

Confluent Heun functions and the Coulomb problem for spin $1/2$ particle in Minkowski space

V. Balan, A.M. Manukyan, E.M. Ovsiyuk, V.M. Red'kov and O.V. Veko

Abstract. The quantum mechanical problem for a spin $1/2$ particle in external Coulomb potential, reduced to a system of two first-order differential equations, is reconsidered from the point of view of solving this system by using the Heun function theory. It is shown that, besides the standard approach of solving the problem in terms of confluent hypergeometric functions, there are several other possibilities, which rely on using the confluent Heun functions. We consider two new methods to construct the solutions of the problem: the first implies that only one component of the pair of relevant functions is expressed in terms of the Heun functions, and in the second approach both functions of the system are expressed in terms of the Heun functions. In this context, certain relations between the two classes of involved functions are established. It is shown that *all* the considered cases lead to the same energy spectrum, which validates the correctness of the approaches.

M.S.C. 2010: 33E30, 34B30.

Key words: Quantum mechanics; Dirac equation; Coulomb problem; confluent Heun functions; confluent hypergeometric functions.

1 Introduction

The general Heun equation is a second-order linear differential equation which has four regular singularities and takes different confluent forms [28, 64, 67]. This equation and all its confluent forms turn out to be of primary significance in physical applications, for instance, in quantum mechanics and in field theory on the background of curved space-time models, and in optics (see the extensive reference list [1]–[74]). A comprehensive list of references can be found on the *the Heun Project website* [75].

In this paper, the well-known quantum mechanical problem of a spin $1/2$ particle in external Coulomb potential, reduced to a system of two first-order differential equations, is studied using the Heun function theory for solving this system. We show that, besides the standard way [5] of solving the problem in terms of confluent hypergeometric functions, there exist several other possibilities, which rely on applying

confluent Heun functions. We consider two new methods to construct the solutions of the problem: in the first one, only one component of the pair of relevant functions is expressed in terms of the Heun functions, while in the second approach both functions of the system are expressed in terms of the Heun functions. Both approaches lead to a unique energy spectrum, which confirms their correctness.

2 The Coulomb problem: solutions constructed by both hypergeometric and Heun functions

In a spherical space endowed with the metric $dS^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$, a diagonal tetrad has the form [61, 53]

$$(2.1) \quad e_{(0)}^\alpha = (1, 0, 0, 0), \quad e_{(1)}^\alpha = (0, 0, r^{-1}, 0), \quad e_{(2)}^\alpha = (0, 0, 0, r^{-1} \sin^{-1} \theta), \quad e_{(3)}^\alpha = (0, 1, 0, 0).$$

By using the substitution $\Psi = \frac{1}{r} \tilde{\Psi}$, the covariant Dirac equation¹

$$\left[i\gamma^c \left(e_{(c)}^\alpha \partial_\alpha + \frac{1}{2} j^{ab} \gamma_{abc} \right) - m \right] \Psi = 0$$

can equivalently be written as

$$\left(i\gamma^0 \frac{\partial}{\partial t} + i\gamma^3 \frac{\partial}{\partial r} + \frac{1}{r} \Sigma_{\theta\phi} - m \right) \tilde{\Psi} = 0, \quad \text{where} \quad \Sigma_{\theta,\phi} = i\gamma^1 \partial_\theta + \gamma^2 \frac{i\partial_\phi + i\sigma^{12}}{\sin \theta}.$$

In order to diagonalize the operators $i\partial_t$, \vec{J}^2 , and J_3 , one takes the wave function of the form [61]

$$\tilde{\Psi} = e^{-iEt} \begin{pmatrix} f_1(r) D_{-1/2} \\ f_2(r) D_{+1/2} \\ f_3(r) D_{-1/2} \\ f_4(r) D_{+1/2} \end{pmatrix},$$

where $D_\sigma = D_{-m,\sigma}^j(\phi, \theta, 0)$ are the Wigner functions [71]. The separation of variables leads to four *radial equations*²

$$(2.2) \quad \begin{cases} E f_3 - i \frac{d}{dr} f_3 - i \frac{\nu}{\sin r} f_4 - m f_1 = 0, & E f_4 + i \frac{d}{dr} f_4 + i \frac{\nu}{\sin r} f_3 - m f_2 = 0, \\ E f_1 + i \frac{d}{dr} f_1 + i \frac{\nu}{\sin r} f_2 - m f_3 = 0, & E f_2 - i \frac{d}{dr} f_2 - i \frac{\nu}{\sin r} f_1 - m f_4 = 0. \end{cases}$$

In a spherical tetrad, the space reflection operator is given by [61]

$$\hat{\Pi}_{sph} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \otimes \hat{P}.$$

The spectral equations $\hat{\Pi}_{sph} \Psi_{jm} = \Pi \Psi_{jm}$ lead to

$$\Pi = \delta (-1)^{j+1}, \quad \text{where } \delta = \pm 1; \quad f_4 = \delta f_1, \quad f_3 = \delta f_2,$$

¹We further use the terminology and notations from [61, 53].

²We consider here the case $\nu = j + 1/2$.

which brings (2.2) to the simplified form

$$(2.3) \quad \left(\frac{d}{dr} + \frac{\nu}{r}\right) f + (E + \delta m) g = 0, \quad \left(\frac{d}{dr} - \frac{\nu}{r}\right) g - (E - \delta m) f = 0,$$

where we performed the change of variables $\{f_1, f_2\} \rightsquigarrow \{f, g\}$ given by

$$f = (f_1 + f_2)/\sqrt{2}, \quad g = (f_1 - f_2)/i\sqrt{2}.$$

For definiteness, in the particular case when $\delta = 1$, the equations (2.3) become

$$(2.4) \quad \left(\frac{d}{dr} + \frac{\nu}{r}\right) f + (E + m) g = 0, \quad \left(\frac{d}{dr} - \frac{\nu}{r}\right) g - (E - m) f = 0,$$

and we note that by performing the replacement $m \rightsquigarrow -m$, we obtain the equations for $\delta = -1$.

The presence of the external Coulomb field is taken into account in (2.4) by the formal change $\varepsilon \rightsquigarrow \varepsilon + e/r$. Thus, the quantum Coulomb problem for a Dirac particle is described by the following radial system

$$(2.5) \quad \left(\frac{d}{dr} + \frac{\nu}{r}\right) f + \left(E + \frac{e}{r} + m\right) g = 0, \quad \left(\frac{d}{dr} - \frac{\nu}{r}\right) g - \left(E + \frac{e}{r} - m\right) f = 0.$$

We further perform the following linear transformation³ of the functions $\{f(r), g(r)\} \rightsquigarrow \{F(r), G(r)\}$

$$(2.6) \quad \begin{cases} f(r) = aF(r) + cG(r), \\ g(r) = dF(r) + bG(r) \end{cases} \Leftrightarrow \begin{cases} F(r) = bf(r) - cg(r), \\ G(r) = -df(r) + ag(r). \end{cases}$$

By linearly combining the equations (2.5) rewritten in terms of F and G with the coefficients $b, -c$ and then with $-d, a$, we get the system

$$(2.7) \quad \begin{cases} \left[\frac{d}{dr} - b'a + c'd + \frac{\nu}{r}(ba + cd) + \left(E + \frac{e}{r} + m\right)bd + \left(E + \frac{e}{r} - m\right)ca \right] F \\ \quad = \left[b'c - bc' - \frac{\nu}{r}2bc - \left(E + \frac{e}{r} + m\right)b^2 - \left(E + \frac{e}{r} - m\right)c^2 \right] G \\ \left[\frac{d}{dr} + d'c - a'b - \frac{\nu}{r}(dc + ab) - \left(E + \frac{e}{r} + m\right)bd - \left(E + \frac{e}{r} - m\right)ca \right] G \\ \quad = \left[-d'a + da' + \frac{\nu}{r}2ad + \left(E + \frac{e}{r} + m\right)d^2 + \left(E + \frac{e}{r} - m\right)a^2 \right] F. \end{cases}$$

To simplify their form, let us assume that the transformation (2.6) does not depend on r , and that it is orthogonal, i.e., the coefficients of the linear combinations have the particular form:

$$S := \begin{pmatrix} a & c \\ d & b \end{pmatrix} = \begin{pmatrix} \cos A/2 & \sin A/2 \\ -\sin A/2 & \cos A/2 \end{pmatrix},$$

Then (2.7) become

$$(2.8) \quad \begin{cases} \left(\frac{d}{dr} + \frac{\nu}{r} \cos A - m \sin A\right) F = \left(-\frac{\nu}{r} \sin A - \frac{e}{r} - E - m \cos A\right) G, \\ \left(\frac{d}{dr} - \frac{\nu}{r} \cos A + m \sin A\right) G = \left(-\frac{\nu}{r} \sin A + \frac{e}{r} + E - m \cos A\right) F. \end{cases}$$

³The coefficients of this transformation may depend on the radial variable; we assume that its determinant satisfies the identity $a(r)b(r) - c(r)d(r) = 1$.

There exist four possibilities (of which only two are distinct):

$$\begin{aligned}
1) \quad -\frac{\nu}{r} \sin A + \frac{e}{r} = 0 &\Rightarrow \begin{cases} \sin A = \frac{e}{\nu}, \cos A = \sqrt{1 - e^2/\nu^2}, \\ \cos \frac{A}{2} = \sqrt{\frac{\nu + \sqrt{\nu^2 - e^2}}{2\nu}}, \sin \frac{A}{2} = \sqrt{\frac{\nu - \sqrt{\nu^2 - e^2}}{2\nu}}. \end{cases} \\
1') \quad -\frac{\nu}{r} \sin A - \frac{e}{r} = 0 &\Rightarrow \begin{cases} \sin A = -\frac{e}{\nu}, \cos A = \sqrt{1 - e^2/\nu^2}, \\ \cos \frac{A}{2} = \sqrt{\frac{\nu - \sqrt{\nu^2 - e^2}}{2\nu}}, \sin \frac{A}{2} = \sqrt{\frac{\nu + \sqrt{\nu^2 - e^2}}{2\nu}}. \end{cases} \\
2) \quad E - m \cos A = 0 &\Rightarrow \begin{cases} \cos A = +\frac{E}{m}, \sin A = \sqrt{1 - E^2/m^2}, \\ \cos \frac{A}{2} = \sqrt{\frac{m+E}{2m}}, \sin \frac{A}{2} = \sqrt{\frac{m-E}{2m}}. \end{cases} \\
2') \quad -E - m \cos A = 0 &\Rightarrow \begin{cases} \cos A = -\frac{E}{m}, \sin A = \sqrt{1 - E^2/m^2}, \\ \cos \frac{A}{2} = \sqrt{\frac{m-E}{2m}}, \sin \frac{A}{2} = \sqrt{\frac{m+E}{2m}}. \end{cases}
\end{aligned}$$

We first consider the case 1); then the system (2.8) takes the form

$$(2.9) \quad \begin{cases} \left(\frac{d}{dr} + \frac{\nu}{r} \cos A - m \sin A \right) F = \left(-\frac{2e}{r} - E - m \cos A \right) G, \\ \left(\frac{d}{dr} - \frac{\nu}{r} \cos A + m \sin A \right) G = (E - m \cos A) F. \end{cases}$$

After eliminating the function F , we get the following second order equation for G

$$\left(\frac{d^2}{dr^2} + E^2 - m^2 + \frac{\nu \cos A - \nu^2 \cos^2 A}{r^2} + \frac{2eE - 2em \cos A + 2m\nu \sin A \cos A}{r} \right) G = 0.$$

By using the identity $\sin A = e/\nu$, this equation reduces to

$$\left(\frac{d^2}{dr^2} + E^2 - m^2 + \frac{\nu \cos A - \nu^2 \cos^2 A}{r^2} + \frac{2eE}{r} \right) G = 0,$$

and after changing the variable, $r \rightsquigarrow x = 2\sqrt{m^2 - E^2} r$, it becomes

$$\frac{d^2 G}{dx^2} + \left(-\frac{1}{4} - \frac{\nu \cos A (\nu \cos A - 1)}{x^2} + \frac{eE}{\sqrt{m^2 - E^2} x} \right) G = 0.$$

Further, by the use of the substitution $G(x) = x^a e^{bx} \bar{G}(x)$, for \bar{G} we get

$$x \frac{d^2 \bar{G}}{dx^2} + (2a + 2bx) \frac{d\bar{G}}{dx} + \left[\left(b^2 - \frac{1}{4} \right) x + \sigma + 2ab + \frac{eE}{\sqrt{m^2 - E^2}} \right] \bar{G} = 0.$$

where $\sigma = \frac{a^2 - a - \nu \cos A (\nu \cos A - 1)}{x}$. For $a = +\nu \cos A = \sqrt{\nu^2 - e^2}$ and $b = -\frac{1}{2}$, this equation reduces to

$$(2.10) \quad x \frac{d^2 \bar{G}}{dx^2} + (2a - x) \frac{d\bar{G}}{dx} - \left(a - \frac{eE}{\sqrt{m^2 - E^2}} \right) \bar{G} = 0,$$

which is a *confluent hypergeometric equation* of the form

$$(2.11) \quad xF''(x) + (\gamma - x)F'(x) - \alpha F(x) = 0, \text{ where } \alpha = a - \frac{eE}{\sqrt{m^2 - E^2}}, \gamma = 2a.$$

The known condition for (2.11) to have polynomial solutions, $\alpha = -n$ with $n = 0, 1, 2, \dots$, gives the energy quantization rule

$$(2.12) \quad E = \frac{m}{\sqrt{1 + e^2/(n + \sqrt{\nu^2 - e^2})^2}}.$$

Moreover, from (2.9) it follows a second-order equation for F

$$\left[\frac{d^2}{dr^2} + \left(\frac{1}{r} - \frac{1}{r-R} \right) \frac{d}{dr} + E^2 - m^2 + \frac{2eE}{r} + \frac{e^2 - \nu^2}{r^2} + \frac{mR \sin A - \nu \cos A}{r(r-R)} \right] F = 0,$$

where we denoted $-\frac{2e}{E+m \cos A} = R$. After changing the variable $r \rightsquigarrow y = r/R$, the equation reads

$$\begin{aligned} \frac{d^2 F}{dy^2} + \left(\frac{1}{y} - \frac{1}{y-1} \right) \frac{dF}{dy} + \left((E^2 - m^2) R^2 - \frac{\nu^2 - e^2}{y^2} \right. \\ \left. + \frac{-\nu \cos A + mR \sin A}{y-1} + \frac{2eRE - mR \sin A + \nu \cos A}{y} \right) F = 0. \end{aligned}$$

We further search for solutions of the form $F = y^a e^{by} \bar{F}(y)$. Then the function \bar{F} obeys

$$\begin{aligned} \frac{d^2 \bar{F}}{dy^2} + \left(\frac{2a+1}{y} + 2b - \frac{1}{y-1} \right) \frac{d\bar{F}}{dy} + \left[b^2 + (E^2 - m^2) R^2 + \frac{a^2 - \nu^2 + e^2}{y^2} - \frac{a+b+\nu \cos A - mR \sin A}{y-1} \right. \\ \left. + \frac{a+b+2ab+R(2eE-m \sin A)+\nu \cos A}{y} \right] \bar{F} = 0. \end{aligned}$$

For a, b chosen ⁴ as: $a \in \{+\sqrt{\nu^2 - e^2}, -\sqrt{\nu^2 - e^2}\}$, $b \in \{+\sqrt{m^2 - E^2} R, -\sqrt{m^2 - E^2} R\}$, the above equation becomes simpler,

$$\begin{aligned} \frac{d^2 \bar{F}}{dy^2} + \left(2b + \frac{2a+1}{y} - \frac{1}{y-1} \right) \frac{d\bar{F}}{dy} + \left(\frac{a+b+2ab+2eRE-mR \sin A+\nu \cos A}{y} \right. \\ \left. - \frac{a+b+\nu \cos A-mR \sin A}{y-1} \right) \bar{F} = 0. \end{aligned}$$

This can be easily recognized as a confluent Heun equation for $H(\alpha, \beta, \gamma, \delta, \eta, z)$,

$$\begin{aligned} H'' + \left(\alpha + \frac{1+\beta}{z} + \frac{1+\gamma}{z-1} \right) H' + \left(\frac{1}{2} \frac{\alpha+\alpha\beta-\beta-\beta\gamma-\gamma-2\eta}{z} \right. \\ \left. + \frac{1}{2} \frac{\alpha+\alpha\gamma+\beta+\beta\gamma+\gamma+2\delta+2\eta}{z-1} \right) H = 0 \end{aligned}$$

with the parameters

$$\alpha = 2b, \quad \beta = 2a, \quad \gamma = -2, \quad \delta = 2eER, \quad \eta = 1 + mR \sin A - 2eER - \nu \cos A.$$

We note that the known condition [64] determining the so called *transcendental confluent Heun functions*:

$$\delta = - \left(n + \frac{\beta + \gamma + 2}{2} \right) \alpha, \quad n = 0, 1, 2, \dots,$$

leads to the energy quantization rule

$$a = +\sqrt{\nu^2 - e^2}, \quad b = -\sqrt{m^2 - E^2} R, \quad eER = (n + \sqrt{\nu^2 - e^2}) \sqrt{m^2 - E^2} R,$$

whence we infer

$$E = \frac{m}{\sqrt{1 + e^2/(n + \sqrt{\nu^2 - e^2})^2}},$$

which coincides with the known formula for energy levels (2.12).

It should be emphasized that, as follows from (2.9), the function F (being constructed in terms of the confluent Heun functions) can be related to the function G (which is determined in terms of confluent hypergeometric functions) by means of the following differential operators:

$$G = \left(-\frac{2e}{r} - E - m \cos A \right)^{-1} \left(\frac{d}{dr} + \frac{\nu}{r} \cos A - m \sin A \right) F,$$

⁴We shall select and further use the underlined values.

and

$$F = \frac{1}{(E - m \cos A)} \left(\frac{d}{dr} - \frac{\nu}{r} \cos A + m \sin A \right) G.$$

Let us examine now the case 2) described on page 12. The equations (2.8) take the form

$$(2.13) \quad \begin{cases} \left(\frac{d}{dr} + \frac{\nu}{r} \cos A - m \sin A \right) F = \left(-\frac{\nu \sin A + e}{r} - 2m \cos A \right) G, \\ \left(\frac{d}{dr} - \frac{\nu}{r} \cos A + m \sin A \right) G = \frac{e - \nu \sin A}{r} F. \end{cases}$$

Taking into account the identity $\cos A = E/m$, we obtain the following second-order equation for $G(r)$:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + E^2 - m^2 + \frac{e^2 - \nu^2}{r^2} + \frac{2eE}{r} + \frac{\sqrt{m^2 - E^2}}{r} \right) G = 0.$$

and after performing the change of variables $r \rightsquigarrow x = 2\sqrt{m^2 - E^2}r$, we get

$$\frac{d^2 G}{dx^2} + \frac{1}{x} \frac{dG}{dx} + \left(-\frac{1}{4} - \frac{\nu^2 - e^2}{x^2} + \frac{1}{2} \frac{m^2 - E^2 + 2Ee\sqrt{m^2 - E^2}}{(m^2 - E^2)x} \right) G = 0.$$

Let $G(x) = x^a e^{bx} \bar{G}(x)$; then the function \bar{G} satisfies

$$x \frac{d^2 \bar{G}}{dx^2} + (2a + 1 + 2bx) \frac{d\bar{G}}{dx} + \left[\left(b^2 - \frac{1}{4} \right) x + \frac{a^2 - \nu^2 + e^2}{x} + 2ab + b + \frac{1}{2} \frac{m^2 - E^2 + 2Ee\sqrt{m^2 - E^2}}{m^2 - E^2} \right] \bar{G} = 0.$$

For $a = \sqrt{\nu^2 - e^2}$ and $b = -\frac{1}{2}$, we get

$$x \frac{d^2 \varphi}{dx^2} + (2a + 1 - x) \frac{d\varphi}{dx} - \left(a - \frac{Ee}{\sqrt{m^2 - E^2}} \right) \varphi = 0,$$

which is a *confluent hypergeometric equation*

$$(2.14) \quad xF'' + (\gamma - x)F' - \alpha F = 0, \quad \text{with } \alpha = a - \frac{Ee}{\sqrt{m^2 - E^2}}, \quad \gamma = 2a + 1.$$

The equation admits polynomial solutions if $\alpha = -n$, $n = 0, 1, 2, \dots$; this provides the energy spectrum (2.12). Moreover, (2.13) leads to a second-order equation for $F(r)$,

$$\left[\frac{d^2}{dr^2} + \left(\frac{1}{r} - \frac{1}{r-D} \right) \frac{d}{dr} + \frac{m \sin A}{r-D} - \frac{\nu \cos A}{D} \left(\frac{1}{r-D} - \frac{1}{r} \right) + E^2 - m^2 + \frac{e^2 - \nu^2}{r^2} + \frac{2eE - m \sin A}{r} \right] F = 0,$$

where $D = -\frac{e + \nu \sin A}{2m \cos A}$. Relative to the variable $y = r/D$, the equation looks simpler,

$$\frac{d^2 F}{dy^2} + \left[\frac{1}{y} - \frac{1}{y-1} \right] \frac{dF}{dy} + \left[(E^2 - m^2) D^2 - \frac{\nu^2 - e^2}{y^2} + \frac{-\nu \cos A + mD \sin A}{y-1} + \frac{D(2eE - m \sin A) + \nu \cos A}{y} \right] F = 0.$$

Let $F = y^a e^{by} \bar{F}(y)$; the function \bar{F} satisfies

$$\frac{d^2 \bar{F}}{dy^2} + \left(\frac{2a+1}{y} + 2b - \frac{1}{y-1} \right) \frac{d\bar{F}}{dy} + \left[b^2 + (E^2 - m^2) D^2 + \frac{a^2 - \nu^2 + e^2}{y^2} - \frac{a+b+\nu \cos A - mD \sin A}{y-1} + \frac{a+b+2ab+D(2eE - m \sin A) + \nu \cos A}{y} \right] \bar{F} = 0.$$

For a, b chosen ⁵ as: $a \in \{+\sqrt{\nu^2 - e^2}, -\sqrt{\nu^2 - e^2}\}$, $b \in \{+\sqrt{m^2 - E^2} D, -\sqrt{m^2 - E^2} D\}$, the equation becomes

$$\frac{d^2 \bar{F}}{dy^2} + \left(2b + \frac{2a+1}{y} - \frac{1}{y-1} \right) \frac{d\bar{F}}{dy} + \left(\frac{a+b+2ab+D(2eE-m \sin A)+\nu \cos A}{y} - \frac{a+b+\nu \cos A-mD \sin A}{y-1} \right) \bar{F} = 0,$$

which is a confluent Heun equation for $H(\alpha, \beta, \gamma, \delta, \eta, y)$

$$H'' + \left(\alpha + \frac{1+\beta}{y} + \frac{1+\gamma}{y-1} \right) H' + \left(\frac{1}{2} \frac{\alpha+\alpha\beta-\beta-\beta\gamma-\gamma-2\eta}{y} + \frac{1}{2} \frac{\alpha+\alpha\gamma+\beta+\beta\gamma+\gamma+2\delta+2\eta}{y-1} \right) G = 0,$$

with the parameters

$$\alpha = 2b, \quad \beta = 2a, \quad \gamma = -2, \quad \delta = 2eED, \quad \eta = 1 + mD \sin A - 2eED - \nu \cos A.$$

By imposing restriction [64] defining the transcendental confluent Heun functions: $\delta = -(n + \frac{\beta+\gamma+2}{2}) \alpha$, for $n = 0, 1, 2, \dots$, we produce the energy quantization rule

$$a = +\sqrt{\nu^2 - e^2}, \quad b = -\sqrt{m^2 - E^2} D, \quad eED = (n + \sqrt{\nu^2 - e^2}) \sqrt{m^2 - E^2} D,$$

whence the energy E given by (2.12) is immediately inferred.

It should be noted that though the confluent Heun equations from the cases 1) and 2) formally coincide, their parameters still essentially differ, namely:

- 1) $\alpha = 2b, \quad \beta = 2a, \quad \gamma = -2, \quad \delta = 2eER, \quad \eta = 1 + mR \sin A - 2eER - \nu \cos A;$
- 2) $\alpha = 2b, \quad \beta = 2a, \quad \gamma = -2, \quad \delta = 2eED, \quad \eta = 1 + mD \sin A - 2eED - \nu \cos A,$

where

- 1) $R = -\frac{2e}{E+m \cos A}, \quad \sin A = \frac{e}{\nu}, \quad \cos A = \sqrt{1 - \frac{e^2}{\nu^2}};$
- 2) $D = -\frac{e+\nu \sin A}{2E}, \quad \cos A = \frac{E}{m}, \quad \sin A = \sqrt{1 - \frac{E^2}{m^2}}.$

3 The standard approach to the Coulomb problem

It should be emphasized that the two proposed approaches to the Coulomb problem for the Dirac equation differ from the well known, established one⁶. We briefly recall this standard approach. To this end, in the original radial system (2.5)

$$\begin{cases} \left(\frac{d}{dr} + \frac{\nu}{r} \right) f + \left(E + \frac{e}{r} + m \right) g = 0, \\ \left(\frac{d}{dr} - \frac{\nu}{r} \right) g - \left(E + \frac{e}{r} - m \right) f = 0. \end{cases}$$

one should introduce the new functions $(f, g) \rightsquigarrow (F_1, F_2)$, via

$$f = \sqrt{m+E} (F_1 + F_2), \quad g = \sqrt{m-E} (F_1 - F_2).$$

This change leads to the equivalent system

$$\begin{cases} r \left(\frac{d}{dr} + \frac{\nu}{r} \right) (F_1 + F_2) + r \sqrt{m^2 - E^2} (F_1 - F_2) + e \frac{\sqrt{m-E}}{\sqrt{m+E}} (F_1 - F_2) = 0, \\ r \left(\frac{d}{dr} - \frac{\nu}{r} \right) (F_1 - F_2) + r \sqrt{m^2 - E^2} (F_1 + F_2) - e \frac{\sqrt{m+E}}{\sqrt{m-E}} (F_1 + F_2) = 0. \end{cases}$$

⁵We shall use below the underlined values.

⁶This was firstly given by G. Darwin and W. Gordon (1928), cf. [5].

By summing and subtracting the equations, we obtain, respectively

$$\begin{cases} r \frac{d}{dr} F_1 + \nu F_2 + r\sqrt{m^2 - E^2} F_1 - \frac{eE}{\sqrt{m^2 - E^2}} F_1 - \frac{em}{\sqrt{m^2 - E^2}} F_2 = 0, \\ r \frac{d}{dr} F_2 + \nu F_1 - r\sqrt{m^2 - E^2} F_2 + \frac{em}{\sqrt{m^2 - E^2}} F_1 + \frac{eE}{\sqrt{m^2 - E^2}} F_2 = 0. \end{cases}$$

By changing the variable $r \rightsquigarrow x = \lambda r$, where $\lambda = \sqrt{m^2 - E^2}$ and using the notations $\frac{em}{\lambda} = \mu$, $\frac{eE}{\lambda} = \varepsilon$, these equations considerably simplify,

$$(3.1) \quad \left(x \frac{d}{dx} + x - \varepsilon \right) F_1 + (\nu - \mu) F_2 = 0, \quad \left(x \frac{d}{dx} - x + \varepsilon \right) F_2 + (\nu + \mu) F_1 = 0.$$

The system (3.1) can be solved in terms of *hypergeometric functions* [5]. To detail this technique, we write (3.1) in terms of the new variable $y = 2x$:

$$\left(y \frac{d}{dy} + \frac{y}{2} - \varepsilon \right) F_1 + (\nu - \mu) F_2 = 0, \quad \left(y \frac{d}{dy} - \frac{y}{2} + \varepsilon \right) F_2 + (\nu + \mu) F_1 = 0.$$

This leads to the second-order differential equations for F_1 and F_2 :

$$(3.2) \quad \left(y \frac{d^2}{dy^2} + \frac{d}{dy} + \varepsilon + \frac{1}{2} - \frac{y}{4} + \frac{\mu^2 - \nu^2 - \varepsilon^2}{y} \right) F_1 = 0,$$

$$(3.3) \quad \left(y \frac{d^2}{dy^2} + \frac{d}{dy} + \varepsilon - \frac{1}{2} - \frac{y}{4} + \frac{\mu^2 - \nu^2 - \varepsilon^2}{y} \right) F_2 = 0.$$

Let us study the first equation (3.2). The substitution $F_1 = y^A e^{By} f_1$ gives

$$(3.4) \quad \left[y \frac{d^2}{dy^2} + (2A + 1 + 2By) \frac{d}{dy} + \varepsilon + \frac{1}{2} + B(1 + 2A) - y \left(\frac{1}{4} - B^2 \right) + \frac{A^2 + \mu^2 - \nu^2 - \varepsilon^2}{y} \right] f_1 = 0.$$

The choice of parameters $A = +\sqrt{\varepsilon^2 - \mu^2 + \nu^2}$ and $B = -\frac{1}{2}$ allows us to write (3.4) as a *confluent hypergeometric type ODE*

$$\left[y \frac{d^2}{dy^2} + (2A + 1 - y) \frac{d}{dy} + \varepsilon - A \right] f_1 = 0$$

with the parameters $\alpha_1 = A - \varepsilon$ and $\gamma_1 = 2A + 1$. By imposing the polynomial condition $\alpha_1 = -n_1$, we obtain for ε the quantization rule: $-n_1 = -\varepsilon + \sqrt{\varepsilon^2 - \mu^2 + \nu^2}$, and taking into account that $\lambda = \sqrt{m^2 - E^2}$, $\frac{em}{\lambda} = \mu$, $\frac{eE}{\lambda} = \varepsilon$, and $\sqrt{\varepsilon^2 - \mu^2 + \nu^2} = \sqrt{\nu^2 - e^2}$, we further derive a formula for the energy levels,

$$(3.5) \quad \frac{eE}{\sqrt{m^2 - E^2}} = \sqrt{\nu^2 - e^2} + n_1 \equiv N_1 \quad \Rightarrow \quad E = \frac{m}{\sqrt{1 + e^2/N_1^2}},$$

which coincides with (2.12).

Now, let us consider the second equation (3.3). With the use of the substitution $F_2 = y^a e^{by} f_2$, this becomes:

$$\left[y \frac{d^2}{dy^2} + (2a + 1 + 2by) \frac{d}{dy} + \varepsilon - \frac{1}{2} + b(1 + 2a) - y \left(\frac{1}{4} - b^2 \right) + \frac{a^2 + \mu^2 - \nu^2 - \varepsilon^2}{y} \right] f_2 = 0.$$

For $a = \pm\sqrt{\varepsilon^2 - \mu^2 + \nu^2}$ and $b = -\frac{1}{2}$, we obtain an equation of the *confluent hypergeometric type*,

$$\left[y \frac{d^2}{dy^2} + (2a + 1 - y) \frac{d}{dy} + \varepsilon - a - 1 \right] f_2 = 0$$

with the parameters $\alpha_2 = a + 1 - \varepsilon$ and $\gamma_2 = 2a + 1$. By imposing the polynomial restriction $\alpha_2 = -n_2$, we get for ε the quantization rule: $-n_2 = -\varepsilon + 1 + \sqrt{\varepsilon^2 - \mu^2 + \nu^2}$, and so we get the formula for energy levels:

$$\frac{eE}{\sqrt{m^2 - E^2}} = 1 + \sqrt{\nu^2 - e^2} + n_2 \equiv N_2 \quad \Rightarrow \quad E = \frac{m}{\sqrt{1 + e^2/N_2^2}}.$$

We further find the *relative coefficient* between two solutions of the system

$$(3.6) \quad \left(y \frac{d}{dy} + \frac{y}{2} - \varepsilon \right) F_1 + (\nu - \mu) F_2 = 0, \quad \left(y \frac{d}{dy} - \frac{y}{2} + \varepsilon \right) F_2 + (\nu + \mu) F_1 = 0.$$

by considering the relations

$$(3.7) \quad F_1 = C_1 y^A e^{-y/2} F(-n_1, \gamma, y), \quad F_2 = C_2 y^A e^{-y/2} F(-n_2, \gamma, y),$$

where $A = +\sqrt{\varepsilon^2 - \mu^2 + \nu^2}$, $\gamma = 2A = 1$, and $-n_2 = -n_1 + 1$. We substitute the expressions for the functions F_1, F_2 into the first equation of the system (3.6),

$$y C_1 \frac{d}{dy} F(-n_1, \gamma, y) + C_1 (A - \varepsilon) F(-n_1, \gamma, y) + (\nu - \mu) C_2 F(-n_1 + 1, \gamma, y) = 0,$$

and by applying the rule of differentiation of the confluent hypergeometric function

$$\frac{d}{dy} F(-n_1, \gamma, y) = -\frac{n_1}{y} F(-n_1 + 1, \gamma, y) + \frac{n_1}{y} F(-n_1, \gamma, y),$$

we get the relation

$$\begin{aligned} -C_1 n_1 F(-n_1 + 1, \gamma, y) + C_1 n_1 F(-n_1, \gamma, y) + C_1 (A - \varepsilon) F(-n_1, \gamma, y) \\ + (\nu - \mu) C_2 F(-n_1 + 1, \gamma, y) = 0. \end{aligned}$$

Taking into account the relation $-n_1 = A - \varepsilon$, we obtain

$$(3.8) \quad \frac{C_1}{C_2} = \frac{\nu - \mu}{n_1} = -\frac{\nu - \mu}{A - \varepsilon}.$$

Now, we substitute the expressions for the functions F_1, F_2 from (3.7) into the second equation of the system (3.6), and infer

$$\begin{aligned} y C_2 \frac{d}{dy} F(-n_1 + 1, \gamma, y) + C_2 (A + \varepsilon) F(-n_1 + 1, \gamma, y) - \\ - y C_2 F(-n_1 + 1, \gamma, y) + (\nu + \mu) C_1 F(-n_1, \gamma, y) = 0. \end{aligned}$$

Then, we apply the rule of differentiation of the confluent hypergeometric functions

$$\frac{d}{dy} F(-n_1 + 1, \gamma, y) = \left(\frac{-n_1 + 1}{\gamma} - 1 \right) F(-n_1 + 1, \gamma + 1, y) + F(-n_1 + 1, \gamma, y),$$

and use the formula for *contiguous confluent hypergeometric functions*

$$y F(-n_1 + 1, \gamma + 1, y) = \gamma F(-n_1 + 1, \gamma, y) - \gamma F(-n_1, \gamma, y),$$

which finally yields

$$C_2 (-n_1 + 1 - \gamma) F(-n_1 + 1, \gamma, y) + C_2 (A + \varepsilon) F(-n_1 + 1, \gamma, y) - \\ - C_2 (-n_1 + 1 - \gamma) F(-n_1, \gamma, y) + (\nu + \mu) C_1 F(-n_1, \gamma, y) = 0.$$

Taking into account that $-n_1 = A - \varepsilon$, we obtain

$$(3.9) \quad \frac{C_1}{C_2} = \frac{-n_1 - 2A}{\nu + \mu} = -\frac{A + \varepsilon}{\nu + \mu}.$$

Then the two expressions for relative coefficients, (3.8) and (3.9), coincide⁷.

4 The spin $1/2$ particle in Coulomb field. Solutions completely constructed by using Heun functions

The promising idea of constructing spectra within the framework of Heun equation theory can be accomplished by considering the known problem of a spin $1/2$ particle in the presence of an external Coulomb field. To this end, we turn back to the equations (2.5) which assume the presence of the Coulomb potential, for $\delta = +1$:

$$\left(\frac{d}{dr} + \frac{\nu}{r}\right)f + \left(E + \frac{e}{r} + m\right)g = 0, \quad \left(\frac{d}{dr} - \frac{\nu}{r}\right)g - \left(E + \frac{e}{r} - m\right)f = 0.$$

After eliminating the function g , one gets the second order ODE

$$(4.1) \quad \frac{d^2 f}{dr^2} + \frac{e}{r(Er+e+mr)} \frac{df}{dr} + \left[\frac{e(e^2-\nu^2)}{r^2(Er+e+mr)} + \frac{E(3e^2-\nu^2)-\nu(m+E)+m(e^2-\nu^2)}{r(Er+e+mr)} \right. \\ \left. + \frac{e(E+m)(3E-m)}{Er+e+mr} + \frac{r(E-m)(E+m)^2}{Er+e+mr} \right] f = 0.$$

After changing the variable $r \rightsquigarrow x = -\frac{(E+m)r}{e}$, this equation takes the form

$$x \frac{d^2 f}{dx^2} - \frac{1}{x-1} \frac{df}{dx} + \left[\frac{e^2(Ex - mx - 2E)}{E+m} + \frac{e^2 - \nu^2}{x} - \frac{\nu}{x-1} \right] f = 0.$$

We separate the two factors by means of the replacement $f(x) = x^A e^{Cx} F(x)$, and derive for F the ODE

$$(4.2) \quad \frac{d^2 F}{dx^2} + \left(2C + \frac{2A+1}{x} - \frac{1}{x-1}\right) \frac{dF}{dx} + \left[C^2 + \frac{e^2(E-m)}{E+m} \right. \\ \left. + \frac{A^2+e^2-\nu^2}{x^2} + \frac{A+C+2AC-2Ee^2/(E+m)+\nu}{x} - \frac{A+C+\nu}{x-1} \right] F = 0.$$

Since the *bound states* are of primary interest, we fix A, C as follows:

$$C^2 + \frac{e^2(E-m)}{E+m} = 0 \quad \Rightarrow \quad C = +e\sqrt{\frac{m-E}{m+E}}, \\ A^2 + e^2 - \nu^2 = 0 \quad \Rightarrow \quad A = +\sqrt{\nu^2 - e^2},$$

and then equation (4.2) becomes simpler,

$$\frac{d^2 F}{dx^2} + \left(2C + \frac{2A+1}{x} - \frac{1}{x-1}\right) \frac{dF}{dx} + \left[\frac{A+C+\nu+2AC-2Ee^2/(E+m)}{x} - \frac{A+C+\nu}{x-1} \right] F = 0,$$

⁷Indeed, one can easily check that $\frac{C_1}{C_2} = -\frac{\nu-\mu}{A-\varepsilon} = -\frac{A+\varepsilon}{\nu+\mu} \Leftrightarrow \nu^2 - \mu^2 = A^2 - \varepsilon^2$, which infers the identity $e^2 \equiv e^2$.

which is a *confluent Heun equation* for $F(\alpha, \beta, \gamma, \delta, \eta; x)$

$$\frac{d^2}{dx^2}F + \left(a + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1} \right) \frac{dF}{dx} + \left(\frac{1}{2} \frac{a+a\gamma+\beta+\beta\gamma+\gamma+2\delta+2\eta}{x-1} + \frac{1}{2} \frac{a\beta+a-\beta\gamma-\beta-\gamma-2\eta}{x} \right) F = 0,$$

with the parameters $a = 2C = +2e \sqrt{\frac{m-E}{m+E}}$, $\beta = 2A = +2 \sqrt{\nu^2 - e^2}$, $\gamma = -2$, $\delta = -\frac{2Ee^2}{E+m}$, and $\eta = 1 - \nu + \frac{2Ee^2}{E+m}$. The known quantization condition (which determines the transcendental confluent Heun functions) is $\delta = -a \left(n + \frac{\gamma+\beta+2}{2} \right)$, which can be written as $\frac{Ee}{\sqrt{m^2-E^2}} = N$, where $N = n + \sqrt{\nu^2 - e^2}$, which lead to $E = \frac{m}{\sqrt{1+e^2/N^2}}$, the exact energy spectrum for the hydrogen atom in Dirac theory.

Due to the symmetry between the functions $f(r)$ and $g(r)$, the second function $g(r)$ will be expressed in terms of confluent Heun functions as well.

5 Conclusions

In the paper, the well-known quantum mechanical problem of a spin 1/2 particle in external Coulomb potential, reduced to a system of two first-order differential equations, is studied by using the Heun function theory to solve the system. It is shown that, in addition to the standard approach which solves the problem in terms of confluent hypergeometric functions, there exist several other opportunities, which rely on applying confluent Heun functions. Namely, in the paper there are elaborated two combined possibilities to construct solutions: the first applies when one equation of the pair of relevant functions is expressed through hypergeometric functions, and the other function is constructed in terms of confluent Heun functions. In the second case, both functions of the system may be expressed in terms of confluent Heun functions only. All the applied methods lead to a single energy spectrum, which validates their correctness.

6 Acknowledgements

This work was supported by the Fund for Basic Researches of Belarus (Grant No. F14ARM-021) and by the Armenian State Committee of Science (Grant No. 13RB-052), within the cooperation framework between Belarus and Armenia. The research has been partially conducted within the scope of the International Associated Laboratory (CNRS-France & SCS-Armenia) IRMAS. The work by A. Manukyan has received funding from the European Union Seventh Framework Programme, grant No. 295025-IPERA). The present work was developed under the auspices of the project AR-FRBCF 2565 - F14RA-006-2014, part of the cooperation framework between Romanian Academy and Belarusian Republican Foundation for Fundamental Research.

References

- [1] A. Al-Badawi, I. Sakalli, *Solution of the Dirac equation in the rotating Bertotti-Robinson spacetime*, J. Math. Phys. **49** (2008), 052501; arXiv:0805.4485 [gr-qc].
- [2] A. Arda, O. Aydogdu, R. Sever, *Scattering of Woods-Saxon potential in Schrodinger equation*, J. Phys. A: Math Theor. **43** (2010), 425204; arXiv:1009.5181 [quant-ph]
- [3] W.G. Baber, H.R. Hassé, *The two centre problem in wave mechanics*, Proc. Cambridge Philos. Soc. **31** (1935), 564–581.

- [4] D. Batic, R. Williams, M. Nowakowski, *Potentials of the Heun class*, J. Phys. A: Math. Theor. **A 46**, 24 (2013), 245204; <http://arxiv.org/abs/1212.0448v1> [math-ph].
- [5] V.B. Berestetzkiy, E.M. Lifshitz, L.B. Pitaevskiy, *Quantum Electrodynamics*, Moscow, Nauka, 1989 (in Russian).
- [6] T. Birkandan, M. Hortacsu, *Examples of Heun and Mathieu functions as solutions of wave equations in curved spaces*, J. Phys. A: Math. Theor. **40** (2007), 1105–1116; arXiv: 0607108 [gr-qc]
- [7] A.A. Bogush, G.G. Krylov, E.M. Ovsyuk, and V.M. Red'kov, *Maxwell equations in complex form of Majorana-Oppenheimer, solutions with cylindric symmetry in Riemann S_3 and Lobachevsky H_3 spaces*, Ricerche di Matematica. **59** (2010), 59–96.
- [8] R.S. Borissov, P.P. Fiziev, *Exact solutions of Teukolsky master equation with continuous spectrum*, Bulg. J. Phys. **37** (2010), 065–089; arXiv:0903.3617 [gr-qc].
- [9] D. Bouaziz, M. Bawin, *Regularization of the singular inverse square potential in quantum mechanics with a minimal length*, Phys. Rev. **A76** (2007), 032112, arXiv:0711.0599 [quant-ph].
- [10] F. Caruso, J. Martins, V. Oguri, *Solving a two-electron quantum dot model in terms of polynomial solutions of a biconfluent Heun equation*, Ann. Phys. **347** (2014), 130-140; arXiv:1308.0815v1 [quant-ph].
- [11] R.N. Chaudhuri, S. Mukherjee, *On the $\mu x^2 + \lambda x^4 + \eta x^6$ interaction*, J. Phys. A: Math. Gen. **A 17** (1984), 3327.
- [12] A. Chychuryn, V. Red'kov, *Quantum mechanical scalar particle with polarisability in the Coulomb field, analytical and numerical consideration*, Europejska Wyzsza Szkola Informtyczno-Ekonomiczna (Warszawa), Studia i Materialy, **6** (2013), 73–89.
- [13] I. Costa, N. Derruelle, M. Novello, N. F. Svaiter, *Quantum fields in cosmological spacetimes: a soluble example*, Class. Quantum Grav. **6** (1989), 1893–1907.
- [14] M.S. Cunha, H.R. Christiansen, *Confluent Heun functions in gauge theories on thick braneworlds*, Phys. Rev. **D84** (2011), 085002; arXiv:1109.3486 [hep-th].
- [15] S.V. Dhurandhar, C.V. Vishveshwara, J.M. Cohen, *Electromagnetic, neutrino and gravitational fields in the Kasner space-time with rotational symmetry*, Class. Quantum Grav. **1** (1984), 61–69.
- [16] C.A. Downing, *On a solution of the Schrodinger equation with a hyperbolic double-well potential*, J. Math. Phys. **54** (2013), 072101; arXiv:1211.0913,
- [17] P.S. Epstein., *The stark effect from the point of view of Schrödinger quantum theory*, Phys. Rev. **28** (1926), 695.
- [18] G. Esposito, R. Roychowdhury, *On the complete analytic structure of the massive gravitino propagator in four-dimensional de Sitter space*, Gen. Rel. Grav. **42**, (2010), 1221–1238; arXiv:0907.3634 [hep-th].
- [19] H. Exton, *The exact solution of two new types of Schrodinger equation*, J. Phys. A: Math. Gen. **A 28** (1995), 6739–6741.
- [20] E.R. Figueiredo Medeiros, E.R. Bezerra de Mello, *Relativistic quantum dynamics of a charged particle in cosmic string spacetime in the presence of magnetic field and scalar potential*, Eur. Phys. J. **C 72** (2012), 2051; arXiv:1108.3786 [hep-th]
- [21] P.P. Fiziev, *Exact solutions of Regge-Wheeler equation and quasi-normal modes of compact objects*, Class. Quantum Grav. **23** (2006), 2447–2468.
- [22] P.P. Fiziev, *Exact Solutions of Regge-Wheeler Equation*, J. Phys. Conf. Ser. **66** (2007), 012016; arXiv:gr-qc/0702014v1.
- [23] P.P. Fiziev, *Classes of exact solutions to the Teukolsky master equations*, Class. Quantum Grav. **27** (2010), 135001; arXiv:gr-qc/0908.4234.

- [24] P. Fiziev, D. Staicova, *Application of the confluent Heun functions for finding the quasinormal modes of nonrotating black holes*, Phys. Rev. D **84** (2011), 127502; arXiv:1109.1532 [gr-qc]
- [25] R. Giachetti, E. Sorace, *States of the Dirac equation in confining potentials*, Phys. Rev. Lett. **101** (2008), 190401; arXiv:0706.0127 [hep-th].
- [26] R.L. Hall, N. Saad, K.D. Sen, *Soft-core Coulomb Potential and Heun's Differential Equation*, J. Math. Phys. **51** (2010), 022107. arXiv:0912.3445
- [27] M. Hamzavi, A.A. Rajabi, *Scalar-vector-pseudoscalar Cornell potential for a spin-1/2 particle under spin and pseudospin symmetries: 1+1 dimensions*, Ann. Phys. **334** (2013), 316-320.
- [28] K. Heun, *Zur Theorie der Riemann'schen Functionen Zweiter Ordnung mit Verzweigungspunkten*, Math. Ann. **33** (1889), 161-179.
- [29] J.M. Hoff da Silva, S. H. Pereira, *Exact solutions to Elko spinors in spatially flat Friedmann-Robertson-Walker spacetimes*, Journal of Cosmology and Astroparticle Physics **3** (2014), 009; arXiv:1401.3252 [hep-th].
- [30] M. Hortacsu, *Heun Functions and their uses in Physics*, in: Proc. 13-th Regional Conf. on Math. Phys., Antalya, Turkey, Oct. 27-31, 2010, (Eds.: U. Camci and I. Semiz), World Scientific (2013), 23-39; arXiv:1101.0471 [math-ph].
- [31] A.M. Ishkhanyan, K.-A. Suominen, *Analytic treatment of the polariton problem for a smooth interface*, J. Phys. A: Math. Gen. **34** (2001), L591-L598.
- [32] A.M. Ishkhanyan, K.-A. Suominen, *Solutions of the two-level problem in terms of biconfluent Heun functions*, J. Phys. A: Math. Gen. **A 34** (2001), 6301-6306.
- [33] A.M. Ishkhanyan, R. Sokhoyan, B. Joulakian, and K.-A. Suominen, *Rosen-Zener model in cold molecule formation*, Opt. Commun. **282** (2009), 218-226.
- [34] A. Ishkhanyan, R. Sokhoyan, K.-A. Suominen, C. Leroy, and H.-R. Jauslin, *Quadratic-nonlinear Landau-Zener transition for association of an atomic Bose-Einstein condensate with inter-particle elastic interactions included*, Eur. Phys. J. D **56** (2010), 421-429.
- [35] Lea Jaccoud El-Jaick, B. D. B. Figueiredo, *A limit of the confluent Heun equation and the Schrödinger equation for an inverted potential and for an electric dipole*, J. Math. Phys. **50** (2009), 123511.
- [36] J. Karwowski, H.A. Witek, *Biconfluent Heun equation in quantum chemistry: Harmonium and related systems*, Theor. Chem. Accounts. **133** (2014), Paper 1494, 11pp.
- [37] V.V. Kisel, G.G. Krylov, E.M. Ovsyuk, M. Amirfakchriyan, V.M. Red'kov, *Wave functions of the particle with polarizability in Coulomb field*, Nonlinear Dynamics and Applications, **18**, Minsk 2011, 168-179; Proc. of XVIII Annual Seminar NPCS-2011, May 17-20, 2011, Minsk, Belarus; 13 pp.
- [38] V.V. Kisel, E.M. Ovsyuk, V.M. Red'kov, *On the wave functions and energy spectrum for a spin 1 particle in external Coulomb field*, NPCS. **13** (2010), 352-367.
- [39] A. Lamieux and A.K. Bose, *Construction de potentiels pour lesquelles l'équation de Schrödinger est soluble*, Ann. Inst. Henri Poincaré **A 10** (1969), 259-270.
- [40] B. Léauté, G. Marcihacy, *On the Schrödinger equations of rotating harmonic, three-dimensional and doubly anharmonic oscillators and a class of confinement potentials in connection with the biconfluent Heun differential equation*, J. Phys. A: Math. Gen. **A 19** (1986), 3527-3533.
- [41] E.L. Leaver, *Solutions to a generalized spheroidal wave equation: Teukolsky equations in general relativity, and the two-center problem in molecular quantum mechanics*, J. Math. Phys. **27** (1986), 1238-1265.

- [42] M.F. Manning, *Exact solutions of the Schrödinger equation*, Phys. Rev. **48** (1935), 161–164.
- [43] D. Momeni, K. Yerzhanov, R. Myrzakulov, *Quantized black hole and Heun function*, Can. J. Phys. **90** (2012), 877–881, arXiv:1009.0130 [physics.gen-ph]
- [44] T. Oota, Y. Yasui, *Toric Sasaki-Einstein manifolds and Heun equation*, Nucl. Phys. **B 742** (2006), 275–294; arXiv:hep-th/0512124.
- [45] V.I. Osherov, V.G. Ushakov, *Stark problem in terms of the Stokes multipliers for the triconfluent Heun equation*, Phys. Rev. **A 88** (2013), 053414.
- [46] V.S. Otchik, *Analytic solutions of the Teukolsky equations*, In: "Quant. Sys.: New Trends and Methods" (Eds: A. O. Barut et al.), World Scientific, Singapore 1995.
- [47] E.M. Ovsyuk, *On solutions of Maxwell equations in the space-time of Schwarzschild black hole*, NPCS. **15** (2012), 81–91.
- [48] E.M. Ovsyuk, O. Florea, A. Chichurin, V.M. Red'kov, *Electromagnetic field in Schwarzschild black hole background. Analytical treatment and numerical simulation*, Proc. 7-th Int. Workshop CASTR-2013, Sept. 22–25, 2013, Siedlce, Poland, Publ. Collegium Mazovia, IV, 1 (2013), 204–214; <http://arxiv.org/abs/1410.8300v1> [math-ph].
- [49] E.M. Ovsyuk, O.V. Veko and M. Amirfakhrian, *On Schrodinger equation with potential $U = -\alpha r^{-1} + \beta r + kr^2$ and the biconfluent Heun functions theory*, Nonlinear Phenomena in Complex Systems **15** (2012), 163-170; .
- [50] E.M. Ovsyuk, O.V. Veko, V.M. Red'kov, *On simulating a medium with special reflecting properties by Lobachevsky geometry*, NPCS **16** (2013), 331–344.
- [51] E. Ovsyuk, O. Veko, K. Kazmerchuk, V. Kisel, V. Red'kov, *Quantum mechanics of a spin 1 particle in the magnetic monopole potential, in spaces of Euclid, Lobachevsky, and Riemann: nonrelativistic approximation*, Ukr. Phys. J. **58** (2013), 1073–1083.
- [52] E.M. Ovsyuk, O.V. Veko, V.V. Kisel, V.M. Red'kov, *New problems of quantum mechanics and Heun equation*, Naucno-tekhnicheskie vedomosti, St.Petersburg Univ., Ser. Fiz.-Mat. **1(141)** (2012), 137–145.
- [53] E.M. Ovsyuk, V.V. Kisel, V.M. Red'kov, *Maxwell Electrodynamics and Boson Fields in Spaces of Constant Curvature*, Nova Science Publishers Inc., New York 2014.
- [54] D. Petroff, *Slowly rotating homogeneous stars and the Heun equation*, Class. Quant. Grav. **24** (2007), 1055–1068; arXiv:gr-qc/0701081v2 6 Feb 2007
- [55] R. Pons, G. Marçilhac, *Exact solutions for electromagnetic, neutrino and gravitational fields in Kasner spacetime*, Class. Quantum Grav. **4** (1987), 171–179.
- [56] A.A. Rajabi, M. Hamzavi, *Relativistic effect of external magnetic and Aharonov-Bohm fields on the unequal scalar and vector Cornell model*, Eur. Phys. J. Plus **128** (2013), 5-6.
- [57] A. Ralko, T.T. Truong, *Heun Functions and the energy spectrum of a charged particle on a sphere under magnetic field and Coulomb force*, J. Phys. A: Math. Gen. **35** (2002), 9573-9584; arXiv:quant-ph/0209152
- [58] P.P. Ray, K. Mahata, *Bounded states of the potential $V(r) = \frac{Ze^2}{r+\beta}$* , J. Phys. A: Math. Gen. **22** (1989), 3161.
- [59] V.M. Red'kov, A.V. Chichurin, *A symbolic-numerical method for solving the differential equation describing the states of polarizable particle in Coulomb potential*, Programming and Computer Software **40**, 2 (2014), Pleiades Publishing, Ltd., 2014; 86-92.
- [60] V.M. Red'kov, E.M. Ovsyuk, *On exact solutions for quantum particles with spin $S = 0, 1/2, 1$ and de Sitter event horizon*, Ricerche di Matematica **60** (2011), 57–88.
- [61] V.M. Red'kov, E.M. Ovsyuk, *Quantum mechanics in spaces of constant curvature*, Nova Science Publishers. Inc., New York, 2012.

- [62] V.M. Red'kov, E.M. Ovsyuk, O.V. Veko, *Spin 1/2 particle in the field of Dirac string on the background of de Sitter space-time*, Uzhhorod University Scientific Herald. Series Physics. **32** (2012), 141–150.
- [63] T. Regge, J.A. Wheeler, *Stability of a Schwarzschild singularity*, Phys. Rev. **108** (1957), 1063–1069.
- [64] A. Ronveaux (ed.), *Heun's Differential Equation*, Oxford University Press, 1995.
- [65] J. Sesma, *The generalized quantum isotonic oscillator*, J. Phys. A: Math. Theor. **43** (2010), 185303; arXiv:1005.1227 [quant-ph]
- [66] T.A. Shahverdyan, D.S. Mogilevtsev, A.M. Ishkhanyan, V.M. Red'kov, *Complete-return spectrum for a generalized Rosen-Zener two-state term-crossing model*, Non-linear Phenomena in Complex Systems **16** (2013), 86–92.
- [67] Y.S. Slavyanov, W. Lay, *Special Functions. A Unified Theory Based on Singularities*, Oxford Univ. Press, 2000.
- [68] H. Suzuki, E. Takasugi, H. Umetsu, *Perturbations of Kerr-de Sitter black holes and Heun's equations*, Progr. Theoret. Phys. **100** (1998), 491–505; arXiv:gr-qc/9805064.
- [69] T. Takemura, *Finite-gap potential, Heun differential equation and WKB analysis*, arXiv:math/0703256v1 [math.CA]
- [70] T. T. Truong and D. Bazzali, *Exact low-lying states of two interacting equally charged particles in a magnetic field*, Phys. Lett. **A 269** (2000), 186–193.
- [71] D.A. Varshalovich, A.N. Moskalev, V.K. Hersonskiy, *Quantum Theory of Angular Momentum* (in Russian), Nauka, Leningrad, 1975.
- [72] H.S. Vieira, V.B. Bezerra, C.R. Muniz, *Exact solutions of the Klein-Gordon equation in the Kerr-Newman background and Hawking radiation*, Ann. Phys. **350** (2014), 14–28.
- [73] S.Q. Wu, X. Cai, *Massive complex scalar field in a Kerr-Sen Black Hole Background: exact solution of wave equation and Hawking radiation*, J. Math. Phys. **44** (2003), 1084–1088; arXiv:gr-qc/0303075
- [74] V.H. Zaveri, *Quarkonium and hydrogen spectra with spin-dependent relativistic wave equation*, Pramana J. Phys. **75** (2010), 579–598.
- [75] ***, Sofia University, The Heun Project: Heun functions, their generalizations and applications, <http://theheunproject.org>

*Authors' addresses:**Vladimir Balan*Univ. Politehnica of Bucharest, Romania. E-mail vladimir.balan@upb.ro*Ashot M. Manukyan*Inst. Phys. Research of NAS of Armenia. E-mail: manookian@gmail.com*Elena M. Ovsyuk*Mosyr State Pedagog. Univ., Belarus. E-mail: e.ovsyuk@mail.ru*Victor M. Red'kov*"B.I. Stepanov" Inst. of Physics, NAS of Belarus. E-mail: redkov@dragon.bas-net.by*Olga V. Veko*Kalinkovich Gymnasium, Belarus. E-mail: vekoolga@mail.ru