Improvement of bounds for the Poisson-binomial relative error

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Abstract. In this article, the Stein-Chen method and the binomial w-function are used to determine new uniform and non-uniform bounds on two forms of the relative error of the binomial cumulative distribution function with parameters $n \in \mathbb{N}$ and $p \in (0,1)$ and the Poisson cumulative distribution function with mean $\lambda = np$. The bounds obtained in the present study are sharper than those reported in Teerapabolarn [14, 15]. Finally, some numerical examples are provided to illustrate the goodness of these bounds.

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Key words: binomial w-function; cumulative distribution function; Poisson approximation; relative error; Stein-Chen method.

1 Introduction

Let a non-negative integer-valued random variable X have the binomial distribution with parameters $n \in \mathbb{N}$ and $p \in (0,1)$. This distribution is a well-known discrete distribution that can be applied in topics related to probability and statistics. The probability mass function of X, or binomial probability function, is of the form

$$p_X(x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, ..., n, \tag{1.1}$$

where $q = 1-p$ and the mean and variance of X are $\mu = np$ and $\sigma^2 = npq$, respectively. In particular case, $n = 1$, the random variable is the Bernoulli random variable with parameter p. In addition, from the experimental point of view, the random variable X can be thought of as the number of successes in a sequence of n independent Bernoulli trials, where each trial results in the success or the failure with probabilities p and q. It is well-known that if the number of trials $n \to \infty$ and the probability of success $p \to 0$ while $\lambda = np$ remains a constant ($0 < \lambda < \infty$) then $\binom{n}{x} p^x q^{n-x} \to \frac{e^{-\lambda}\lambda^x}{x!}$ for every $x = 0, 1, ..., n$, which is a Poisson limit theorem. Therefore the Poisson distribution with mean $\lambda = np$ can be used as an approximation of the binomial distribution with parameters n and p when n is sufficiently large and p is sufficiently

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small. In the past, there have been a lot of studies related to Poisson approximation of binomial distribution. For example, in the case of pointwise approximation was examined by Anderson and Samuels [1], Feller [6] and Johnson et al. [8] also [3] and [2]. In the case of cumulative probability approximation, Anderson and Samuels [1] provided that

$$\mathbb{P}_\lambda(x_0) - \mathbb{B}_{n,p}(x_0) \begin{cases} > 0 & \text{if } x_0 \le \dfrac{\lambda n}{n+1}, \\ < 0 & \text{if } x_0 \ge \lambda, \end{cases} \tag{1.2}$$

where $\mathbb{P}_\lambda(x_0) = \sum_{k=0}^{x_0} \frac{e^{-\lambda}\lambda^k}{k!}$ and $\mathbb{B}_{n,p}(x_0) = \sum_{k=0}^{x_0} \binom{n}{k} p^k q^{n-k}$ are the Poisson and binomial cumulative distribution functions at $x_0 \in \{0, 1, ..., n\}$, respectively. Ivchenko [7] gave the asymptotic relation on the ratio of the binomial and Poisson cumulative distribution functions

$$\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} = 1 + o(1), \tag{1.3}$$

which is fulfilled uniformly in $x_0 < \lambda$. Very similar criteria for measuring the accuracy of Poisson approximation were used by Teerapabolarn [12], who applying the Stein-Chen method gave both non-uniform and uniform bounds for the relation error of the considered distributions. His results presented in [12] are as follows:

$$\left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1 \right| \le \frac{p(e^\lambda - 1)\Delta(x_0)}{x_0 + 1}, \ x_0 = 0, 1, ..., n, \tag{1.4}$$

where

$$\Delta(x_0) = \begin{cases} e^{-\lambda} q^{-n} & \text{if } x_0 < \lambda, \\ 1 & \text{if } x_0 \ge \lambda, \end{cases} \tag{1.5}$$

and he also gave a non-uniform bound for the another form of the relative error of two such cumulative distribution functions,

$$\left| \frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1 \right| \le \frac{(e^\lambda - 1)p}{x_0 + 1}, \ x_0 = 0, 1, ..., n. \tag{1.6}$$

and those from [13] are of the form:

$$\sup_{0 \le x_0 \le n} \left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1 \right| \le \frac{(1 - e^{-\lambda})(1 - q^n)}{nq^n} \tag{1.7}$$

and

$$\sup_{0 \le x_0 \le n} \left| \frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1 \right| \le \frac{(e^\lambda - 1)(1 - q^n)}{n}. \tag{1.8}$$

The bounds presented above were then improved by the same author in [14, 15] to the sharper ones:

$$\sup_{0 \le x_0 \le n} \left| \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1 \right| \le \max\left\{ e^{-\lambda} q^{-n} - 1, \frac{1 - (1+\lambda)e^{-\lambda}}{nq^n} \min\left(1, \frac{2(1 - q^n)}{\lambda}\right) \right\} \tag{1.9}$$

and

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right|\le \max\left\{1-e^\lambda q^n, \frac{e^\lambda-\lambda-1}{n}\min\left(1,\frac{2(1-q^n)}{\lambda}\right)\right\}, \tag{1.10}$$

for the uniform case and

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\frac{2(e^\lambda-\lambda-1)\Delta(x_0)}{n(x_0+1)},\ x_0=0,1,...,n \tag{1.11}$$

and

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right|\le\frac{2(e^\lambda-\lambda-1)}{n(x_0+1)},\ x_0=0,1,...,n \tag{1.12}$$

in the non-uniform one.

The aim of this article is further improvement of the latter bounds. It will be achieved by using the Stein-Chen method and the binomial w-function and the obtained results presented in Sections 2 and 3. In Section 4, some numerical examples are provided to show the goodness of new bounds. Concluding remarks are presented in the last section.

2 The method

In this study, we use as our main tools the Stein-Chen method and the binomial w-function.

2.1 The binomial w-function

The w-functions were studied by many authors, among others by Cacoullos and Papathanasiou [4], Papathanasiou and Utev [10], and Majsnerowska [9]. The following lemma presents another form of the w-function associated with the binomial random variable given in [9] and [10], which we are called the binomial w-function throughout this study.

Lemma 2.1. *Let $w(X)$ be the w-function associated with the binomial random variable X, then*

$$w(x)=\frac{(n-x)p}{\sigma^2},\ x=0,1,...,n, \tag{2.1}$$

where $\sigma^2=npq$.

The next relation stated by Cacoullos and Papathanasiou [4] is crucial for obtaining our main results.

If a non-negative integer-valued random variable Y has probability mass function $p_Y(y)>0$ for every $y\in\mathcal{S}(Y)$, the support of Y, and $\sigma^2=Var(Y)$ is finite, then

$$E[(Y-\mu)f(Y)]=\sigma^2E[w(Y)\Delta f(Y)], \tag{2.2}$$

for any function $f:\mathbb{N}\cup\{0\}\to\mathbb{R}$ for which $E|w(Y)\Delta f(Y)|<\infty$, where $\mu=E(Y)$ and $\Delta f(y)=f(y+1)-f(y)$.

2.2 The Stein-Chen method

The classical Stein method introduced by Stein in [11] was developed for Poisson case by Chen [5]. The resulted version is referred to as the Stein-Chen method. Following Teerapabolarn [12], Stein's equation of the Poisson cumulative distribution function with parameter $\lambda > 0$ is of the form

$$h_{x_0}(x) - \mathbb{P}_\lambda(x_0) = \lambda f_{x_0}(x+1) - x f_{x_0}(x), \tag{2.3}$$

where $x_0, x \in \mathbb{N} \cup \{0\}$ and function $h_{x_0} : \mathbb{N} \cup \{0\} \to \mathbb{R}$ is defined by

$$h_{x_0}(x) = \begin{cases} 1 & \text{if } x \le x_0, \\ 0 & \text{if } x > x_0 \end{cases}$$

and

$$f_{x_0}(x) = \begin{cases} (x-1)!\lambda^{-x} e^{\lambda}[\mathbb{P}_\lambda(x-1)[1-\mathbb{P}_\lambda(x_0)]] & \text{if } x \le x_0, \\ (x-1)!\lambda^{-x} e^{\lambda}[\mathbb{P}_\lambda(x_0)[1-\mathbb{P}_\lambda(x-1)]] & \text{if } x > x_0, \\ 0 & \text{if } x = 0. \end{cases} \tag{2.4}$$

Lemma 2.2. *For $x_0, x \in \mathbb{N}$, let $\check{\lambda} = \left\lfloor \frac{n\lambda}{n+1} \right\rfloor + 1$ and $\hat{\lambda} = \lceil \lambda \rceil$, we have*

1. *For $x_0 > \frac{n\lambda}{n+1}$,*

$$\sup_{x \ge 1} |\Delta f_{x_0}(x)| \le \frac{(x_0+2)\mathbb{P}_\lambda(x_0)}{(x_0+1)(x_0+2-\lambda)} \tag{2.5}$$

and

$$\sup_{x \ge 1} |\Delta f_{x_0}(x)| \le \frac{(\check{\lambda}+2)\mathbb{P}_\lambda(x_0)}{\check{\lambda}+2-\lambda} \min\left\{ \frac{1}{\check{\lambda}+1}, \frac{1}{x} \right\}. \tag{2.6}$$

2. *For $x_0 \ge \lambda$,*

$$\sup_{x \ge 1} |\Delta f_{x_0}(x)| \le \frac{(\hat{\lambda}+2)\mathbb{P}_\lambda(x_0)}{\hat{\lambda}+2-\lambda} \min\left\{ \frac{1}{\hat{\lambda}+1}, \frac{1}{x} \right\}. \tag{2.7}$$

Proof. 1. For $x \le x_0$, it follows from [15] that

$$\begin{aligned} \Delta f_{x_0}(x) &\le \frac{\mathbb{P}_\lambda(x_0)}{x_0+1} \left\{ 1 + \frac{\lambda}{x_0+2} + \frac{\lambda^2}{(x_0+2)(x_0+3)} + \cdots \right\} \\ &\le \frac{\mathbb{P}_\lambda(x_0)}{x_0+1} \left\{ 1 + \frac{\lambda}{x_0+2} + \left(\frac{\lambda}{x_0+2} \right)^2 + \cdots \right\} \\ &= \frac{(x_0+2)\mathbb{P}_\lambda(x_0)}{(x_0+1)(x_0+2-\lambda)}, \end{aligned} \tag{2.8}$$

which also implies that

$$\Delta f_{x_0}(x) \le \frac{(\check{\lambda}+2)\mathbb{P}_\lambda(x_0)}{\check{\lambda}+2-\lambda} \min\left\{ \frac{1}{\check{\lambda}+1}, \frac{1}{x} \right\}. \tag{2.9}$$

For $x > x_0$, by following [15],

$$
\begin{aligned}
& 0 < -\Delta f_{x_0}(x) \\
(2.10) \qquad & \leq \frac{\mathbb{P}_\lambda(x_0)}{x}\left\{\frac{1}{x+1}+\frac{2\lambda}{(x+1)(x+2)}+\frac{3\lambda^2}{(x+1)(x+2)(x+3)}+\cdots\right\} \\
& \leq \frac{\mathbb{P}_\lambda(x_0)}{x_0+1}\left\{\frac{1}{x_0+2}+\frac{2\lambda}{(x_0+2)(x_0+3)}+\frac{3\lambda^2}{(x_0+2)(x_0+3)(x_0+4)}+\cdots\right\} \\
& \leq \frac{\mathbb{P}_\lambda(x_0)}{x_0+1}\left\{1+\frac{\lambda}{x_0+2}+\left(\frac{\lambda}{x_0+2}\right)^2+\cdots\right\} \\
(2.11) \qquad & = \frac{(x_0+2)\mathbb{P}_\lambda(x_0)}{(x_0+1)(x_0+2-\lambda)},
\end{aligned}
$$

and by (2.10) and (2.11), we can obtain

(2.12)

$$-\Delta f_{x_0}(x) \leq \frac{(\check{\lambda}+2)\mathbb{P}_\lambda(x_0)}{\check{\lambda}+2-\lambda}\min\left\{\frac{1}{\check{\lambda}+1},\frac{1}{x}\right\}.$$

Hence, the inequality (2.5) is obtained from (2.8) and (2.11) and the inequality (2.6) follows from (2.9) and (2.12).

2. The arguments derived in the proof of (2.6) lead also to the result in (2.7). □

The next lemma is obtained from Teerapabolarn [12].

Lemma 2.3. *For $x_0 \in \{0,1,...,n\}$, the following relation holds:*

$$\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} \leq e^{-\lambda}q^{-n}. \tag{2.13}$$

3 Main results

We now present the main results of the study, i.e. new new non-uniform and uniform bounds on two forms of the relative error of the binomial and Poisson cumulative distribution functions.

Theorem 3.1. *For $x_0 \in \{0,...,n\}$, the following inequality holds:*

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right| \leq \begin{cases} e^{-\lambda}q^{-n}\min\left\{1-e^\lambda q^n, \frac{2(e^\lambda-\lambda-1)}{n(x_0+1)}\right\} & \text{if } x_0 \leq \frac{n\lambda}{n+1}, \\ \frac{\Delta(x_0)}{x_0+1}\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^\lambda-\lambda-1)}{n}\right\} & \text{if } x_0 > \frac{n\lambda}{n+1}, \end{cases} \tag{3.1}$$

where $\Delta(x_0)$ is defined in (1.5).

Proof. For $x_0 \leq \frac{n\lambda}{n+1}$, by combining the inequalities in (1.2) and (2.13), we have that $0 < \frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1 \leq e^{-\lambda}q^{-n} - 1$ or $\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \leq e^{-\lambda}q^{-n} - 1$, which yields the first bound. For the second one, not that it is the result in (1.11) applied to considered x_0. Therefore, we obtain we obtain the first inequality of (3.1).

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right| \leq e^{-\lambda}q^{-n}\min\left\{1-e^\lambda q^n, \frac{2(e^\lambda-\lambda-1)}{n(x_0+1)}\right\}. \tag{3.2}$$

For $x_0 > \frac{n\lambda}{n+1}$, substituting x by X and taking expectation in (2.3) yields

$$\begin{aligned}\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0) &= E[\lambda f(X+1) - Xf(X)] \\ &= \lambda E[f(X+1)] - E[(X-\mu)f(X)] - \mu E[f(X)] \\ &= \lambda E[\Delta f(X)] - E[(X-\mu)f(X)],\end{aligned}$$

where $f = f_{x_0}$ is defined in (2.4). Because $E|w(X)\Delta f(X)| = E[w(X)|\Delta f(X)|] < \infty$, we have by (2.2),

$$\begin{aligned}|\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)| &= |\lambda E[\Delta f(X)] - \sigma^2\mathbb{E}[w(X)\Delta f(X)]| \\ &\le E\{|\lambda - \sigma^2 w(X)||\Delta f(X)|\} \\ &= E\{|\lambda - (n-X)p||\Delta f(X)|\} \quad \text{(by Lemma 2.1)} \qquad (3.3)\\ &\le \sup_{x\ge 1}|\Delta f(x)|\, E(X)p \\ &\le \frac{(x_0+2)\mathbb{P}_\lambda(x_0)\lambda p}{(x_0+1)(x_0+2-\lambda)} \quad \text{(by (2.5))} \qquad (3.4)\end{aligned}$$

and dividing the inequality (3.4) by $\mathbb{B}_{n,p}(x_0)$, we obtain

$$\begin{aligned}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| &\le \frac{(x_0+2)\mathbb{P}_\lambda(x_0)\lambda p}{(x_0+1)(x_0+2-\lambda)\mathbb{B}_{n,p}(x_0)} \\ &\le \frac{(x_0+2)\lambda p\Delta(x_0)}{(x_0+1)(x_0+2-\lambda)} \quad \text{(by (1.2))}, \qquad (3.5)\end{aligned}$$

which gives the first bound. The second one follows immediately from (1.11), that is, $\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \le \frac{2(e^\lambda-\lambda-1)\Delta(x_0)}{n(x_0+1)}$. Thus, we also obtain

$$\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \le \frac{\Delta(x_0)}{x_0+1}\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^\lambda-\lambda-1)}{n}\right\}. \qquad (3.6)$$

Hence, by (3.2) and (3.6), the inequality (3.1) holds. □

Corollary 3.1. *For $x_0 \in \{0, ..., n\}$, then*

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \le \begin{cases}\min\left\{1 - e^\lambda q^n, \frac{2(e^\lambda-\lambda-1)}{n(x_0+1)}\right\} & \text{if } x_0 \le \frac{n\lambda}{n+1}, \\ \frac{1}{x_0+1}\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^\lambda-\lambda-1)}{n}\right\} & \text{if } x_0 > \frac{n\lambda}{n+1}.\end{cases} \qquad (3.7)$$

Proof. If $x_0 \le \frac{n\lambda}{n+1}$, using the same inequalities in (1.2) and (2.13), we have that $0 < 1 - \frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} \le 1 - e^\lambda q^n$ or $\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \le 1 - e^\lambda q^n$. For $\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \le \frac{2(e^\lambda-\lambda-1)}{n(x_0+1)}$, which together with (1.12) gives the bounds in the first case of (3.7).

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \le \min\left\{1 - e^\lambda q^n, \frac{2(e^\lambda-\lambda-1)}{n(x_0+1)}\right\}. \qquad (3.8)$$

For $x_0 > \frac{n\lambda}{n+1}$, i.e. for the second case of (3.7), the inequality follows from dividing the inequality (3.4) by $\mathbb{P}_\lambda(x_0)$ and using (1.12).

$$\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \le \frac{1}{x_0+1} \min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^\lambda - \lambda - 1)}{n}\right\}. \tag{3.9}$$

Hence, the result in is obtained from (3.8) and (3.9). □

The following corollary is an immediately consequence of the Theorem 3.1 and Corollary 3.1.

Corollary 3.2. *We have the following relation:*

$$\sup_{0\le x_0 \le \frac{n\lambda}{n+1}} \left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \le e^{-\lambda}q^{-n} - 1 \tag{3.10}$$

and

$$\sup_{0\le x_0 \le \frac{n\lambda}{n+1}} \left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \le 1 - e^{\lambda}q^{n}. \tag{3.11}$$

Corollary 3.3. *The following inequalities hold:*

$$\sup_{\lambda\le x_0 \le n} \left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \le \begin{cases} \frac{e^\lambda-\lambda-1}{n} \min\left\{1, \frac{2(1-q^n)}{\lambda}\right\} & \text{if } \lambda \le 1, \\ \frac{(\hat\lambda+2)p}{\hat\lambda+2-\lambda} \min\left\{\frac{\lambda}{\hat\lambda+1}, 1-q^n\right\} & \text{if } \lambda > 1, \end{cases} \tag{3.12}$$

and

$$\sup_{\lambda\le x_0 \le n} \left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)} - 1\right| \le \begin{cases} \frac{e^\lambda-\lambda-1}{n} \min\left\{1, \frac{2(1-q^n)}{\lambda}\right\} & \text{if } \lambda \le 1, \\ \frac{(\hat\lambda+2)p}{\hat\lambda+2-\lambda} \min\left\{\frac{\lambda}{\hat\lambda+1}, 1-q^n\right\} & \text{if } \lambda > 1. \end{cases} \tag{3.13}$$

Proof. For $x_0 \ge \lambda$, the bound in (3.12) and (3.13) are the same bound. Then it suffices to show that (3.12) holds. For $\lambda \le 1$, Teerapabolarn [14] showed that

$$\sup_{\lambda\le x_0 \le n} \left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)} - 1\right| \le \frac{e^\lambda - \lambda - 1}{n} \min\left\{1, \frac{2(1-q^n)}{\lambda}\right\}. \tag{3.14}$$

For $\lambda > 1$, from (3.3), we have

$$\begin{aligned} |\mathbb{B}_{n,p}(x_0) - \mathbb{P}_\lambda(x_0)| &\le \sum_{x=1}^{n} xp|\Delta f(x)|p_X(x) \\ &\le \sum_{x=1}^{n} \frac{(\hat\lambda+2)\mathbb{P}_\lambda(x_0)xp_X(x)p}{\hat\lambda+2-\lambda} \min\left\{\frac{1}{\hat\lambda+1}, \frac{1}{x}\right\} \text{ (by (2.7))} \\ &= \frac{(\hat\lambda+2)\mathbb{P}_\lambda(x_0)p}{\hat\lambda+2-\lambda} \sum_{x=1}^{n} xp_X(x) \min\left\{\frac{1}{\hat\lambda+1}, \frac{1}{x}\right\} \\ &= \frac{(\hat\lambda+2)\mathbb{P}_\lambda(x_0)p}{\hat\lambda+2-\lambda} \min\left\{\frac{\lambda}{\hat\lambda+1}, 1-q^n\right\}. \end{aligned}$$

Dividing the last inequality by $\mathbb{B}_{n,p}(x_0)$, we obtain

$$\sup_{\lambda\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\frac{(\hat\lambda+2)p}{\hat\lambda+2-\lambda}\min\left\{\frac{\lambda}{\hat\lambda+1},1-q^n\right\}. \tag{3.15}$$

Therefore, from (3.14) and (3.15), the inequality (3.12) holds. □

Theorem 3.2. *For* $\delta(\lambda)=\begin{cases}e^{-\lambda}q^{-n} & \text{if } \check\lambda<\hat\lambda,\\ 1 & \text{if } \check\lambda=\hat\lambda.\end{cases}$, *we have*

1. *If* $\check\lambda=1$, *then*

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\max\left\{e^{-\lambda}q^{-n}-1,\frac{(e^\lambda-\lambda-1)\delta(\lambda)}{n}\times\min\left\{1,\frac{2(1-q^n)}{\lambda}\right\}\right\}. \tag{3.16}$$

2. *If* $\check\lambda>1$, *then*

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\max\left\{e^{-\lambda}q^{-n}-1,\frac{(\check\lambda+2)p\delta(\lambda)}{\check\lambda+2-\lambda}\times\min\left\{\frac{\lambda}{\check\lambda+1},1-q^n\right\}\right\}. \tag{3.17}$$

Proof. 1. The inequality (3.16) directly follows from the result in [14].

2. For $x_0>\frac{n\lambda}{n+1}$, using (2.6) and the arguments derived in the proof of (3.12), we have

$$|\mathbb{B}_{n,p}(x_0)-\mathbb{P}_\lambda(x_0)|\le\frac{(\check\lambda+2)\mathbb{P}_\lambda(x_0)p}{\check\lambda+2-\lambda}\min\left\{\frac{\lambda}{\check\lambda+1},1-q^n\right\}.$$

Dividing the inequality by $\mathbb{B}_{n,p}(x_0)$, we get

$$\begin{aligned}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|&\le\frac{(\check\lambda+2)p\mathbb{P}_\lambda(x_0)}{(\check\lambda+2-\lambda)\mathbb{B}_{n,p}(x_0)}\min\left\{\frac{\lambda}{\check\lambda+1},1-q^n\right\}\\&\le\frac{(\check\lambda+2)p\delta(\lambda)}{\check\lambda+2-\lambda}\min\left\{\frac{\lambda}{\check\lambda+1},1-q^n\right\}\end{aligned}$$

which gives

$$\sup_{\frac{n\lambda}{n+1}<x_0\le n}\left|\frac{\mathbb{P}_\lambda(x_0)}{\mathbb{B}_{n,p}(x_0)}-1\right|\le\frac{(\check\lambda+2)p\delta(\lambda)}{\check\lambda+2-\lambda}\min\left\{\frac{\lambda}{\check\lambda+1},1-q^n\right\}. \tag{3.18}$$

Therefore, by combining (3.10) and (3.18), we have (3.17). □

The following corollary is a consequence of the Theorem 3.2.

Corollary 3.4. *We have the following inequalities.*

1. *If* $\check{\lambda} = 1$, *then*

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right| \le \max\left\{1-e^{\lambda}q^n, \frac{e^{\lambda}-\lambda-1}{n}\min\left\{1, \frac{2(1-q^n)}{\lambda}\right\}\right\}. \tag{3.19}$$

2. *If* $\check{\lambda} > 1$, *then*

$$\sup_{0\le x_0\le n}\left|\frac{\mathbb{B}_{n,p}(x_0)}{\mathbb{P}_\lambda(x_0)}-1\right| \le \max\left\{1-e^{\lambda}q^n, \frac{(\check{\lambda}+2)p}{\check{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\}\right\}. \tag{3.20}$$

Remark. 1. Let us consider the results in Theorem 3.1, Corollary 3.1, Theorem 3.2 and Corollary 3.4. Note that, if p or λ is small, then all bounds presented in the study approach zero. It indicates that the results in approximating the binomial cumulative distribution function by the Poisson cumulative distribution function are more accurate when p or λ is small.

2. Because

$$\min\left\{1-e^{\lambda}q^n, \frac{2(e^{\lambda}-\lambda-1)}{n(x_0+1)}\right\} \le \frac{2(e^{\lambda}-\lambda-1)}{n(x_0+1)} \quad \text{for } x_0 \le \frac{n\lambda}{n+1}$$

and

$$\min\left\{\frac{(x_0+2)\lambda p}{x_0+2-\lambda}, \frac{2(e^{\lambda}-\lambda-1)}{n}\right\} \le \frac{2(e^{\lambda}-\lambda-1)}{n} \quad \text{for } x_0 > \frac{n\lambda}{n+1}$$

and

$$\begin{aligned}&\max\left\{1-e^{\lambda}q^n, \frac{(\check{\lambda}+2)p}{\check{\lambda}+2-\lambda}\min\left\{\frac{\lambda}{\check{\lambda}+1}, 1-q^n\right\}\right\}\\ &\quad\le \max\left\{1-e^{\lambda}q^n, \frac{e^{\lambda}-\lambda-1}{n}\min\left\{1, \frac{2(1-q^n)}{\lambda}\right\}\right\} \text{ for} \check{\lambda} > 1,\end{aligned}$$

hence the bounds given in Theorem 3.1 and Corollary 3.1 are sharper than those in (1.11) and (1.12), and for $\check{\lambda} > 1$, the bounds in Theorem 3.2 and Corollary 3.4 are sharper than the bounds in (1.9) and (1.10).

4 Numerical examples

This section presents some numerical examples of each result in approximating the binomial cumulative distribution function by a Poisson cumulative distribution function using Theorems 3.1 and Corollary 3.1 for non-uniform bounds and using Theorems 3.2 and Corollaries 3.2−3.4 for uniform bounds.

Example 4.1. Let $n = 100$ and $p = 0.01$, then $\lambda = 1.0$ and the numerical results are as follows.

• For non-uniform bounds, the numerical result of Theorem 3.1 is of the form

$$\left|\frac{\mathbb{P}_{1.0}(x_0)}{\mathbb{B}_{100,0.01}(x_0)}-1\right| \leq \begin{cases} 0.00504628 & \text{if } x_0=0, \\ 0.00718282 & \text{if } x_0=1, \\ \frac{0.01(x_0+2)}{(x_0+1)^2} & \text{if } x_0=2,...,100, \end{cases}$$

which is better than the numerical result obtained from (1.11),

$$\left|\frac{\mathbb{P}_{1.0}(x_0)}{\mathbb{B}_{100,0.01}(x_0)}-1\right| \leq \begin{cases} 0.01443813 & \text{if } x_0=0, \\ \frac{0.01436564}{x_0+1} & \text{if } x_0=1,...,100. \end{cases}$$

The numerical result of Corollary 3.1 is of the form

$$\left|\frac{\mathbb{B}_{100,0.01}(x_0)}{\mathbb{P}_{1.0}(x_0)}-1\right| \leq \begin{cases} 0.00504628 & \text{if } x_0=0, \\ 0.00718282 & \text{if } x_0=1, \\ \frac{0.01(x_0+2)}{(x_0+1)^2} & \text{if } x_0=2,...,100, \end{cases}$$

which is also better than the numerical result obtained from (1.12),

$$\left|\frac{\mathbb{B}_{100,0.01}(x_0)}{\mathbb{P}_{1.0}(x_0)}-1\right| \leq \frac{0.01436564}{x_0+1},\ x_0=0,1,...,100.$$

• For uniform bounds, the numerical results of Corollaries 3.2 and 3.3 are the following

$$\sup_{0\leq x_0\leq 0.99009901}\left|\frac{\mathbb{P}_{1.0}(x_0)}{\mathbb{B}_{100,0.01}(x_0)}-1\right| \leq 0.00504628,$$
$$\sup_{0\leq x_0\leq 0.99009901}\left|\frac{\mathbb{B}_{100,0.01}(x_0)}{\mathbb{P}_{1.0}(x_0)}-1\right| \leq 0.00502094,$$
$$\sup_{1.0\leq x_0\leq 100}\left|\frac{\mathbb{P}_{1.0}(x_0)}{\mathbb{B}_{100,0.01}(x_0)}-1\right| \leq 0.00718282$$

and

$$\sup_{1.0\leq x_0\leq 100}\left|\frac{\mathbb{B}_{100,0.01}(x_0)}{\mathbb{P}_{1.0}(x_0)}-1\right| \leq 0.00718282.$$

The numerical results of Theorem 3.2 and Corollary 3.4 are the following

$$\sup_{0\leq x_0\leq 100}\left|\frac{\mathbb{P}_{1.0}(x_0)}{\mathbb{B}_{100,0.01}(x_0)}-1\right| \leq 0.00718282,$$

which is the same numerical results obtained from (1.9) and (1.10).

Example 4.2. Let $n = 250$ and $p = 0.01$, then $\lambda = 2.5$ and the numerical results are as follows.

• For non-uniform bounds, the numerical result of Theorem 3.1 is of the form

$$\left|\frac{\mathbb{P}_{2.5}(x_0)}{\mathbb{B}_{250,0.01}(x_0)}-1\right| \le \begin{cases} 0.01266347 & \text{if } x_0=0,1,2, \\ \frac{0.025(x_0+2)}{(x_0-0.5)(x_0+1)} & \text{if } x_0=3,...,250, \end{cases}$$

which is better than the numerical result obtained from (1.11),

$$\left|\frac{\mathbb{P}_{2.5}(x_0)}{\mathbb{B}_{250,0.01}(x_0)}-1\right| \le \begin{cases} \frac{0.07033956}{x_0+1} & \text{if } x_0=0,1,2, \\ \frac{0.06945995}{x_0+1} & \text{if } x_0=3,...,250. \end{cases}$$

The numerical result of Corollary 3.1 is of the form

$$\left|\frac{\mathbb{B}_{250,0.01}(x_0)}{\mathbb{P}_{2.5}(x_0)}-1\right| \le \begin{cases} 0.01250512 & \text{if } x_0=0,1,2, \\ \frac{0.025(x_0+2)}{(x_0-0.5)(x_0+1)} & \text{if } x_0=3,...,250, \end{cases}$$

which is also better than the numerical result obtained from (1.12),

$$\left|\frac{\mathbb{B}_{250,0.01}(x_0)}{\mathbb{P}_{2.5}(x_0)}-1\right| \le \frac{0.06945995}{x_0+1},\ x_0=0,1,...,250.$$

• For uniform bounds, the numerical results of Corollaries 3.2 and 3.3 are the following

$$\sup_{0\le x_0\le 2.49003984}\left|\frac{\mathbb{P}_{2.5}(x_0)}{\mathbb{B}_{250,0.01}(x_0)}-1\right| \le 0.01266347,$$
$$\sup_{0\le x_0\le 2.49003984}\left|\frac{\mathbb{B}_{250,0.01}(x_0)}{\mathbb{P}_{2.5}(x_0)}-1\right| \le 0.01250512,$$
$$\sup_{2.5\le x_0\le 250}\left|\frac{\mathbb{P}_{2.5}(x_0)}{\mathbb{B}_{250,0.01}(x_0)}-1\right| \le 0.01250000$$

and

$$\sup_{2.5\le x_0\le 250}\left|\frac{\mathbb{B}_{250,0.01}(x_0)}{\mathbb{P}_{2.5}(x_0)}-1\right| \le 0.01250000.$$

The numerical result of Theorem 3.2 is of the form

$$\sup_{0\le x_0\le 250}\left|\frac{\mathbb{P}_{2.5}(x_0)}{\mathbb{B}_{250,0.01}(x_0)}-1\right| \le 0.01266347,$$

which is better than the numerical result obtained from (1.9),

$$\sup_{0\le x_0\le 250}\left|\frac{\mathbb{P}_{2.5}(x_0)}{\mathbb{B}_{250,0.01}(x_0)}-1\right| \le 0.02585517.$$

The numerical result of Corollary 3.4 is of the form

$$\sup_{0\le x_0\le 250}\left|\frac{\mathbb{B}_{250,0.01}(x_0)}{\mathbb{P}_{2.5}(x_0)}-1\right| \le 0.01250512,$$

which is also better than the numerical result obtained from (1.10),

$$\sup_{0\leq x_0\leq 250}\left|\frac{\mathbb{B}_{250,0.01}(x_0)}{\mathbb{P}_{2.5}(x_0)}-1\right|\leq 0.02553185.$$

Example 4.3. Let $n=1000$ and $p=0.005$, then $\lambda=5.0$ and the numerical results are as follows.

• For non-uniform bounds, the numerical result of Theorem 3.1 is of the form

$$\left|\frac{\mathbb{P}_{5.0}(x_0)}{\mathbb{B}_{1000,0.005}(x_0)}-1\right|\leq\begin{cases}0.01262080 & \text{if } x_0=0,1,2,3,4,\\ \frac{0.025(x_0+2)}{(x_0-3)(x_0+1)} & \text{if } x_0=5,...,1000,\end{cases}$$

which is better than the numerical result obtained from (1.11),

$$\left|\frac{\mathbb{P}_{5.0}(x_0)}{\mathbb{B}_{1000,0.005}(x_0)}-1\right|\leq\begin{cases}\frac{0.28842105}{x_0+1} & \text{if } x_0=0,1,2,3,4,\\ \frac{0.28482632}{x_0+1} & \text{if } x_0=5,...,1000.\end{cases}$$

The numerical result of Corollary 3.1 is of the form

$$\left|\frac{\mathbb{B}_{1000,0.005}(x_0)}{\mathbb{P}_{5.0}(x_0)}-1\right|\leq\begin{cases}0.01246350 & \text{if } x_0=0,1,2,3,4,\\ \frac{0.025(x_0+2)}{(x_0-3)(x_0+1)} & \text{if } x_0=5,...,1000,\end{cases}$$

which is also better than the numerical result obtained from (1.12),

$$\left|\frac{\mathbb{B}_{1000,0.005}(x_0)}{\mathbb{P}_{5.0}(x_0)}-1\right|\leq\frac{0.28482632}{x_0+1},\ x_0=0,1,...,1000.$$

• For uniform bounds, the numerical results of Corollaries 3.2 and 3.3 are the following

$$\sup_{0\leq x_0\leq 4.99500500}\left|\frac{\mathbb{P}_{5.0}(x_0)}{\mathbb{B}_{1000,0.005}(x_0)}-1\right|\leq 0.01262080,$$
$$\sup_{0\leq x_0\leq 4.99500500}\left|\frac{\mathbb{B}_{1000,0.005}(x_0)}{\mathbb{P}_{5.0}(x_0)}-1\right|\leq 0.01246350,$$
$$\sup_{5.0\leq x_0\leq 1000}\left|\frac{\mathbb{P}_{5.0}(x_0)}{\mathbb{B}_{1000,0.005}(x_0)}-1\right|\leq 0.01458333$$

and

$$\sup_{5.0\leq x_0\leq 1000}\left|\frac{\mathbb{B}_{1000,0.005}(x_0)}{\mathbb{P}_{5.0}(x_0)}-1\right|\leq 0.01458333.$$

The numerical result of Theorem 3.2 is of the form

$$\sup_{0\leq x_0\leq 1000}\left|\frac{\mathbb{P}_{5.0}(x_0)}{\mathbb{B}_{1000,0.005}(x_0)}-1\right|\leq 0.01458333,$$

which is better than the numerical result obtained from (1.9),

$$\sup_{0\le x_0\le 1000}\left|\frac{\mathbb{P}_{5.0}(x_0)}{\mathbb{B}_{1000,0.005}(x_0)}-1\right|\le 0.05730038.$$

The numerical result of Corollary 3.4 is of the form

$$\sup_{0\le x_0\le 1000}\left|\frac{\mathbb{B}_{1000,0.005}(x_0)}{\mathbb{P}_{5.0}(x_0)}-1\right|\le 0.01458333,$$

which is also better than the numerical result obtained from (1.10),

$$\sup_{0\le x_0\le 1000}\left|\frac{\mathbb{B}_{1000,0.005}(x_0)}{\mathbb{P}_{5.0}(x_0)}-1\right|\le 0.05658622.$$

Example 4.4. Let $n = 2000$ and $p = 0.005$, then $\lambda = 10.0$ and the numerical results are as follows.

• For non-uniform bounds, the numerical result of Theorem 3.1 is of the form

$$\left|\frac{\mathbb{P}_{10.0}(x_0)}{\mathbb{B}_{2000,0.005}(x_0)}-1\right|\le\begin{cases}0.02540089 & \text{if } x_0=0,1,...,9,\\ \frac{0.05(x_0+2)}{(x_0-8)(x_0+1)} & \text{if } x_0=10,...,2000,\end{cases}$$

which is better than the numerical result obtained from (1.11),

$$\left|\frac{\mathbb{P}_{10.0}(x_0)}{\mathbb{B}_{2000,0.005}(x_0)}-1\right|\le\begin{cases}\frac{22.57467819}{x_0+1} & \text{if } x_0=0,1,...,9,\\ \frac{22.01546579}{x_0+1} & \text{if } x_0=10,...,2000.\end{cases}$$

The numerical result of Corollary 3.1 is of the form

$$\left|\frac{\mathbb{B}_{2000,0.005}(x_0)}{\mathbb{P}_{10.0}(x_0)}-1\right|\le\begin{cases}0.02477167 & \text{if } x_0=0,1,...,9,\\ \frac{0.05(x_0+2)}{(x_0-8)(x_0+1)} & \text{if } x_0=10,...,2000,\end{cases}$$

which is also better than the numerical result obtained from (1.12),

$$\left|\frac{\mathbb{B}_{2000,0.005}(x_0)}{\mathbb{P}_{10.0}(x_0)}-1\right|\le\frac{22.01546579}{x_0+1},\ x_0=0,1,...,2000.$$

• For uniform bounds, the numerical results of Corollaries 3.2 and 3.3 are the following

$$\sup_{0\le x_0\le 9.99500250}\left|\frac{\mathbb{P}_{10.0}(x_0)}{\mathbb{B}_{2000,0.005}(x_0)}-1\right|\le 0.02540089,$$
$$\sup_{0\le x_0\le 9.99500250}\left|\frac{\mathbb{B}_{2000,0.005}(x_0)}{\mathbb{P}_{10.0}(x_0)}-1\right|\le 0.02477167,$$
$$\sup_{10.0\le x_0\le 2000}\left|\frac{\mathbb{P}_{10.0}(x_0)}{\mathbb{B}_{2000,0.005}(x_0)}-1\right|\le 0.02727273$$

and

$$\sup_{10.0 \leq x_0 \leq 2000} \left| \frac{\mathbb{B}_{2000,0.005}(x_0)}{\mathbb{P}_{10.0}(x_0)} - 1 \right| \leq 0.02727273.$$

The numerical result of Theorem 3.2 is of the form

$$\sup_{0 \leq x_0 \leq 2000} \left| \frac{\mathbb{P}_{10.0}(x_0)}{\mathbb{B}_{2000,0.005}(x_0)} - 1 \right| \leq 0.02727273,$$

which is better than the numerical result obtained from (1.9),

$$\sup_{0 \leq x_0 \leq 2000} \left| \frac{\mathbb{P}_{10.0}(x_0)}{\mathbb{B}_{2000,0.005}(x_0)} - 1 \right| \leq 2.25736787.$$

The numerical result of Corollary 3.4 is of the form

$$\sup_{0 \leq x_0 \leq 2000} \left| \frac{\mathbb{B}_{2000,0.005}(x_0)}{\mathbb{P}_{10.0}(x_0)} - 1 \right| \leq 0.02727273,$$

which is also better than the numerical result obtained from (1.10),

$$\sup_{0 \leq x_0 \leq 2000} \left| \frac{\mathbb{B}_{2000,0.005}(x_0)}{\mathbb{P}_{10.0}(x_0)} - 1 \right| \leq 2.20144911.$$

Considering the Examples 4.1−4.4, we see that the numerical results in Poisson approximation to binomial cumulative distribution are more accurate if p is small even with λ is relatively large. In addition, numerical comparison shows that the bounds in Theorem 3.1, Corollary 3.1, Theorem 3.2 and Corollary 3.4 are sharper than the corresponding bounds in (1.9), (1.10), (1.11) and (1.12).

5 Conclusion

In this study, the uniform and non-uniform bounds in Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.4 provide new general criteria for measuring the accuracy in approximating a binomial cumulative distribution with parameter n and p by the Poisson cumulative distribution with mean $\lambda = np$. With the bounds, it is pointed out that each result in the theorems and corollaries gives a good Poisson approximation when p is small, and all bounds obtained in this study are sharper than those reported in Teerapabolarn [14, 15], including both theoretical and numerical results.

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