Magnetic geometric dynamics around one infinite rectilinear circuit

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Abstract. In this paper we compare the trajectories of particles described by the Lorentz Law and the trajectories of geometric dynamics generated by the magnetic flow around an infinite rectilinear circuit.

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1 Magnetic field around an electric circuit

We consider the space $\mathbb{R}^3$ endowed with the standard scalar product $< \cdot, \cdot >$, the induced Euclidean norm $||\cdot||$ and the vector product $\times$. An electric wire is represented by a smooth curve $\alpha : [a, b] \subset \mathbb{R} \to \alpha([a, b]) \subset \mathbb{R}^3$ and a constant current intensity J. Then magnetic field B created by the electric circuit (α, J), at the point $q = (x, y, z) \in \mathbb{R}^3 \setminus \alpha([a, b])$, is given by the Biot-Savart law

$$B(q) = \frac{\mu_0 J}{4\pi} \int_a^b \frac{\dot{\alpha}(t) \times (q - \alpha(t))}{||q - \alpha(t)||^3}\, dt,$$

where μ_0 is the permeability constant. The magnetic vector field B does not depend on the parametrization of the circuit. We take $\frac{\mu_0}{4\pi} = 1$ for simplicity.

Theory of magnetic fields generated around the piecewise rectilinear circuits is developed in the papers and books of C. Udrişte [27-32]. Meanwhile the magnetic flow theory around piecewise rectilinear circuits was taken in the works [1-26], not mentioning the original versions in [27-32].

1.1 Magnetic field around Oz

Particularly, we consider the case when the electric circuit is rectilinear, and it is no loss in generality supposing that the curve α is just the axis $OZ : x = 0, y = 0, z = t$. Fixing the Cartesian frame $\{i, j, k\}$, we find the magnetic field

$$B(q) = \frac{2(-yi + xj)}{x^2 + y^2}. \tag{1.1}$$

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This magnetic vector field has the symmetries $X_1 = \frac{\partial}{\partial z}$ (translation) and $X_2 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ (rotation), and its integral curves (field lines) are circles around the Oz-axis.

2 Motion in the magnetic field of an infinite rectilinear wire

The equations of motion of a (non-relativistic) unit-mass, unit-charge particle in the presence of a magnetic field B are given by the Lorentz Law

$$\ddot{q}(t) = \dot{q}(t) \times B(q),$$

where the dot over $q(t)$, as usual, is the time derivative. We remark that the previous (magnetic) Lorentz force exerted on a charged particle by a magnetic field is always perpendicular to its instantaneous direction of motion. Also, a trajectory in magnetic dynamics has constant speed, i.e., $||\dot{q}(t)|| = c$. Hence, the curvature of a trajectory is

$$k = \frac{||\dot{q} \times \ddot{q}||}{||\dot{q}||^3} = \frac{|| < \dot{q}, B > \dot{q} - c^2 B||}{c^3}.$$

Similarly, we can compute the torsion of a trajectory.

Generally, a charged particle placed in a magnetic field executes a circular orbit in the plane perpendicular to the direction of the field. But this is not the most general motion of a charged particle. Indeed, we can also add an arbitrary drift along the direction of the magnetic field. This follows because the force acting on the particle only depends on the component of the particle's velocity which is perpendicular to the direction of magnetic field. The combination of circular motion in the plane perpendicular to the magnetic field, and uniform motion along the direction of the field, gives rise to a spiral trajectory (a curve that turns around an axis at a constant or continuously varying distance while moving parallel to the axis) of a charged particle in a magnetic field, where the field forms the axis of the spiral (charged particles spiral back and forth along field lines). This statement can be confirmed taking $\tau(t)$ as a vector collinear to the velocity $\dot{q}(t)$; then $< \ddot{q}(t), \tau(t) >= 0$.

The Lorentz Law is represented also by Euler-Lagrange ODEs associated to the Lagrangian

$$L_1 = \frac{1}{2}\, ||\dot{q}(t)||^2 + < \dot{q}(t), A(q(t)) >,$$

where A is the *vector potential*, i.e., $\operatorname{rot} A = B$. This gives the Hamiltonian (first integral of movement)

$$H_1 = \frac{1}{2}\, ||\dot{q}(t)||^2.$$

2.1 Lorenz law for the magnetic field around Oz

Suppose the magnetic field has the formula (1.1). Since $\dot{q} = (\dot{x}, \dot{y}, \dot{z})$ and $\ddot{q} = (\ddot{x}, \ddot{y}, \ddot{z})$, the Lorentz Law is transformed into the ODE system

$$\ddot{x}(t) = -\frac{2x(t)}{x(t)^2 + y(t)^2}\, \dot{z}(t)$$

$$\ddot{y}(t) = -\frac{2y(t)}{x(t)^2 + y(t)^2}\,\dot{z}(t)$$
$$\ddot{z}(t) = \frac{2x(t)}{x(t)^2 + y(t)^2}\,\dot{x}(t) + \frac{2y(t)}{x(t)^2 + y(t)^2}\,\dot{y}(t).$$

3 Magnetic geometric dynamics around a rectilinear infinite circuit

The Geometric Dynamics, the dynamics generated by a flow and a Riemannian metric, was introduced and studied by C. Udriste ([27-28], [32]). The key findings on magnetic geometric dynamics are included in [29-31].

Let A be the *vector potential* of the magnetic field B, i.e., $\operatorname{rot} A = B$. The magnetic flow of the vector field $-A$, i.e.,

$$\dot{q}(t) = -A(q(t))$$

and the Euclidean norm determine the associated least squares Lagrangian

$$L_2 = \frac{1}{2}\,||\dot{q}(t) + A(q(t))||^2.$$

Let $f = \frac{1}{2}||A||^2$ be the magnetic energy density. Then the vector Euler-Lagrange equation is

$$\ddot{q}(t) = \dot{q}(t) \times \operatorname{rot} A(q(t)) + \nabla f(q(t)) \tag{3.1}$$

or

$$\ddot{q}(t) = \dot{q}(t) \times B(q(t)) + \nabla f(q(t)).$$

Proposition *Along each trajectory in magnetic geometric dynamics, we have*

$$||\dot{q}(t)||^2 = f(q(t)) + c.$$

Proof Multiplying the relation (3.1), in the scalar sense, by $\dot{q}$, we find

$$< \ddot{q}, \dot{q} >=< \nabla f, \dot{q} > .$$

Consequently,

$$\frac{d}{dt}||\dot{q}(t)||^2 = \frac{d}{dt}f(q(t)).$$

The relation from previous Proposition can be written in three ways

$$||\dot{q}(t)||^2 = ||A(q(t))||^2,$$
$$||\dot{q}(t)||^2 = ||A(q(t))||^2 + k^2,\ \ ||\dot{q}(t)||^2 = ||A(q(t))||^2 - \ell^2,$$

which reflect either collinearity, or orthogonality (Pythagora's Theorem).

The associated Hamiltonian (first integral of the movement)

$$H_2(q, \dot{q}) = \frac{1}{2}\,||\dot{q}(t)||^2 - f(q(t))$$

shows that the trajectories in geometric dynamics splits in three categories: (i) magnetic field lines $H_2(q, \dot{q}) = 0$; (ii) trajectories for which $H_2(q, \dot{q}) > 0$ (transversal to field lines); (iii) trajectories for which $H_2(q, \dot{q}) < 0$ (transversal to field lines).

3.1 Magnetic geometric dynamics of the magnetic field around the Oz axis

Suppose the vector potential A has the components $(0, 0, -\ln(x^2 + y^2))$ and the magnetic vector field is given by the formula (2). Then $f_A = \frac{1}{2}\ln^2(x^2 + y^2)$ and the magnetic geometric dynamics is described by the ODE system

$$\ddot{x}(t) = -\frac{2x(t)}{x(t)^2 + y(t)^2}\left(\dot{z}(t) - \ln(x(t)^2 + y(t)^2)\right)$$

$$\ddot{y}(t) = -\frac{2y(t)}{x(t)^2 + y(t)^2}\left(\dot{z}(t) - \ln(x(t)^2 + y(t)^2)\right)$$

$$\ddot{z}(t) = \frac{2x(t)}{x(t)^2 + y(t)^2}\,\dot{x}(t) + \frac{2y(t)}{x(t)^2 + y(t)^2}\,\dot{y}(t).$$

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