On (m, n)-quasi-ideals in LA-semigroups

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Abstract. Within the framework of the theory of (m, n)-quasi-ideals in LA-semigroups, we investigate the relations satisfied by (m, n)-quasi-ideals in regular LA-semigroups.

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1 Introduction

The concept of (m, n)-quasi-ideal of a semigroup was introduced by R.Chinaram [8]. The notion of left almost semigroup (LA-semigroup) was first introduced by Kazin and Naseerudin [2]. In the present note, we define and study the (m, n)-ideals of an LA-semigroup. We discuss a well some properties of (m, n)-quasi-ideals and investigate the relations of (m, n)-quasi-ideals in regular LA-semigroups.

2 Preliminaries and basic definitions

Definition 2.1. [2, p.2188] A groupoid $(S, \cdot)$ is called an *LA-semigroup* or an *AG-groupoid*, if its satisfies left inversion law

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a, \quad \text{for all} \quad a, b, c \in S.$$

Lemma 2.1. *[4, p.1] In an LA-semigroup, a subset S satisfies the* medial law *if*

$$(ab)(cd) = (ac)(bd), \quad \textit{for all} \quad a, b, c, d \in S.$$

Definition 2.2. [6, p.1759] An element $e \in S$ is called *left identity* if $ea = a$ for all $a \in S$.

Lemma 2.2. *[2, p.2188] If S is an LA-semigroup with left identity, then*

$$a(bc) = b(ac), \quad \textit{for all} \quad a, b, c \in S.$$

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Lemma 2.3. *[4, p.1] An LA-semigroup S with left identity satisfies the* paramedial law *if*

$$(ab)(cd) = (dc)(ba), \quad \textit{for all} \quad a, b, c, d \in S.$$

Definition 2.3. [2, p.2188] An LA-semigroup S is called a *locally associative* LA-semigroup if its satisfies

$$(aa)a = a(aa), \quad \text{for all} \quad a \in S.$$

If A and B are any subsets of a locally associative LA-semigroup S by Lemmas 2.1 and 2.3 infers that $(AB)^n = A^n B^n$ for $n \geq 1$.

In [2, p. 2188], the authors define powers of an element in a locally associative LA-semigroup S as follows: $a^1 = a, a^{n+1} = a^n a$, for $n \geq 1$. In a locally associative LA-semigroup S with left identity, the results $a^m a^n = a^{m+n}, (a^m)^n = a^{mn}$ and $(ab)^n = a^n b^n$ hold for all $a, b \in S$, where m and n are positive integers.

Definition 2.4. [3, p. 2] a) Let S be an LA-semigroup. A non-empty subset A of S is called an *LA-subsemigroup of S* if $AA \subseteq A$.

b) A non-empty subset A of an LA-semigroup S is called a *left (right) ideal* of S if $SA \subseteq A (AS \subseteq A)$. As usual, A is called an *ideal* if it is both left and right ideal.

c) A non-empty subset A of an LA-semigroup S is called a *quasi-ideal* of S if $SA \cap AS \subseteq A$.

3 (m, n)-quasi-ideals in LA-semigroups

In this section we define and study (m, n)-quasi-ideals of an LA-semigroup in a similar manner to (m, n)-quasi-ideals of semigroups.

Definition 3.1. A subset A of an LA-semigroup S is called an $(m, n)-$*quasi-ideal* of S if $S^m A \cap AS^n \subseteq A$, where m and n are positive integers.

Lemma 3.1. *Let S be an LA-semigroup and let T_i be an LA-subsemigroup of S for all $i \in I$. If $\bigcap_{i \in I} T_i \neq \varnothing$, then $\bigcap_{i \in I} T_i$ is an LA-subsemigroup.*

Proof. Assume that $\bigcap_{i \in I} T_i \neq \varnothing$. Let $a, b \in \bigcap_{i \in I} T_i$ for all $i \in I$. Since T_i is an LA-subsemigroup for all $i \in I$, we have $ab \in T_i$ for all $i \in I$. Hence $ab \in \bigcap_{i \in I} T_i$. Thus $\bigcap_{i \in I} T_i$ is an LA-subsemigroup. □

Theorem 3.2. *Let S be an LA-semigroup and let Q_i be an (m, n)-quasi-ideal of S for all $i \in I$. If $\bigcap_{i \in I} Q_i \neq \varnothing$, then $\bigcap_{i \in I} Q_i$ is an (m, n)-quasi-ideal.*

Proof. Assume that $\bigcap_{i \in I} Q_i \neq \varnothing$. By Lemma 3.1, we have that $\bigcap_{i \in I} Q_i$ is an LA-semigroup of S. Further, let $c \in S^m(\bigcap_{i \in I} Q_i) \cap (\bigcap_{i \in I} Q_i)S^n$. Then $c = xq = py$ for some $x \in S^m, y \in S^n$ and $p, q \in \bigcap_{i \in I} Q_i$. Thus $p, q \in Q_i$ for all $i \in I$. So $c \in S^m Q_i \cap Q_i S^n$, for all $i \in I$. Since Q_i is an (m, n)-quasi ideal of S for all $i \in I$, we have $c \in Q_i$ for all $i \in I$. Thus $c \in \bigcap_{i \in I} Q_i$. Hence $\bigcap_{i \in I} Q_i$ is an (m, n)-quasi-ideal of S. □

Definition 3.2. [2, p.2189] A subset A of an LA-semigroup S is called an $(m, 0)$-*ideal*($(0, n)$-ideal) of S if $SA^m \subseteq A(A^n S \subseteq A)$ for $m, n \in \mathbb{N}$.

Lemma 3.3. *Let S be an LA-semigroup and $a \in S$. Then the following statements hold true:*

a) $S^m a$ is an m-left ideal of S.

b) aS^n is an n-left ideal of S.

c) $S^m a \cap aS^n$ is am (m,n)-quasi-ideal.

Proof. a) We have $(S^m a)(S^m a) \subseteq S^m a$ and $S^m(S^m a) \subseteq S^m a$. Then (1) holds.

b) We have $(aS^n)(aS^n) \subseteq aS^n$ and $(aS^n a)S^n \subseteq aS^n$. Then (2) holds.

c) We have $(S^m a \cap aS^n)(S^m a \cap aS^n) \subseteq S^m a \cap aS^n$ and $S^m(S^m a \cap aS^n)S^n \subseteq S^m a \cap aS^n$. Then (3) holds. □

Theorem 3.4. *Let S be an LA-semigroup. The following statements are true:*

a) Let L_i be an m-left ideal of S for all $i \in I$. If $\bigcap_{i\in I} L_i \neq \varnothing$, then $\bigcap_{i\in I} L_i$ is m-left ideal of S.

b) Let R_i be an n-right ideal of S for all $i \in I$. If $\bigcap_{i\in I} R_i \neq \varnothing$, then $\bigcap_{i\in I} R_i$ is n-right ideal of S.

Proof. Since L_i is an m-left ideal of S for all $i \in I$, we have $S^m L_i \subseteq L_i$. We will show that $\bigcap_{i\in I} L_i$ is m-left ideal of S. Assume that $\bigcap_{i\in I} L_i \neq \varnothing$. Let $c\ inS^m \bigcap_{i\in I} L_i$; then $c \in S^m$ and $c \in \bigcap_{i\in I} L_i$. Thus $S^m(\bigcap_{i\in I} L_i) \subseteq \bigcap_{i\in I} L_i$. Hence $\bigcap_{i\in I} L_i$ is an m-left of S. In a similar way one can show that $\bigcap_{i\in I} R_i$ is an n-right ideal of S □

Lemma 3.5. *Let S be an LA-semigroup The following statements are true:*

a) Every m-left ideal of S is an (m,n)-quasi-ideal of S.

b) Every n-right ideal of S is an (m,n)-quasi-ideal of S.

Proof. a) Let A be an m-left ideal of S; then $S^m A \subseteq A$ and $A \subseteq S$. By considering

$$S^m A \cap AS^n \subseteq S^m A \subseteq A.$$

we infer that A is an (m,n)-quasi ideal of S.

b) Let B be an n-right ideal of S; then $BS^n \subseteq B$ and $B \subseteq S$. By considering

$$BS^n \cap BS^n \subseteq BS^m \subseteq B.$$

we infer that B is an (m,n)-quasi-ideal of S. □

Theorem 3.6. *Let S be an LA-semigroup and let a A be an m-left ideal and B a right-ideal of S. Then $A \cap B$ is an (m,n)-quasi-ideal of S.*

Proof. By properties of A and B, we have $A^m B^n \subseteq S^m A \cap BS^n \subseteq A \cap B$. Thus $A \cap B$ is non-empty. By Lemma 3.1, we get that $A \cap B$ is a LA-subsemigroup of S.

We further show that $A \cap B$ is an (m,n)-quasi-ideal of S. Since A is an m-left ideal and B is an right-ideal of S, we have $S^m A \subseteq A$ and $AS^n \subseteq A$. By considering

$$(S^m(A \cap B)) \cap ((A \cap B)S^n \subseteq S^m A \cap BS^n \subseteq A \cap B.$$

we get that $A \cap B$ is an (m,n)-quasi-ideal of S. □

Theorem 3.7. *Every (m,n)-quasi-ideal Q of an LA-semigroup S is the intersection of some m-left ideal and some n-right ideal of S.*

Proof. Let Q be an (m,n)-quasi ideal of S. Let $L = Q \cap S^mQ$ and $R = Q \cap QS^n$. In order to show that L is an LA-semigroup of S, we consider $a, b \in L$.

Case 1: $a, b \in Q$. Since Q is an LA-semigroup of S, we have $ab \in Q \subseteq L$.
Case 2: $a \in Q$ and $b \in S^mQ$. Then $ab \in QS^mQ \subseteq S^mQ \subseteq L$.
Case 3: $a \in S^mQ$ and $b \in Q$. Then $ab \in S^mQ^2 \subseteq S^mQ \subseteq L$.
Case 4: $a \in S^mQ$ and $b \in S^mQ$. Then $ab \in S^mQS^mQ \subseteq S^mQ \subseteq L$.

Then L is an LA-subsemigroup of S. Next, we have

$$S^mL = S^m(Q \cap S^mQ) = S^mQ \cap S^{2m}Q \subseteq S^mQ \subseteq L.$$

Hence L is an m-left ideal of S. Similarly, R is an n-right ideal of S. Since $S^mQ \cap QS^n \subseteq Q$, we have $L \cap R = (Q \cap S^mQ) \cap (Q \cap QS^n) = Q \cap (S^mQ \cap QS^n) = Q$, which infers $Q = L \cap R$. □

We further study the relation of (m,n)-quasi-ideals in regular LA-semigroups.

Definition 3.3. [4, p.11] An element a of an LA-semigroup S is called a *regular* element of S, if there exists $x \in S$ such that $a = (ax)a$; S is called *regular* if all its elements are regular.

Now we will state and prove the intersection property of regular LA-semigroups with (m,n)-quasi-ideals.

Lemma 3.8. *Let S be an locally associative LA-semigroup. If S is regular and $\varnothing \neq Q \subseteq S$, then the following statements hold:*

a) $Q \subseteq S^mQ$ where $m \in \mathbb{Z}^+$.
b) $Q \subseteq QS^n$ where $n \in \mathbb{Z}^+$.

Proof. a) Let $P(n)$ be the statement $Q \subseteq S^mQ$, where $m \in \mathbb{Z}^+$, and let $x \in Q$. Then there exists $y \in S$ such that $x = (xy)x$, since S is a regular LA-semigroup. Thus $(xy)x \in SQ$. So $x \in SQ$. Therefore $Q \subseteq SQ$. Hence $P(1)$ holds true.

We further show that $P(k+1)$ holds true. Let $P(k)$ hold true for all $k \in \mathbb{Z}^+$. Then $Q \subseteq S^kQ$. Since S is a locally associative LA-semigroup, we have $SQ \subseteq S(S^kQ) = (SS^k)Q = S^{k+1}Q$. So $Q \subseteq S^{k+1}Q$. Therefore $P(k+1)$ is true. Hence $Q \subseteq SQ$ where $m \in \mathbb{Z}^+$.

b) Let $P(n)$ be the statement $Q \subseteq QS^n$, where $m \in \mathbb{Z}^+$, and let $x \in Q$. Then there exists $y \in S$ such that $x = x(yx)$, since S is a regular LA-semigroup. Thus $x(yx) \in QS$. So $x \in SQ$. Therefore $Q \subseteq QS$. Hence $P(1)$ holds true. Now we show that $P(k+1)$ is true. To this aim, let $P(k)$ be true for all $k \in \mathbb{Z}^+$. Then $Q \subseteq S^kQ$. Since S is a locally associative LA-semigroup, we have $QS \subseteq (QS)S^k = Q(SS^k) = QS^{k+1}$. So $Q \subseteq QS^{k+1}$, and therefore $P(k+1)$ is true. Hence $Q \subseteq QS^n$, where $n \in \mathbb{Z}^+$. □

Theorem 3.9. *Let S be an locally associative LA-semigroup. Then every regular LA-semigroup has the intersection property of (m,n)-quasi-ideals for any positive integers $m, n \in N$.*

Proof. Let Q be an (m,n)-quasi-ideal of a regular LA-semigroup S. Lemma 3.8 infers $Q \subseteq S^n$, and thus $Q \cap QS^n = QS^n$. Hence $S^m Q \cap (Q \cap S^n) = S^m Q \cap QS^n$. Therefore Q has the intersection property. □

Theorem 3.10. *Let S be a locally associative LA-semigroup and let S be a regular LA-semigroup. Then a non-empty subset A of S is an (m,n)-quasi-ideal of S if and only if it is the intersection of an m-left ideal and an n-right ideal.*

Proof. ($\Rightarrow$) Let A be an (m,n)-quasi ideal of S; then $S^m A \cap AS^n \subseteq A$. Lemma 3.5 infers that $S^m A$ is an m-left ideal and that AS^n is an n-right ideal of S. By Lemma 3.8, we have $A \subseteq S^m A$ and $A \subseteq AS^n$. Then $A \subseteq S^m A \cap AS^n$. Hence $A = S^m A \cap AS^n$. Therefore A is the intersection of m-left ideal and an n-right ideal.

($\Leftarrow$) Let A be an intersection of an m-left ideal and an n-right ideal. By Theorem 3.6, we get that A is an (m,n)-quasi-ideal of S. □

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