On the oscillation of certain odd order nonlinear neutral difference equations

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Abstract. Our main aim of this paper is to investigate some oscillation criteria for solutions of certain odd order nonlinear neutral difference equation. We present some sufficient conditions that guarantee for all solutions of odd order neutral difference equation are oscillatory. Some examples are given to illustrate the main results.

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1 Introduction

In this paper, we are concerned with the following higher order neutral difference equation of the form

$$\Delta^m \left[x(n) + p(n)x\left(\tau(n)\right)\right] + q(n)f\left(x\left(\sigma(n)\right)\right) = 0, n \in \mathbb{N} = \{0, 1, ...\}, \tag{1.1}$$

where m is odd and $m \geq 1$. Δ is the forward difference operator defined by

$$\Delta x(n) = x\left(n+1\right) - x(n).$$

Throughout this paper, we assume the following conditions to hold:

(H1) $\{q(n)\}$ is a real-valued sequence with $q(n) \geq 0, n \in \mathbb{N}$ and $\{q(n)\}$ is not identically zero.

(H2) $\{p(n)\}$ is a real-valued sequence with $0 \leq p(n) < 1, n \in \mathbb{N}$.

(H3) $\{\tau(n)\}$ and $\{\sigma(n)\}$ are non-decreasing sequences such that $\tau(n) < n$ with $\lim\limits_{n\to+\infty} \tau(n) = +\infty$ and $\sigma(n) < n$ with $\lim\limits_{n\to+\infty} \sigma(n) = +\infty$.

(H4) $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing continuous function such that $xf\left(x\right) > 0$ for $x \neq 0$ and

$$-f\left(-xy\right) \geq f\left(xy\right) \geq f\left(x\right) f\left(y\right). \tag{1.2}$$

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In recent years, the oscillation behavior of neutral difference equations has been studied vigorously, for example, see (1-16) and the references cited therein. This is because of the fact that neutral difference equations find various applications in some variational problems, in natural science and technology.

In [5], Agarwal and Grace considered the higher order difference equation

$$\Delta\left(\Delta^{m-1}x(n)\right)^{\alpha}+q(n)x^{\alpha}\left(n-\tau\right)=0 \tag{1.3}$$

and obtained some sufficient conditions for the oscillation of all solutions of (1.3).

Yasar Bolat et. al. [8] have taken even order non-linear neutral difference equation

$$\Delta^{m}\left[y\left(k\right)+p\left(k\right)y\left(\tau\left(k\right)\right)\right]+q\left(k\right)y\left(\sigma\left(k\right)\right)=0, \tag{1.4}$$

and established some criteria for oscillation of bounded solutions only.

Therefore, it is to be noted that, to the best of our knowledge, there is no paper for higher order nonlinear neutral difference equations which ensures that all the solutions are oscillatory when m is odd. Following this notion, in this paper, we provide sufficient conditions which ensure that all solutions of (1.1) are oscillatory.

2 Preliminaries

Definition 2.1. The factorial expression is defined as $(r)^{(s)}=\prod_{i=0}^{s-1}(r-i)$ with $(r)^{(0)}=1$, for all $r\in\mathbb{R}=(-\infty,\infty)$ and s, a non-negative integer.

Definition 2.2. Let N_0 be a fixed non-negative integer. By a solution of equation (1.1), we mean a nontrivial real sequence $\{x(n)\}$ which is defined for all $n\geq\min_{i\geq0}\{\tau\left(i\right),\sigma\left(i\right)\}$ and satisfies equation (1.1) for $n\geq N_0$.

Definition 2.3. A solution $\{x(n)\}$ of equation (1.1) is said to be oscillatory if for every $n_1\geq N_0$, there exists $n\geq n_1$ such that $x_nx_{n+1}\leq 0$. Otherwise it is called non-oscillatory.

Definition 2.4. A difference equation is said to be oscillatory if all of its solutions are oscillatory. Otherwise, it is non-oscillatory.

To obtain the main results, we shall use the following notations.
For all large $n\geq n_0>0$, let

$$\begin{aligned}R_j(n)=&f\left(\frac{(\sigma(n)-m+j)^{(j-i)}}{j!}\right)\sum_{r=n}^{\infty}\frac{(r-n+m-j-3)^{(m-j-3)}}{(m-j-3)!}\\&\left(\sum_{j=r}^{\infty}q(j)\right)f\left(1-p\left(\sigma\left(r\right)\right)\right),\ j\in\{1,2,...,m-3\};\end{aligned}$$

$$R_{m-1}(n)=q(n)f\left(1-p\left(\sigma(n)\right)\right)f\left(\frac{(\sigma(n)-1)^{(m-2)}}{(m-1)!}\right);$$

$$R_0(n) = q(n) f\left(1 - p\left(\sigma(n)\right)\right) f\left(\sum_{r=\sigma(n)}^{\eta(n)} \frac{\left(r - \sigma(n) + m - 2\right)^{(m-2)}}{(m-2)!}\right),$$

for some non-decreasing function $\eta(n)$ with $\sigma(n) < \eta(n) \leq n$, $n \geq n_0$.

We need the following lemma to obtain our results.

Lemma 2.1. *(see [1]) Let $x(n)$ be defined for $n \geq n_0 \in \mathbb{N}$ and $x(n) > 0$ with $\Delta^n x(n)$ be of constant sign for $n \geq n_0$ and not identically zero. Then there exists an integer l, $0 \leq l \leq m$ with $(m+l)$ odd for $\Delta^m x(n) \leq 0$ and $(m+l)$ even for $\Delta^m x(n) \geq 0$ such that*

(i) $l \leq m-1$ implies $(-1)^{l+k} \Delta^k x(n) > 0$ for all $n \geq n_0$, $l \leq k \leq m-1$;

(ii) $l \geq 1$ implies $\Delta^k x(n) > 0$ for all large $n \geq n_0$, $1 \leq k \leq l-1$.

3 Main results

In this section, we discuss the following theorems.

Theorem 3.1. *Assume that the conditions (H1)-(H4) hold and*

$$\sum_{n=n_0}^{\infty} \left(\sum_{j=n}^{\infty} q(j)\right) \left(\sigma(n)\right)^{(m-2)} = \infty. \tag{3.1}$$

Let m be odd. If all the second order equations

$$\Delta^2 y(n) + R_j(n) f\left(y\left(\sigma(n)\right)\right) = 0,\ j \in \{2, 4, ..., m-1\} \tag{3.2}$$

for $n \geq n_0$ are oscillatory and if there exists a non-decreasing sequence $\{\eta(n)\}$ with $\sigma(n) < \eta(n) \leq n$, $n \geq n_0$ such that the first order difference equation

$$\Delta v(n) + R_0(n) f\left(v\left(\eta(n)\right)\right) = 0 \tag{3.3}$$

is oscillatory, then every solution of equation (1.1) oscillates.

Proof. Let $\{x(n)\}$ be a non-oscillatory solution of (1.1). Without loss of generality, assume that $x(n) > 0, x\left(\tau(n)\right) > 0, x\left(\sigma(n)\right) > 0$, for all $n \geq n_0 \geq 0$.
Let

$$z(n) = x(n) + p(n) x\left(\tau(n)\right) \geq x(n) > 0. \tag{3.4}$$

Then (1.1) becomes

$$\Delta^m z(n) = -q(n) f\left(x\left(\sigma(n)\right)\right) \leq 0, \text{ for } n \geq n_1 \geq n_0. \tag{3.5}$$

From Lemma 2.1, it is evident that

$$\Delta^{m-1} z(n) > 0, \text{ for } n \geq n_1. \tag{3.6}$$

Also from (3.5), we have $\Delta^m z(n) \leq 0$.
So, $z(n)$ satisfies Lemma 2.1, for some $l \in \{1, 2, ..., m-3\}$ and $(l+m)$ odd. Also by Lemma 2.1, $\Delta z(n) > 0$. Since $z(n)$ is increasing, we have

$$\begin{aligned} (1-p(n))\, z(n) &\leq z(n) - p(n) z\,(\tau(n)) \\ &= x(n) - p(n) p\,(\tau(n))\, x\,(\tau\,(\tau(n))) \\ &\leq x(n), \text{ for } n \geq n_1. \end{aligned}$$

That is,

$$(1-p(n))\, z(n) \leq x(n), \text{ for } n \geq n_1. \tag{3.7}$$

Now the following two cases are considered:
$l \in \{1, 2, ..., m-3\}$ and $l = 0$.

Case(i). Let $l \in \{1, 2, ..., m-3\}$.
From the discrete Taylor's formula, we have
(3.8)

$$\begin{aligned} -\Delta^{l+1} z(n) = &\sum_{j=l+1}^{m-2} \frac{(s-n+j-l-2)^{(j-l-1)}}{(j-l-1)!} (-1)^{j-l}\, \Delta^j z\,(s) \\ &+ (-1)^{m-l-3} \sum_{r=n}^{s-1} \frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!} \Delta^{m-1} z\,(r)\,, \end{aligned}$$

for $s \geq n \geq n_1$. Using Lemma 2.1 in (3.8), we obtain

$$-\Delta^{l+1} z(n) \geq \sum_{r=n}^{\infty} \frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!} \Delta^{m-1} z\,(r)\,. \tag{3.9}$$

Summing up the equation (1.1) from r to $u-1$ and letting $u \to \infty$, we have

$$\Delta^{m-1} z\,(r) \geq \sum_{j=r}^{\infty} q(j) f\,(x\,(\sigma\,(r)))\,, \text{ for } n \geq n_2 \geq n_1. \tag{3.10}$$

Substituting (3.10) in (3.9), we have

$$-\Delta^{l+1} z(n) \geq \sum_{r=n}^{\infty} \frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!} \left(\sum_{j=r}^{\infty} q(j) \right) f\,(x\,(\sigma\,(r)))\,. \tag{3.11}$$

Using (3.7) and (1.2) in (3.11), we get
(3.12)

$$\begin{aligned} -\Delta^{l+1} z(n) \geq &\sum_{r=n}^{\infty} \frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!} \\ &\left(\sum_{j=r}^{\infty} q(j) \right) f\,(1-p\,(\sigma\,(r)))\, f\,(z\,(\sigma\,(r)))\,. \end{aligned}$$

From (3.10), we can see that

$$(n)^{(m-l-1)}\Delta^{m-1}z(n) \geq (n)^{(m-l-1)}\left(\sum_{j=n}^{\infty} q(j)\right) x\left(\sigma(n)\right)$$
$$\geq (n)^{(m-l-1)}\left(\sum_{j=n}^{\infty} q(j)\right) \left(\sigma(n)\right)^{(l-1)}$$
$$\geq \sum_{j=n}^{\infty} q(j)\left(\sigma(n)\right)^{(m-l-2)}.$$

Hence from (3.1), we get

$$\sum^{\infty} (s)^{(m-l-1)}\,\Delta^{m-1}z\,(s) = \infty. \tag{3.13}$$

Consider the equality

$$\sum_{j=l-1}^{m-2} (-1)^{(j+l+1)}\,\frac{(n-m+j+1)^{(j-l+1)}}{(j-l+1)!}\Delta^{j}z(n)$$
$$= \sum_{j=l-1}^{m-2} (-1)^{(j+l+1)}\,\frac{(n_2)^{(j-l+1)}}{(j-l+1)!}\Delta^{j}z\left(n_1+m-j-2\right)$$
$$+ (-1)^{m+l-1}\sum_{s=n_2}^{m-2} (s)^{(m-l-1)}\,\Delta^{m-1}z\,(s)\,.$$

with $l \in \{1, 2, ..., m-1\}$ and $(l+m)$ is odd.
Now from the above, as in ([9],[16]), there exists an integer $n \geq n_3 \geq n_2$ such that

$$\Delta^{l-1}z(n) \geq (n-m+l+1)\,\Delta^{l}z(n) \tag{3.14}$$

and

$$z(n) \geq \frac{(n-m+l)^{(l-1)}}{l!}\Delta^{l-1}z(n). \tag{3.15}$$

Then we can find an integer $N \geq n_3$ such that

$$z\left(\sigma(n)\right) \geq \frac{\left(\sigma(n)-m+l\right)^{(l-1)}}{l!}\Delta^{l-1}z\left(\sigma(n)\right), \text{ for } n \geq N \tag{3.16}$$

Using (3.16) in (3.12), we have

$$-\Delta^{l+1}z(n) \geq \sum_{r=n}^{\infty} \frac{(r-n+m-l-3)^{(m-l-3)}}{(m-l-3)!}\left(\sum_{j=r}^{\infty} q(j)\right) f\left(1-p\left(\sigma\left(r\right)\right)\right)$$
$$f\left(\frac{\left(\sigma(n)-m+l\right)^{(l-1)}}{l!}\right) f\left(\Delta^{l-1}z\left(\sigma(n)\right)\right).$$

That is,

$$-\Delta^{l+1}z(n) \geq R_l(n)f\left(\Delta^{l-1}z\left(\sigma(n)\right)\right).$$

Let $y(n) = \Delta^{l-1}z(n)$.
Then $y(n) > 0$ for $n \geq N$ and above inequality becomes,

$$\Delta^2 y(n) + R_l(n)f\left(y\left(\sigma(n)\right)\right) \leq 0, \text{ for } n \geq N.$$

Thus the last inequality has a positive solution. By a well-known result in [10, 12], we can see that the equation

$$\Delta^2 y(n) + R_l(n)f\left(y\left(\sigma(n)\right)\right) = 0, \text{ for } n \geq N$$

also has a positive solution, which contradicts our assumption.

Case(ii). Let $l = 0$.

From discrete Taylor's formula, we have

$$\begin{aligned} z(n) = &\sum_{j=0}^{m-2} \frac{(s-n+j-1)^{(j)}}{j!}(-1)^j\,\Delta^j z\left(s\right) \\ &+ \sum_{r=n}^{s-1} \frac{(r-n+m-2)^{(m-2)}}{(m-2)!}\Delta^{m-1}z\left(r\right), \end{aligned}$$

for $s \geq n$. Considering Lemma 2.1 with $l = 0$ and using this in the above equation, we get

$$z(n) \geq \sum_{r=n}^{s-1} \frac{(r-n+m-2)^{(m-2)}}{(m-2)!}\Delta^{m-1}z\left(r\right),$$

for $n \geq n_1 \geq n_0$. Then we can find an integer $n_2 \geq n_1$ and a non-decreasing function $\eta(n)$ with $\sigma(n) < \eta(n) \leq n$ such that

$$z\left(\sigma(n)\right) \geq \sum_{r=\sigma(n)}^{\eta(n)} \frac{(r-\sigma(n)+m-2)^{(m-2)}}{(m-2)!}\Delta^{m-1}z\left(\eta(n)\right), \tag{3.17}$$

for $n \geq n_2 \geq n_1$. From (1.1), (1.2), (3.7) and (3.17), we have

$$\begin{aligned} -\Delta\left(\Delta^{m-1}z(n)\right) &= q(n)f\left(x\left(\sigma(n)\right)\right) \\ &\geq q(n)f\left(1-p\left(\sigma(n)\right)\right)f\left(z\left(\sigma(n)\right)\right) \\ &\geq q(n)f\left(1-p\left(\sigma(n)\right)\right)f\left(\sum_{r=\sigma(n)}^{\eta(n)} \frac{(r-\sigma(n)+m-2)^{(m-2)}}{(m-2)!}\right) \\ &\times f\left(\Delta^{m-1}z\left(\eta(n)\right)\right) \\ &= R_0(n)f\left(\Delta^{m-1}z\left(\eta(n)\right)\right). \end{aligned}$$

Let $v(n) = \Delta^{m-1} z(\eta(n))$. Then $v(n) > 0$, for $n \geq n_2$ and the above inequality becomes,

$$\Delta v(n) + R_0(n) f(v(\sigma(n))) \leq 0,$$

for which a positive solution exists. By a well-known result in [10, 12], we have equation (3.3) also has a positive solution, which contradicts our assumption. This completes the proof. □

Example 3.1. Consider the third order difference equation

$$\Delta^3 \left[x(n) + \frac{1}{2} x(n-1) \right] + 4x(n-2) = 0. \tag{E1}$$

Here $0 \leq p(n) = \frac{1}{2} < 1$, $q(n) = 4n$, $\tau(n) = n-1 < n$, $\sigma(n) = n-2 < n$ and $f(u) = \frac{u}{n}$. Also

$$\sum_{n=n_0}^{\infty} \left(\sum_{j=n}^{\infty} q(j) \right) (\sigma(n))^{(m-2)} = \sum_{n=n_0}^{\infty} \left(\sum_{j=n}^{\infty} 4j \right) (n-2) = \infty.$$

We can easily see that all conditions of Theorem 3.1 are satisfied and hence all the solutions of equation (E1) are oscillatory.
One of such solutions is $x(n) = (-1)^n$.

Next we consider the following theorem.

Theorem 3.2. *Assume that the conditions (H1)-(H4) and (3.1) hold. Let m be odd. If*
(3.18)

$$f\left(\frac{(\sigma(n) - m + j)^{(j-i)}}{j!} \right) \sum_{r=n_0}^{\infty} \frac{(r - n + m - j - 2)^{(m-j-2)}}{(m-j-2)!} \left(\sum_{j=r}^{\infty} q(j) \right) f(1 - p(\sigma(r))) = \infty,$$

for $j \in \{2, 4, ..., m-1\}$ and if there exists a non-decreasing sequence $\{\eta(n)\}$ with $\sigma(n) < \eta(n) \leq n$, $n \geq n_0$ such that the equation (3.3) is oscillatory, then every solution of equation (1.1) oscillates.

Proof. Assume that $\{x(n)\}$ be a non-oscillatory solution of (1.1). Without loss of generality, assume that $x(n) > 0, x(\tau(n)) > 0, x(\sigma(n)) > 0$, for all $n \geq n_0 \geq 0$.
Let

$$z(n) = x(n) + p(n) x(\tau(n)) \geq x(n) > 0. \tag{3.19}$$

Proceeding as in the proof of Theorem 3.1, we get the following two cases:
$l \in \{1, 2, ..., m-3\}$ and $l = 0$.

Case(i). Let $l \in \{1, 2, ..., m-3\}$.

From the discrete Taylor's formula, we have
(3.20)

$$\Delta^l z(n) = \sum_{j=l}^{m-2} \frac{(s-n+j-l-2)^{(j-l)}}{(j-l)!} (-1)^{j-l} \Delta^j z(s) + (-1)^{m-l-1} \sum_{r=n}^{s-1} \frac{(r-n+m-l-2)^{(m-l-2)}}{(m-l-2)!} \Delta^{m-1} z(r),$$

for $s \geq n$. Using (1.2), (3.7), (3.10) and Lemma 2.1 in (3.20), we obtain
(3.21)

$$\Delta^l z(n) \geq \sum_{r=n}^{\infty} \frac{(r-n+m-l-2)^{(m-l-2)}}{(m-l-2)!} \left(\sum_{j=r}^{\infty} q(j) \right) f(1-p(\sigma(n))) f(z(\sigma(n))).$$

From (3.15), there exists a $n_2 \geq n_1$ and a positive constant $c > 0$ such that

$$z(\sigma(n)) \geq \frac{(\sigma(n)-m+l)^{(l-1)}}{l!} \Delta^{l-1} z(\sigma(n)), \text{ for } n \geq n_2. \tag{3.22}$$

and

$$\Delta^{l-1} z(\sigma(n)) \geq c, \text{ for } n \geq n_2. \tag{3.23}$$

Using (3.22) and (3.23) in (3.21), we get

$$\infty > \Delta^l z(n_2) \geq \sum_{r=n_2}^{\infty} \frac{(r-n+m-l-2)^{(m-l-2)}}{(m-l-2)!} \left(\sum_{j=r}^{\infty} q(j) \right) f(1-p(\sigma(n))) f(c) f\left(\frac{(\sigma(n)-m+l)^{(l-1)}}{l!} \right),$$

which contradicts (3.18).

Case(ii). Let $l = 0$.

The proof for this case is similar to the proof of Case(ii) in Theorem 3.1 and hence omitted. This completes the proof. □

Example 3.2. Consider the first order difference equation

$$\Delta \left[x(n) + \frac{3}{4} x(n-1) \right] + \frac{1}{2} x(n-2) = 0. \tag{E2}$$

Here $0 \leq p(n) = \frac{3}{4} < 1$, $q(n) = \frac{1}{2}$, $\tau(n) = n-1 < n$, $\sigma(n) = n-2 < n$ and $f(u) = u$. We can find that all hypotheses of Theorem 3.2 are fulfilled. Also, equation(E2) has an oscillatory solution given by $x(n) = \frac{(-1)^n}{2}$.

4 Conclusions

In this paper, we have proposed the comparison method for identifying oscillatory solutions of odd order neutral difference equations. This method compares the first and second order equations which are very simple. Moreover, the above examples reveal the efficiency of our method.

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