x,y) = m$, the midpoint between x and y. Let us assume that $bar_{n-1}\ (y_1, y_2, ..., y_{n-1})$ is constructed, then $bar_n(y_1, y_2, ..., y_n)$ is defined in the following way, start with $(y_1, y_2, ..., y_n) = \left(y_1^{(0)}, y_2^{(0)}, ..., y_n^{(0)}\right)$ and replace each y_i by $bar_{n-1}\ (y_1, y_2, ..., y_{i-1}, y_{i+1,...,}y_n)$, which is defined by the inductive hypothesis. Thus results a set $\left(y_1^{(1)}, y_2^{(1)}, ..., y_n^{(1)}\right)$ which applying the same procedure leads to set $\left(y_1^{(2)}, y_2^{(2)}, ..., y_n^{(2)}\right)$. Continuing with the procedure is obtained a Cauchy sequence $\left\{\left(y_1^{(k)}, y_2^{(k)}, ..., y_n^{(k)}\right)\right\}_{k\in\mathbf{N}}$ whose limit is denoted by $(\overline{y_1}, \overline{y_2}, ..., \overline{y_n})$. Then is defined $bar_n\ (y_1, y_2, ...y_n) = (\overline{y_1}, \overline{y_2}, ..., \overline{y_n})$. To prove the convergence in [8] the fact that

$$d\left(bar_n\ (y_1, y_2, ..., y_n), bar_n\ (z_1, z_2, ..., z_n)\right) \le \frac{1}{n}\sum_{i=1}^{n} d(y_i, z_i).$$

is used.

Let $Q = (y_1, y_2, ..., y_n)$ be an arbitrary family of points in Y and let

$$Q^k = (y_1, y_2, ..., y_n, y_1, y_2, ..., y_n, ..., y_1, y_2, ..., y_n),$$

where there are $k-$blocks in Q^k. The sequence $\left\{bar_{nk}\left(Q^k\right)\right\}$ is a Cauchy sequence [8], so is defined

$$bar\left(\frac{1}{N}\sum_{n=0}^{N-1}\delta_{y_i}\right) := \text{limit point of } \left\{bar_{nk}\left(Q^k\right)\right\}.$$

It was proved in [8] that

$$d\left(bar\left(\frac{1}{N}\sum_{n=0}^{N-1}\delta_{y_i}\right),\ bar\left(\frac{1}{N}\sum_{n=0}^{N-1}\delta_{z_i}\right)\right)\leq\frac{1}{n}\min_{\sigma\in S_n}\sum_{i=1}^{n}d\left(y_i,z_{\sigma(i)}\right),$$

where S_n is the group of permutations of $n-$elements. Besides by the construction $bar\left(\frac{1}{N}\sum_{n=0}^{N-1}\delta_{y_i}\right)$ is the convex closure of the set $\{y_1,y_2,...,y_n\}$. The diameter of the convex closure of the set $\{y_1,y_2,...,y_n\}$ equals the diameter of $\{y_1,y_2,...,y_n\}$ [8].

Let $\mathcal{M}_{\mathbf{Q}}(Y)$ be the set of measures on Y whose atoms have rational mass, now by the above construction is defined an application $bar:\mathcal{M}_{\mathbf{Q}}(Y)\rightarrow Y$. This map is $1-$Lipschitz as is controlled by the Wasserstein metric

$$d\left(bar\left(\mu_1\right),bar\left(\mu_2\right)\right)\leq W_1\left(\mu_1,\mu_2\right).$$

This important result was initially proved by Sturn [12] and extended by Navas to Busemann spaces.

Since Y is separable, $\mathcal{M}_{\mathbf{Q}}(Y)$ is W_1-dense in $\mathcal{M}_1(Y)$ the map *bar* is extended to a function $bar:\mathcal{M}_1(Y)\rightarrow Y$ and it is called the *barycenter map.* This function replaces the ergodic averages in the real case.

For real valuated maps Bufetov and Series[3] proved the ergodic convergence of surface groups. This result was obtained from a previous result by Bufetov[2] about convergence of Cesàro averages. Let Γ be the fundamental group of a surface of genus $g\geq 2$ acting on probability space (X,ν), the length of $\gamma\in\Gamma$,denoted $|\gamma|$, is the minimal number of generators needed to represent γ. Let $S(n)$ be the sphere of radius n in this metric, i.e.,

$$S(n)=\{\gamma:|\gamma|=n\}.$$

For $\gamma\in\Gamma$, by T_γ is denoted the transformation on X given by $T_\gamma\ (x)=\gamma x$. Let $\varphi\ \in L^1(X,\nu)$, in [3] was established that the ergodic average

$$\mathcal{S}_{N,}\varphi(x):=\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{cardS(n)}\sum_{\gamma\in S(n)}\varphi\left(T_\gamma(x)\right),$$

converges for $\nu-a,e,\ x\in X$. When Γ acts ergodically on X holds:

$$\mathcal{S}_{N,}\varphi(x)\rightarrow_N\int\varphi d\nu.$$

Actually the Bufetov and Series ergodic theorem can be applied for a wide class of finitely generated Fuchsian groups. For the class of groups for which the real ergodic convergence of [3] is valid we shall consider the empirical measures

$$\mathcal{E}_{N,}\varphi(x):=\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{cardS(n)}\sum_{\gamma\in S(n)}\delta_{\varphi(T_\gamma(x))}.$$

where δ is the point mass Dirac measure. Thus the measure of $E\subset Y$ is

$$\mathcal{E}_{N,}\varphi(x)\left(E\right):=\frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{cardS(n)}\times card\left\{\gamma\in S(n):\varphi\left(T_\gamma(x)\right)\in E\right\}.$$

Recall that our aim is to prove the pointwise L^1convergence of the sequence $\{bar(\mathcal{E}_{N,}\varphi(x))\}$.

3 L^1 barycentric convergence

Before stating and proving our result, we must specify the class of action groups to be considered. As we earlier mentioned the groups should be those for which the Bufetov and Series ergodic theorem works. They are non-elementary Fuchsian groups acting on the hyperbolic disc H^2 with a symmetric set of generators S, and such that the Series coding can be done. For details about Series coding see[11]. By the Series coding, a partition $\mathcal{I}$ of the boundary of the hyperbolic disc H^2 in intervals can be defined, which results a Markov partition. Then a transition matrix $(A_{I,J})_{I,J\in\mathcal{I}}$ can be obtained and a Markov symbolic space given by the sequences $I_{i_0}...I_{i_{n-1}}$, allowed by the matrix A. A matrix $A=(a_{i,j})$, $a_{i,j}\geq 0$, is irreducible if for any i,j there is a $n>0$ such that $a_{i,j}^{(n)}>0$, where $a_{i,j}^{(n)}$ is the $i,j-$entry of A^n. A matrix A is strictly irreducible if A^tA is irreducible (A^t means the transpose of A). Let $I,J\in\mathcal{I}$, and the equivalence relation $I\sim J$ if $f(I)\cap f(J)\neq\varnothing$, where f is the boundary map defined in[11], A is strictly irreducible if there is just one equivalence class. Then is established that when $|\partial\mathcal{R}|\leq 4$ to obtain a symbolic representation it must be imposed that A be strictly irreducible[3]. Here $|\partial\mathcal{R}|$ means the number of sides of $\partial\mathcal{R}$. Another condition on the group is that it has fundamental domain $\mathcal{R}$ with $|\partial\mathcal{R}|\geq 5$ or if $|\partial\mathcal{R}|<5$ then the transition matrix from the Series coding is strictly irreducible. The class of groups considered include fundamental groups of surfaces with genus ≥ 2.

Theorem 3.1. *Let (X,ν) be a probability space and Y a complete, separable, nonpositively curved space in the sense of Busemann. Let Γ be a non-elementary Fuchsian group acting on the hyperbolic disc H^2 with a symmetric set of generators S and admitting a Series coding. Besides Γ there acts on X by measure preserving actions T, which has a fundamental region R satisfying the above conditions. If $\varphi:X\to Y$ is in the class $L^1(X,Y,\nu)$ then there exists a $T-$map $\overline{\varphi}$, such that $bar(\mathcal{E}_{N,}\varphi(x))$ converges in $L^1(X,Y,\nu)$ to $\overline{\varphi}(x)$. The map $\overline{\varphi}$ is constant $\nu-a.e.$ when the group acts ergodically.*

To prove the theorem we follow the idea used in [1] and [8]. Firstly we prove the theorem for maps taking only a finite number of values. Since Y is separable, any map $\varphi:X\to Y$ can be approximate, in the d_1 metric, by as sequence of finite-valued maps $\{\psi_n\}$, let us fix $\psi:=\psi_{n_0}$ such that $d_1(\varphi,\psi)<\alpha$. It can be proved that the measure of set $\left\{x:\sup_{N\geq 1}d\left(bar\left(\mathcal{E}_{N,}\varphi(x)\right),bar(\mathcal{E}_{N,}\psi(x))\right)>\alpha\right\}$ can be bounded by a quantity which be arbitrary small. This will be possible by a maximal ergodic theorem for real valuated maps.

For amenable locally compact groups, a maximal ergodic theorem was proved by Lindenstrauss [6]. Austin and Navas used this result. Here we formulate a maximal ergodic theorem for our setting.

For the proof of the theorem we shall use the following result:

Proposition 3.2. *[7] Let $\alpha>0$, $\varphi\in L^1(X,\nu)$ and $Z_\alpha:=\left\{x:\sup_{N\geq 1}\mathcal{S}_{N,}\varphi(x)>\alpha\right\}$. Then*

$$\nu\left(Z_\alpha\right)\leq\frac{1}{\alpha}\int_{Z_\alpha}\varphi d\nu.$$

Proof of the Theorem. Let $\varphi, \psi \in L^1(X, Y, \nu)$, $\alpha > 0$ and $\Phi(x) = d(\varphi(x), \psi(x))$. By the above proposition applied to the real valuated function Φ, we have

$$\nu\left(\left\{x : \sup_{N\geq 1}\{\mathcal{S}_{N,\Phi}(x)\} > \alpha\right\}\right) \leq \frac{1}{\alpha}\int \Phi d\nu = \frac{1}{\alpha}\int d(\varphi(x), \psi(x))$$

and since $d(bar(\mathcal{E}_{N,}\varphi(x)), bar(\mathcal{E}_{N,}\psi(x))) \leq W_1(\mathcal{E}_{N,}\varphi(x), \mathcal{E}_{N,}\psi(x))$, one gets

$$\left\{x : \sup_{N\geq 1} d(bar(\mathcal{E}_{N,}\varphi(x)), bar(\mathcal{E}_{N,}\psi(x))) > \alpha\right\}$$
$$\subset \left\{x : \sup_{N\geq 1} W_1(\mathcal{E}_{N,}\varphi(x), \mathcal{E}_{N,}\psi(x)) > \alpha\right\}.$$

Let us consider the measure $\mathcal{F}_{N,\varphi,\psi}(x) := \frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{cardS(n)}\sum_{\gamma\in S(n)}\delta_{(\varphi(T_\gamma(x)),\psi(T_\gamma(x)))}$, with $x, y \in X$, which is a joining between $\mathcal{E}_{N,}\varphi(x), \mathcal{E}_{N,}\psi(y)$. Thus by the definition of the W_1 metric

$$W_1(\mathcal{E}_{N,}\varphi(x), \mathcal{E}_{N,}\psi(x)) \leq \sqrt{\int_{Y\times Y} d(\varphi(T_\gamma(x)), \psi(T_\gamma(x)))\, d\mathcal{F}_{N,\varphi,\psi}(x)}$$
$$\leq \frac{1}{N}\sum_{n=0}^{N-1}\frac{1}{cardS(n)}\sum_{\gamma\in S(n)}\Phi(\gamma x),$$

and so

$$\left\{x : \sup_{N\geq 1} d(bar(\mathcal{E}_{N,}\varphi(x)), bar(\mathcal{E}_{N,}\psi(x))) > \alpha\right\} \subset \left\{x : \sup_{N\geq 1}\{\mathcal{S}_{N,\Phi}(x)\} > \alpha\right\}. \tag{3.1}$$

Therefore

$$\nu\left(\left\{x : \sup_{N\geq 1} d(bar(\mathcal{E}_{N,}\varphi(x)), bar(\mathcal{E}_{N,}\psi(x))) > \alpha\right\}\right)$$
$$\leq \nu\left(\left\{x : \sup_{N\geq 1}\{\mathcal{S}_{N,\Phi}(x)\} > \alpha\right\}\right) \leq \frac{1}{\alpha} d_1(\varphi, \psi).$$

Let $\{y_1, y_2, ..., y_m\}$ be the image of φ and $A_i := \varphi^{-1}(y_i)$, $i = 1, 2, ...m$. Let $\{\nu_x\}$ be the ergodic decomposition of ν with respect to T, by the convergence theorem of Bufetov and Series, for any $\varepsilon > 0$, $0 \leq i \leq n$, there is a $N_0 = N_0(\varepsilon, x, i)$ such that for $N \geq N_0$

$$\left|\mathcal{S}_{N,I_{A_i}}(x) - \nu_x(A_i)\right| < \varepsilon, \tag{3.2}$$

for any $\nu-$ a.e., x. Let $m(x) := \sum_{i=1}^{m}\nu_x(A_i)\,\delta_{y_i}$, we have

$$W_1(\mathcal{E}_{N,}\varphi(x), m(x)) \leq \sum_{i=1}^{m}\left|\mathcal{S}_{N,I_{A_i}}(x) - \nu_x(A_i)\right|\left[\max_{i,j} d(y_i, y_j)\right] < \varepsilon\left[\max_{i,j} d(y_i, y_j)\right],$$

for $N \geq N_0$. By the property that the barycentre is dominated by the W_1-metric we have $d\left(bar(\mathcal{E}_{N,}\varphi(x)), bar\left(m(x)\right)\right) < \varepsilon\times$fixed quantity, therefore

$$bar(\mathcal{E}_{N,}\varphi(x)) \rightarrow bar\left(m(x)\right) := \overline{\varphi}(x) \text{ as } N \rightarrow \infty.$$

Thus the sequence $\{bar\left(\mathcal{E}_{N,}\varphi(x)\right)\}$ pointwise converges to $\overline{\varphi}(x)$. Also since φ take finite values $\{y_1, y_2, ..., y_m\}$ the barycenter of $\mathcal{E}_{N,}\varphi(x)$ is in the convex closure of $\{y_1, y_2, ..., y_m\}$, so that $d\left(bar(\mathcal{E}_{N,}\varphi(x)),\ \overline{\varphi}(x)\right) \leq \max_{i,j} d\left(y_i, y_j\right)$. Now the convergence dominated theorem can be applied to the sequence $\{bar(\mathcal{E}_{N,}\varphi(x))\}$ and then the convergence is also in $L^1(X, Y, \nu)$, for maps taken finite values. For the general case, let $\varphi \in L^1(X, Y, \nu)$, since Y is separable φ can be approximate, in the d_1 metric by mean a sequence of finite-valuated maps, i.e. there is a family $\{\psi_n\}$ of finite-valuated maps such that for any $\varepsilon > 0$ there is a $k_0 = k_0(\varepsilon)$ and a $\psi := \psi_{k_0}$with $d_1\left(\varphi, \psi\right) < \varepsilon^2$.By (13) we have

$$\nu\left(\left\{x : \sup_{N\geq 1} d\left(bar\left(\mathcal{E}_{N,}\varphi(x)\right), bar\left(\mathcal{E}_{N,}\psi(x)\right)\right) > \varepsilon^2\right\}\right) \leq \frac{1}{\varepsilon^2} d_1\left(\varphi, \psi\right)^2 < \varepsilon. \tag{3.3}$$

Therefore the sequence $\{bar(\mathcal{E}_{N,}\varphi(x))\}$ oscillates in a set of measure at most ε, with α arbitrary, and since the sequence $bar\left(\mathcal{E}_{N,}\psi(x)\right)$ is almost surely convergent to a map $\overline{\psi}\left(x\right)$, the sequence $\{bar\left(\mathcal{E}_{N,}\varphi(x)\right)\}$ also converges, $\nu - a.e$, for any x. To show the convergence in L^1, let $\varepsilon > 0$, since $\{bar\left(\mathcal{E}_{N,}\psi_m(x)\right)\}$ converges in $L^1(X, Y, \nu)$, when ψ_m takes finite values, there is a $n_0 = n_0(\varepsilon)$ such that for $N, M \geq n_0$

$$\int d\left(bar(\mathcal{E}_{N,}\ \psi_m(x)), bar\left(\mathcal{E}_{M,}\ \psi_m(x)\right)\right) d\nu(x) < \varepsilon/3.$$

Let $\{\psi_m\}$ be a sequence of maps taking finite values converging to φ in L^1. We have

$$\begin{aligned}
&\int d\left(bar(\mathcal{E}_{N,}\varphi(x)), bar\left(\mathcal{E}_{M,}\varphi(x)\right)\right) d\nu(x) \\
&\leq \int d\left(bar(\mathcal{E}_{N,}\varphi(x)), bar\left(\mathcal{E}_{N,}\ \psi_m(x)\right)\right) d\nu(x) \\
&+ \int d\left(bar(\mathcal{E}_{N,}\ \psi_m(x)), bar\left(\mathcal{E}_{M,}\ \psi_m(x)\right)\right) d\nu(x) \\
&+ \int d\left(bar\left(\mathcal{E}_{M,}\ \psi_m(x)\right), bar\left(\mathcal{E}_{M,}\varphi(x)\right)\right) d\nu\left(x\right).
\end{aligned}$$

Thus for $N, M \geq n_0$

$$\int d\left(bar(\mathcal{E}_{N,}\varphi(x)), bar\left(\mathcal{E}_{M,}\varphi(x)\right)\right) d\nu(x) \leq 2d_1\left(\varphi, \psi_m\right) + \varepsilon/3.$$

If m is taken enough large such that $d_1\left(\varphi, \psi_m\right) < \varepsilon/3$ then

$$\int d\left(bar(\mathcal{E}_{N,}\varphi(x)), bar\left(\mathcal{E}_{M,}\varphi(x)\right)\right) d\nu(x) < \varepsilon, \text{ for } N, M \geq n_0.$$

Then the sequence $\{bar(\mathcal{E}_{N,}\varphi(x))\}$ is Cauchy in $L^1(X, Y, \nu)$ so it converges in this space. $\square$

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