On an η-Einstein (k, μ)-contact metric manifold

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Abstract. In [4] it was shown that if k $(k \neq 0)$ is a rational number and μ $(\mu \neq 0)$ is integer, $(2m+1)$-dimensional $(m \geq 2)$ C-Bochner semi-symmetric non Sasakian (k, μ)-contact metric manifolds do not exist. In this paper we consider an η-Einstein (k, μ)-contact metric manifold. And we study the relation between numbers k or μ and C-Bochner semi-symmetries on an η-Einstein (k, μ)-contact metric manifold.

M.S.C. 2010: 53C15, 53C25.
Key words: C-Bochner semi-symmetry; (k, μ)-contact metric manifold; Sasakian manifold; η-Einstein manifold.

1 Introduction

Let R be the Riemannian curvature tensor of a Riemannian manifold M with a positive-definite metric tensor g. M is said to be a locally symmetric if $\nabla R = 0$, where ∇ denotes the Levi-Civita connection. For any tangent vectors X and Y, we consider $R(X, Y)$ as a derivation of the tensor algebra at each point on M. M is said to be semi-symmetric if $R(X, Y).R = 0$ as a proper generalization of locally symmetric manifold. Many geometers have considered semi-symmetric spaces and in turn their generalizations.

On the other hand, M. Matsumoto and G. Chuman [5] defined the contact Bochner curvature tensor B by

$$(1.1)\qquad \begin{aligned} B(X,Y) = R(X,Y) &+ \frac{1}{2(m+2)}[QY \wedge X - QX \wedge Y + Q\phi Y \wedge \phi X \\ &- Q\phi X \wedge \phi Y + 2g(Q\phi X, Y)\phi \\ &+ 2g(\phi X, Y)Q\phi + \eta(Y)QX \wedge \xi + \eta(X)\xi \wedge QY] \\ &- \frac{p+2m}{2(m+2)}[\phi Y \wedge \phi X + 2g(\phi X, Y)\phi] \\ &- \frac{p-4}{2(m+2)}Y \wedge X + \frac{p}{2(m+2)}[\eta(Y)\xi \wedge X + \eta(X)Y \wedge \xi] \end{aligned}$$

on $(2m+1)$-dimensional Sasakian manifold (B is called C-Bochner curvature), where Q is the Ricci operator of M, $p = \frac{2m+r}{2(m+1)}$ (r is the scalar curvature of M) and

Applied Sciences, Vol.18, 2016, pp. 43-49.

$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$. C.D. Uday and G. Sujit [6] defined the C-Bochner semi-symmetry on a (k,μ)-contact metric manifold as follows:

Definition 1.1. A $(2m+1)$-dimensional (k,μ)-contact metric manifold is said to be C-Bochner semi-symmetric if

$$R(X,Y).B = 0 \tag{1.2}$$

for any vector fields X and Y.

2 Preliminaries

Let (M,ϕ,ξ,η,g) be a $(2m+1)$-dimensional contact metric manifold, that is, let M be a differentiable manifold and (ϕ,ξ,η,g) a contact metric structure on M, formed by tensor fields ϕ,ξ,η, of type $(1,1)$, $(1,0)$ and $(0,1)$, respectively, and a Riemannian metric g such that

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1,\\ &\eta(X) = g(X,\xi),\ g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),\\ &d\eta(X,Y) = g(X,\phi Y) \end{aligned} \tag{2.1}$$

for any vector fields X and Y. We denote by ∇ the Riemannian connection defined by g and define a tensor field h on a contact metric manifold M by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where $\mathcal{L}$ denotes the Lie differentiation. Then it is well-known that h is a symmetric operator,

$$\nabla_X\xi = -\phi X - \phi h X$$

is satisfied for any vector field X, h anti-commutes with ϕ and $\mathrm{tr}h = 0$ on a contact metric manifold, where $\mathrm{tr}h$ is the trace of h (see. [1]).

If ξ is Killing vector on a contact metric manifold M, then M is said to be a K-contact Riemannian manifold. If a contact metric manifold M is normal (i.e., $N + 2d\eta \otimes \xi = 0$, where N denotes the Nijenhuis tensor formed with ϕ), then M is called a Sasakian manifold. Every Sasakian manifold is a K-contact Riemannian manifold. On a Sasakian manifold with structure tensors (ϕ,ξ,η,g), we have

$$\nabla_X\xi = -\phi X,\ \ (\nabla_X\phi)Y = R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X$$

(see [1]).

The (k,μ)-nullity distribution of a contact metric manifold for the pair $(k,\mu) \in \mathbb{R}^2$, is a distribution

$$\begin{aligned} &N(k,\mu) : p \to N_p(k,\mu),\\ &N_p(k,\mu) := [W \in T_pM \mid R(X,Y)W = (kI + \mu h)(g(Y,W)X - g(X,W)Y)]. \end{aligned}$$

If M is a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution, i.e.,

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{2.2}$$

then M is called a (k,μ)-contact metric manifold. And the following relations in a (k,μ)-contact manifold are well known (see. [2],[1]) :

$$h^2 = (k-1)\phi^2, \quad k \le 1, \tag{2.3}$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{2.4}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(hX,Y)\xi - \eta(Y)hX], \tag{2.5}$$

for any vector fields X and Y. If $k = 1$, the structure is Sasakian ([2],[1]) and if $k < 1$, the (k,μ)-nullity condition completely determines the curvature of M^{2m+1} (see. [3]). The following theorem is well known:

Theorem 2.1 (e.g.,[3]). *Let (M,ξ,η,ϕ,g) be a (k,μ)-contact metric manifold which is not Sasakian, i.e., $k < 1$. Then its Riemann curvature tensor R is given explicitly in its $(0,4)$-form by*

$$\begin{aligned} & g(R(X,Y)Z,W) \\ &= (1-\frac{\mu}{2})(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \\ &+ g(Y,Z)g(hX,W) - g(X,Z)g(hY,W) \\ &- g(Y,W)g(hX,Z) + g(X,W)g(hY,Z) \\ &+ \frac{1-\frac{\mu}{2}}{1-k}(g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W)) \\ &- \frac{\mu}{2}(g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W)) \\ &+ \frac{k-\frac{\mu}{2}}{1-k}(g(\phi hY,Z)g(\phi hX,W) - g(\phi hY,W)g(\phi hX,Z)) \\ &+ \mu g(\phi X,Y)g(\phi Z,W) \\ &+ \eta(X)\eta(W)((k-1+\frac{\mu}{2})g(Y,Z) + (\mu-1)g(hY,Z)) \\ &- \eta(X)\eta(Z)((k-1+\frac{\mu}{2})g(Y,W) + (\mu-1)g(hY,W)) \\ &+ \eta(Y)\eta(Z)((k-1+\frac{\mu}{2})g(X,W) + (\mu-1)g(hX,W)) \\ &- \eta(Y)\eta(W)((k-1+\frac{\mu}{2})g(X,Z) + (\mu-1)g(hX,Z)) \end{aligned} \tag{2.6}$$

for any vector fields X, Y, Z and W on M.

C.D. Uday and G. Sujit [6] got the following result for (k,μ)-contact metric manifold M^{2m+1} $(2m+1 \ge 5)$.

Lemma 2.2. *Let $(M^{2m+1},\xi,\eta,\phi,g)$ be a (k,μ) contact metric manifold which is not Sasakian. Then the following equations hold:*

$$\begin{aligned} S(X,Y) = {} & [2(m-1) - m\mu]g(X,Y) + [2(m-1)+\mu]g(hX,Y) \\ & + [2(1-m) + m(2k+\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.7}$$

(2.8) $$B(X,Y)\xi = \frac{2(k-1)}{m+2}[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY],$$

(2.9) $$B(X,\xi)Y = \frac{2(k-1)}{m+2}[\eta(Y)X - g(X,Y)\xi] + \mu[\eta(Y)hX - g(hX,Y)\xi],$$

for any vector fields X and Y.

Remark 2.1. In [6], (2.7) $\sim$ (2.9) hold good for the assumption that the dimension $n(=2m+1)$ of M is greater than 5 or equal to 5. However, using (1.1) and (2.6), it is showed that these three equations hold good even if in 3-dimensional (k,μ)-contact metric manifold.

If the Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a and b are smooth functions, then M is called an η-Einstein manifold. Of course if $b = 0$, M is an Einstein manifold.

On the other hand, the condition $R(X,Y)\xi = 0$ for all vector fields X and Y has a strong and interesting implication for a contact metric manifold. The following theorem is well known:

Theorem 2.3 ([1]). *A contact metric manifold M^{2m+1} satisfying $R(X,Y)\xi = 0$ is locally isometric to $E^{m+1} \times S^m(4)$ for $m > 1$ and flat for $m = 1$.*

On the other hand, we got the following three results in [4] ;

Theorem 2.4 ([4]). *If M is a 5-dimensional C-Bochner semi-symmetric non-Sasakian (k,μ)-contact metric manifold, then $k = \mu = 0$. i.e., M is a locally isometric to $E^3 \times S^2(4)$.*

Theorem 2.5 ([4]). *Let M be a $(2m+1)$-dimensional $(m \geq 2)$ C-Bochner semi-symmetric non-Sasakian (k,μ)-contact metric manifold. If k $(k \neq 0)$ is a rational number and μ $(\mu \neq 0)$ is a integer, then there does not exist manifold M satisfying these conditions.*

Theorem 2.6 ([4]). *If M be a $(2m+1)$-dimensional $(m \geq 2)$ non-Sasakian (k,μ)-contact metric manifold satisfying $B(\xi,X).R = 0$ for any vector fields X, then one of the following cases holds:*

(a) $\mu = \frac{(m^2+2m-2)+\sqrt{(m^2+2m-2)^2+4(m+2)(m^2+2m-1)}}{(m+2)(m^2+2m-1)}, \quad k = \frac{4-(m+2)^2\mu^2}{4},$

(b) $\mu = \frac{(m^2+2m-2)-\sqrt{(m^2+2m-2)^2+4(m+2)(m^2+2m-1)}}{(m+2)(m^2+2m-1)}, \quad k = \frac{4-(m+2)^2\mu^2}{4}.$

3 an η-Einstein (k,μ)-contact metric manifold

In this section, we deal with a $(2m+1)$-dimensional η-Einstein (k,μ)-contact metric manifolds. Then we have

(3.1) $$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

for any vector fields X and Y, where a and b are smooth functions.

Before proving our assertions, we give some lemmas.

Lemma 3.1. *Let M^n be an $n(=2m+1)$-dimensional non-Sasakian η-Einstein (k,μ)-contact metric manifold. Then we have*

$$\mu = 3 - n. \tag{3.2}$$

Moreover a and b elements of (3.1) *are constant, i.e.,*

$$a = \frac{1}{2}(n-3)(n+1), \quad b = (n-1)k - \frac{1}{2}(n-3)(n+1). \tag{3.3}$$

Proof. Making use of (2.7) and (3.1), we get

$$\begin{aligned} [(n-3)+\mu]g(hX,Y) &= \left[a-(n-3)+\frac{n-1}{2}\mu\right]g(X,Y) \\ &+ \left[b-(3-n)-\frac{n-1}{2}(2k+\mu)\right]\eta(X)\eta(Y). \end{aligned} \tag{3.4}$$

Let λ be an eigenvalue of h and e_λ an eigenvector corresponding to λ. Since h is anti-commutes with ϕ, we get $h\phi e_\lambda = -\lambda\phi e_\lambda$.

Substituing e_λ into X and Y in (3.4), we have

$$\begin{aligned} \lambda[(n-3)+\mu]g(e_\lambda,e_\lambda) &= [(n-3)+\mu]g(he_\lambda,e_\lambda) \\ &= \left[a-(n-3)+\frac{n-1}{2}\mu\right]g(e_\lambda,e_\lambda). \end{aligned} \tag{3.5}$$

Also, substituing ϕe_λ into X and Y in (3.4) and using $g(\phi e_\lambda,\phi e_\lambda) = g(e_\lambda,e_\lambda)$, it follows that

$$\begin{aligned} -\lambda[(n-3)+\mu]g(e_\lambda,e_\lambda) &= -\lambda[(n-3)+\mu]g(\phi e_\lambda,\phi e_\lambda) \\ &= [(n-3)+\mu]g(h\phi e_\lambda,\phi e_\lambda) \\ &= \left[a-(n-3)+\frac{n-1}{2}\mu\right]g(\phi e_\lambda,\phi e_\lambda), \\ &= \left[a-(n-3)+\frac{n-1}{2}\mu\right]g(e_\lambda,e_\lambda). \end{aligned} \tag{3.6}$$

Subtracting (3.6) from (3.5), it yields (3.2). Substituting (3.2) into (3.4), we get

$$\begin{aligned} S(X,Y) &= \frac{1}{2}(n-3)(n+1)g(X,Y) \\ &+ \left[(n-1)k-\frac{1}{2}(n-3)(n+1)\right]\eta(X)\eta(Y), \end{aligned} \tag{3.7}$$

which implies (3.3). □

From Lemma 3.1, we have

Corollary 3.2. *Let M be a non-Sasakian η-Einstein (k,μ)-contact metric manifold. Then there does not exist M satisfying $\mu > 0$.*

From Lemma 3.1, we get

Lemma 3.3. *Let M^n be an n-dimensional non-Sasakian Einstein (k, μ)-contact metric manifold. Then n is either 3 or 5. Moreover if $n = 3$, then M is flat. If $n = 5$, then we get*

$$k = \frac{3}{2}, \ \mu = -2, \tag{3.8}$$

and

$$S(X, Y) = 6g(X, Y) \tag{3.9}$$

for any tangent vector fields X, Y of M.

Proof. By the assumption we have $b = 0$ in (3.1). By means of (3.3), we get

$$k = \frac{1}{2(n-1)}(n-3)(n+1). \tag{3.10}$$

Making use of (2.3) and (3.10), we obtain

$$(n-2)^2 \leq 5, \tag{3.11}$$

which yields either $n = 3$ or $n = 5$.

Using (3.3) and (3.2) in the case of $n = 3$, we have

$$a = k = \mu = 0, \tag{3.12}$$

or equivalently,

$$R(X, Y)\xi = 0$$

for any tangent vector fields X, Y of M. Applying Theorem 2.3, we see that M is flat.

Also using (3.3) and (3.2) in the case of $n = 5$, we obtain (3.8) and (3.9). □

By virtue of Theorem 2.5 and Lemma 3.1 we have

Theorem 3.4. *Let M^n be an n-dimensional non-Sasakian η-Einstein (k, μ)-contact metric manifold satisfying that k $(k \neq 0)$ is a rational number. Then M is not C-Bochner semi-symmetric.*

Proof. Since M is η-Einstein, by applying Lemma 3.1, we see that k $(k \neq 0)$ is a rational number and μ $(\mu \neq 0)$ is a integer. Hence, by using Theorem 2.5, we infer our result. □

In view of Theorem 2.4, Theorem 2.6 and Lemma 3.3, we conclude the following:

Theorem 3.5. *Let M be an n-dimensional non-Sasakian Einstein (k, μ)-contact metric manifold satisfying $(k, \mu) \neq (0, 0)$. Then M is not C-Bochner semi-symmetric and M does not satisfy $B(\xi, X).R = 0$ for any vector fields X.*

Proof. Since M is Einstein, applying Lemma 3.3, we get $n = 5$ and $\mu = -2$. We assume that M is 5-dimensional C-Bochner semi-symmetric. Using Theorem 2.4, we have $k = \mu = 0$, which yields a contradiction to the fact that $\mu = -2$. Hence we find that M is not C-Bochner semi-symmetric.

On the other hand, we can assume that M is 5-dimensional non-Sasakian (k, μ)-contact metric manifold satisfying $B(\xi, X).R = 0$ for any vector fields X. Making use of Theorem 2.6, we have $\mu = \frac{3 \pm \sqrt{51}}{14} \neq -2$. Hence we conclude that M does not satisfy $B(\xi, X).R = 0$ for any vector fields X. □

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