Coefficient estimate of bi-univalent functions based on $q-$hypergeometric functions

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Abstract. In this paper, we introduce and investigate two new subclasses of the function class Σ of bi-univalent functions of complex order defined in the open unit disk, which are associated with $q-$hypergeometric functions, satisfying subordinate conditions. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Several consequences of the results are also mentioned.

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1 Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \mid z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$.

Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by $f^{-1}(f(z)) = z, \forall z \in \mathbb{U}$, and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

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An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [15] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{z\, f'(z)}{f(z)}$ or $1 + \frac{z\, f''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\mathbb{U}, \phi(0) = 1, \phi'(0) > 0$, and ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{z\, f'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{z\, f''(z)}{f'(z)} \prec \phi(z)$.

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\phi)$ and $\mathcal{K}_\Sigma(\phi)$.In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0).$$

The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where $f(z)$ is given by (1.1) and $h(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$. In terms of the Hadamard product (or convolution), the Dziok-Srivastava linear operator involving the generalized hypergeometric function, was introduced and studied systematically by Dziok and Srivastava [7, 6] and (subsequently) by many other authors.

Euler, Gauss, Riemann and of course many others are few to mention who have used or introduced the notion of hypergeometric in the last centuries. The applications evolving in many different subjects ranging from combinatorics and numerical analysis to dynamical systems and mathematical physics have motivated many researchers to study the behaviour and the properties of the functions. Basically, $q-$ hypergeometric functions are the generalized form of of the classical hypergeometric functions in the sense that by taking the (formal) limit $q \to 1$, it will return to the classical hypergeometric setting. Many of the results for the classical hypergeometric functions can be generalized to the q- hypergeometric level. More recently, Purohit and Raina [18] introduced a generalized $q-$Taylor's formula in fractional $q-$calculus and derived certain $q-$ generating functions for $q-$ hypergeometric functions. Other results related to q-hypergeometric functions can be found in [27]. In this work we proceed to derive a generalized differential operator in the unit disk involving these functions and discuss some of its properties.

For complex parameters $a_1, \ldots, a_l$ and $b_1, \ldots, b_m$ $(b_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *q-hypergeometric function* ${}_l\Psi_m(z)$ is defined by

$$
\begin{aligned}
&{}_l\Psi_m(a_1, \ldots a_l; b_1, \ldots, b_m; q, z) \\
&:= \sum_{n=0}^{\infty} \frac{(a_1, q)_n \ldots (a_l, q)_n}{(q, q)_n (b_1, q)_n \ldots (b_m, q)_n} \left[(-1)^n \, q^{\left(\begin{array}{c} n \\ 2 \end{array} \right)} \right]^{1+m-l} z^n
\end{aligned}
\tag{1.3}
$$

with $\left(\begin{array}{c} n \\ 2 \end{array} \right) = \frac{n(n-1)}{2}$ where $q \neq 0$ when $l > m + 1 (l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathbb{U}$.

The $q-$ shifted factorial is defined for $a, q \in \mathbb{C}$ as a product of n factors by

$$
(a; q)_n = \left\{ \begin{array}{ll} 1 & n = 0 \\ (1-a)(1-aq) \ldots (1-aq^{n-1}) & n \in \mathbb{N} \end{array} \right\}
$$

and in terms of basic analogue of the gamma function

$$
(q^a; q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, n > 0.
$$

The $q-$ derivative of functions f defined on the subset of $\mathbb{C}$ is given by

$$
D_{q,z} f(z) = \frac{f(z) - f(zq)}{z(1-q)}, (z \neq 0, q \neq 1).
$$

It is of interest to note that $\lim_{q \to 1^-} \frac{(q^a; q)_n}{(1-q)^n} = (a)_n = a(a+1)...(a+n-1)$ the familiar Pochhammer symbol and

$$
{}_l\Psi_m(a_1, \ldots a_l; b_1, \ldots, b_m; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_l)_n}{(b_1)_n \ldots (b_m)_n} \frac{z^n}{n!}.
$$

Now for $z \in \mathbb{U}, 0 < |q| < 1$, and $l = m + 1$, the basic hypergeometric function defined in (1.3) takes the form

$$
{}_l\Psi_m(a_1; \ldots a_l; b_1, \ldots, b_m; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, q)_n \ldots (a_l, q)_n}{(q, q)_n (b_1, q)_n \ldots (b_m, q)_n} z^n,
$$

which converges absolutely in the open unit disk $\mathbb{U}$.

We define a new linear operator for $z \in \mathbb{U}$, $|q| < 1$, and $l = m + 1$, as follows:

$$
\begin{aligned}
\mathcal{I}(a_l, b_m; q) f(z) &= {}_l\Psi_m(a_1, \ldots a_l; b_1, \ldots, b_m; q, z) * f(z) \\
&= z + \sum_{n=2}^{\infty} \tfrac{(a_1, q)_{n-1} \ldots (a_l, q)_{n-1}}{(q, q)_{n-1} (b_1, q)_{n-1} \ldots (b_m, q)_{n-1}} a_n z^n.
\end{aligned}
$$

Shortly, we let

$$
\mathcal{I} f(z) = z + \sum_{n=2}^{\infty} \varphi_n a_n z^n
$$

where $\varphi_n = \dfrac{(a_1,q)_{n-1}\dots(a_l,q)_{n-1}}{(q,q)_{n-1}(b_1,q)_{n-1}\dots(b_m,q)_{n-1}}$, unless otherwise stated.

The operator $\mathcal{I}(a_l,b_m;q)f(z)$ was studied recently by Mohammed and Darus [16]. For $a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}$, and $\beta_j \neq 0,-1,-2,...,(i=1,...,l, j=1,..,m)$ and $q \to 1$, we obtain the well-known Dziok-Srivastava linear operator [7, 6] (for $l=m+1$). For $l=1, m=0, a_1=q$, many (well known and new) integral and differential operators can be obtained by specializing the parameters, for example the operators introduced in [1, 5, 11, 12, 21].

Recently, a study on bi-univalent function class Σ has increased. A number of articles discussing on non-sharp coefficient estimates for the first two coefficients $|a_2|$ and $|a_3|$ (eq. 1.1). But the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N}\setminus\{1,2\};\ \ \mathbb{N} := \{1,2,3,\cdots\}$$

is still an open problem (see[2, 3, 4, 13, 17, 24]). Many researchers (see[9, 10, 14, 23, 25, 26]) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

Motivated by the earlier work of Deniz[8] (see[20, 22]), in the present paper we introduce new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C}\backslash\{0\}$,involving $q-$hypergeometric functions $\mathcal{I}f(z)$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of function class Σ. Several related classes are also considered, and connections to earlier known results are made.

Definition 1.1. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{S}^q_\Sigma(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(\mathcal{I}f(z))'}{(1-\lambda)z+\lambda\mathcal{I}f(z)}-1\right) \prec \phi(z) \qquad (\gamma \in \mathbb{C}\backslash\{0\};\ 0 \leqq \lambda \leqq 1;\ z \in \mathbb{U}) \tag{1.4}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathcal{I}g(w))'}{(1-\lambda)w+\lambda\mathcal{I}g(w)}-1\right) \prec \phi(w) \qquad (\gamma \in \mathbb{C}\backslash\{0\};\ 0 \leqq \lambda \leqq 1;\ w \in \mathbb{U}), \tag{1.5}$$

where the function g is given by(1.2).

On specializing the parameters λ and a, b, c, one can state the various new subclasses of Σ as illustrated in the following examples.

Example 1.2. For $\lambda = 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{S}^q_\Sigma(\gamma,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(\mathcal{I}f(z))'}{\mathcal{I}f(z)}-1\right) \prec \phi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathcal{I}g(w))'}{\mathcal{I}g(w)}-1\right) \prec \phi(w),$$

where $z, w \in \mathbb{U}$ and the function g is given by(1.2).

Example 1.3. For $\lambda = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{G}_{\Sigma}^{q}(\gamma, \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left((\mathcal{I}f(z))' - 1\right) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma}\left((\mathcal{I}g(w))' - 1\right) \prec \phi(w),$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

It is of interest to note that for $(a) = (b)$ and $q \to 1$, the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \phi)$ reduces to the following new subclasses:

Example 1.4. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\right) \prec \phi(z) \qquad (\gamma \in \mathbb{C}\backslash\{0\};\ 0 \leqq \lambda \leqq 1; \quad z \in \mathbb{U})$$

and

$$1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)w + \lambda g(w)} - 1\right) \prec \phi(w) \qquad (\gamma \in \mathbb{C}\backslash\{0\};\ 0 \leqq \lambda \leqq 1; \quad w \in \mathbb{U}),$$

where the function g is given by(1.2).

Example 1.5. For $\lambda = 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{S}_{\Sigma}^{*}(\gamma, \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z) \quad \textit{and} \quad 1 + \frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)} - 1\right) \prec \phi(w),$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

Example 1.6. For $\lambda = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{*}(\gamma, \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(f'(z) - 1\right) \prec \phi(z) \quad \textit{and} \quad 1 + \frac{1}{\gamma}\left(g'(w) - 1\right) \prec \phi(w),$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

In the following section we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \phi)$ of the function class Σ by employing the techniques used earlier by Deniz [8].

In order to derive our main results, we shall need the following lemma.

Lemma 1.1. (see [19]) *If $h \in \mathcal{P}$, then $|c_k| \leqq 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\Re\{h(z)\} > 0 \qquad (z \in \mathbb{U}),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

2 Coefficient bounds for the function class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda)$

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \phi)$. Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots$$

or, equivalently,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right].$$

Then $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$. Since $u, v : \mathbb{U} \to \mathbb{U}$, the functions $p(z)$ and $q(z)$ have a positive real part in $\mathbb{U}$, and $|p_i| \leq 2$ and $|q_i| \leq 2$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{S}_{\Sigma}^{q}(\gamma, \lambda, \phi)$. Then*

$$|a_2| \leqq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|[\gamma(\lambda^2 - 2\lambda)B_1^2 + (2-\lambda)^2(B_1 - B_2)]\varphi_2^2 + \gamma(3-\lambda)B_1^2\varphi_3|}} \tag{2.1}$$

and

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{(2-\lambda)^2\varphi_2^2} + \frac{|\gamma| B_1}{(3-\lambda)\varphi_3}.$$

Proof. It follows from (1.4) and (1.5) that

$$1 + \frac{1}{\gamma}\left(\frac{z\big(\mathcal{I}f(z)\big)'}{(1-\lambda)z + \lambda \mathcal{I}f(z)} - 1\right) = \phi(u(z)) \tag{2.2}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w\big(\mathcal{I}g(w)\big)'}{(1-\lambda)w + \lambda \mathcal{I}g(w)} - 1\right) = \phi(v(w)), \tag{2.3}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the following forms:

$$\phi(u(z)) = \phi\left(\tfrac{1}{2}\left[p_1 z + \left(p_2 - \tfrac{p_1^2}{2}\right) z^2 + \cdots\right]\right)$$
$$\phi(v(w)) = \phi\left(\tfrac{1}{2}\left[q_1 w + \left(q_2 - \tfrac{q_1^2}{2}\right) w^2 + \cdots\right]\right).,$$

respectively. Now, equating the coefficients in (2.2) and (2.3), we get

$$\frac{(2-\lambda)}{\gamma}\varphi_2 a_2 = \frac{1}{2} B_1 p_1, \tag{2.4}$$

$$\frac{(\lambda^2-2\lambda)}{\gamma}\varphi_2^2a_2^2+\frac{(3-\lambda)}{\gamma}\varphi_3a_3=\frac{1}{2}B_1(p_2-\frac{p_1^2}{2})+\frac{1}{4}B_2p_1^2, \tag{2.5}$$

$$-\frac{(2-\lambda)}{\gamma}\varphi_2a_2=\frac{1}{2}B_1q_1 \tag{2.6}$$

and

$$\frac{(\lambda^2-2\lambda)}{\gamma}\varphi_2^2a_2^2+\frac{(3-\lambda)}{\gamma}\varphi_3(2a_2^2-a_3)=\frac{1}{2}B_1(q_2-\frac{q_1^2}{2})+\frac{1}{4}B_2q_1^2. \tag{2.7}$$

From (2.4) and (2.6), we find that

$$a_2=\frac{\gamma B_1p_1}{2(2-\lambda)\varphi_2}=\frac{-\gamma B_1q_1}{2(2-\lambda)\varphi_2}, \tag{2.8}$$

which implies

$$p_1=-q_1. \tag{2.9}$$

and

$$8(2-\lambda)^2\varphi_2^2a_2^2=\gamma^2B_1^2(p_1^2+q_1^2).$$

By adding (2.5) and (2.7), by using (2.8) and (2.9), we obtain

$$4\left([\gamma(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3\right)a_2^2=\gamma^2B_1^3(p_2+q_2).$$

Thus,

$$a_2^2=\frac{\gamma^2B_1^3(p_2+q_2)}{4\left([\gamma(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3\right)}.$$

By applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2\leqq\frac{|\gamma|^2B_1^3}{[\gamma(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3}.$$

Hence,

$$|a_2|\leqq\frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|[\gamma(\lambda^2-2\lambda)B_1^2+(2-\lambda)^2(B_1-B_2)]\varphi_2^2+\gamma(3-\lambda)B_1^2\varphi_3|}}.$$

This gives the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.7) from (2.5), we get

$$\frac{2(3-\lambda)}{\gamma}\varphi_3a_3-\frac{2(3-\lambda)}{\gamma}\varphi_3a_2^2=\frac{B_1}{2}(p_2-q_2)+\frac{B_2-B_1}{4}(p_1^2-q_1^2). \tag{2.10}$$

It follows from (2.8), (2.9) and (2.10) that

$$a_3=\frac{|\gamma|^2B_1^2(p_1^2+q_1^2)}{8(2-\lambda)^2\varphi_2^2}+\frac{\gamma B_1(p_2-q_2)}{4(3-\lambda)\varphi_3}.$$

By applying Lemma 1.1 once again for the coefficients p_2 and q_2, we readily get

$$|a_3|\leqq\frac{|\gamma|^2B_1^2}{(2-\lambda)^2\varphi_2^2}+\frac{|\gamma|B_1}{(3-\lambda)\varphi_3}.$$

This completes the proof of Theorem 2.1. □

By putting $\lambda = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{S}^q_\Sigma(\gamma, \phi)$. *Then*

$$|a_2| \leqq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|[(B_1 - B_2) - \gamma B_1^2]\varphi_2^2 + 2\gamma B_1^2 \varphi_3|}}$$

and

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{\varphi_2^2} + \frac{|\gamma| B_1}{2\varphi_3}.$$

Taking $a = c$ and $b = 1$, in Corollary 2.2, we get the following corollary.

Corollary 2.3. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{S}^*_\Sigma(\gamma, \phi)$. *Then*

$$|a_2| \leqq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|[(B_1 - B_2) - \gamma B_1^2] + 2\gamma B_1^2|}}$$

and

$$|a_3| \leqq |\gamma|^2 B_1^2 + \frac{|\gamma| B_1}{2}.$$

By putting $\lambda = 0$ in Theorem 2.1, we have the following

Corollary 2.4. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{G}^q_\Sigma(\gamma, \phi)$. *Then*

$$|a_2| \leqq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|4(B_1 - B_2)\varphi_2^2 + 3\gamma B_1^2 \varphi_3|}}$$

and

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{4\varphi_2^2} + \frac{|\gamma| B_1}{3\varphi_3}.$$

Corollary 2.5. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{S}_\Sigma(\gamma, \lambda, \phi)$. *Then*

$$|a_2| \leqq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|[\gamma(\lambda^2 - 3\lambda + 3)B_1^2 + (2 - \lambda)^2 (B_1 - B_2)]|}}$$

and

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{(2 - \lambda)^2} + \frac{|\gamma| B_1}{(3 - \lambda)}.$$

Corollary 2.6. *Let the function* $f(z)$ *given by* (1.1) *be in the class* $\mathcal{H}^*_\Sigma(\gamma, \phi)$. *Then*

$$|a_2| \leqq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|4(B_1 - B_2) + 3\gamma B_1^2|}}$$

and

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{4} + \frac{|\gamma| B_1}{3}.$$

3 Concluding remarks

For the class of strongly starlike functions, the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \quad (0 < \alpha \leq 1), \tag{3.1}$$

which gives $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$.

On the other hand if we take

$$\phi(z) = \frac{1 + (1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots \quad (0 \leq \beta < 1), \tag{3.2}$$

then $B_1 = B_2 = 2(1-\beta)$.

Remark 3.1. From Corollary 2.3, when $\gamma = 1$, and $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$ for the class $\mathcal{S}^*_\Sigma(\alpha)$[4], we get

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}} \qquad \text{and} \qquad |a_3| \leq 4\alpha^2 + \alpha.$$

and for $B_1 = B_2 = 2(1-\beta)$

$$|a_2| \leq \sqrt{2(1-\beta)} \qquad \text{and} \qquad |a_3| \leq 4(1-\beta)^2 + (1-\beta).$$

Similarly, we can prove the results obtained in the earlier works of Srivastava et al. which we mention them the following remarks.

Remark 3.2. From Corollary 2.6, by taking $\gamma = 1$, when and $\phi(z)$ is given by (3.1) or of the form (3.2) we obtain the corresponding results given earlier by Srivastava et al [23].

Remark 3.3. From Theorem 2.1, by taking $\gamma = 1$, when and $\phi(z)$ is given by (3.1) or of the form (3.2) we obtain the corresponding results given earlier by Srivastava et al [22].

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