Some properties of $(\bar{\alpha}, \bar{\beta})$-fuzzy fantastic ideals in BCH-algebras

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Abstract. The purpose of this paper is to introduce the concept of $(\bar{\alpha}, \bar{\beta})$-fuzzy fantastic ideals in BCH-algebra and investigate some of their related properties.

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1 Introduction

The notion of fantastic ideal in a BCH-algebra was initiated by Saeid and Namdar [17] and studied some of their related properties [24]. The theory of fuzzy set proposed by Zadeh in his classic paper [21], of 1965 provides a natural framework for generalizing some of the basic notions of algebra. Extensive applications of fuzzy set theory have been found in various fields, for example, artificial intelligence, computer science, control engineering, expert system, management science, operation research and many others. The concept was applied to the theory of groupoids and groups by Rosenfeld [16], where he introduced the fuzzy subgroup of a group. In [24], Zulfiqar initiated the concept of fuzzy fantastic ideal in BCH-algebra and discussed some interesting properties.

In 1971, Rosenfeld formulated the elements of theory of fuzzy groups [16]. A new type of fuzzy subgroup, which is, the $(\in, \in \vee\, q)$-fuzzy subgroup, was introduced by Bhakat and Das [4] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [15]. Murali [14] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was found that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee\, q)$-fuzzy subgroup. Bhakat [2-3] initiated the concepts of $(\in \vee\, q)$-level subsets, $(\in, \in \vee\, q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra (see [7-10, 22-25]). In [7-9], Jun defined the notion of (α, β)-fuzzy subalgebras/ideals in BCK/BCI-algebras. The concept of (α, β)-fuzzy positive implicative ideal in BCK-algebras was initiated by Zulfiqar in [22]. Generalizing the concept of quasi-coincident of a fuzzy point with a fuzzy set, in [10], Jun defined $(\in, \in \vee\, q_k)$-fuzzy subalgebras in BCK/BCI-algebras. In [11], Jun et al. discussed $(\in, \in \vee q_k)$-fuzzy ideals in BCK/BCI-algebras. Khan et al. studied ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals [12]. Larimi [13], initiated the concept of $(\in, \in \vee q_k)$-intuitionistic fuzzy ideals of hemirings. Shabir et al. [18], characterized different classes of semigroups by $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy bi-ideals. In [20], Tang and Xie studied $(\in, \in \vee q_k)$-fuzzy ideals of ordered semigroups. Shabir and Mahmood [19], characterized semihypergroups by $(\in, \in \vee q_k)$-fuzzy hyperideals. Currently, Zulfiqar initiated the notions of $(\in, \in \vee q_k)$-fuzzy fantastic ideal in BCI-algebras and investigated some of their related properties [23].

In the present paper, we prove a fuzzy set λ of a BCH-algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X if and only if for any t $\in (\frac{1-k}{2}, 1]$, $\lambda_t = \{x \in \mathrm{X} | \lambda(x) \geq t\}$ is a fantastic ideal of X. We show that if I is a non-empty subset of a BCH-algebra X, then I is a fantastic ideal of X if and only if the fuzzy set λ of X defined by

$$\lambda(x) = \begin{cases} \leq \frac{1-k}{2}, & \text{if } x \in X - I \\ 0, & \text{if } x \in I, \end{cases}$$

is an $(\bar{q}_k, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X. Finally we prove that the intersection of any family of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals of a BCH-algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.

2 Preliminaries

Throughout this paper X always denotes a BCH-algebra without any specification. We also include some basic aspects that are necessary for this paper.

By a BCH-algebra [1, 5, 6], we mean an algebra (X, $*$, 0) of type (2, 0) satisfying the axioms:

(BCH-1) $x * x = 0$;
(BCH-2) $x * y = 0$ and $y * x = 0$ imply $x = y$;
(BCH-3) $(x * y) * z = (x * z) * y$;
for all $x, y, z \in X$.

We can define a partial order $\leq$ on X by $x \leq y$ if and only if $x * y = 0$.

Proposition 2.1. [24] In any BCH-algebra X, the following are true:

(i) $x * (x * y) \leq y$;
(ii) $0 * (x * y) = (0 * x) * (0 * y)$;
(iii) $x \leq 0$ implies $x = 0$;
(iv) $x * 0 = x$;
for all x, y $\in X$.

Definition 2.2. [24] A non-empty subset I of a BCH-algebra X is called an ideal of X if it satisfies the conditions (I1) and (I2), where

(I1) $0 \in I$;
(I2) $x * y \in I$ and $y \in I$ imply $x \in I$;
for all x, y $\in X$.

Definition 2.3. [17] A non-empty subset I of a BCH-algebra X is called a fantastic ideal of X if it satisfies the conditions (I1) and (I3), where

(I1) $0 \in I$;
(I3) (x ∗ y) ∗ $z \in I$ and $z \in I$ imply x ∗ (y ∗ (y ∗ x)) ∈ I;
for all x, y, z X.

We shall further review some fuzzy logic concepts. Recall that the real unit interval [0, 1] with the totally ordered relation $\leq$ is a complete lattice, with $\wedge$ = min and $\vee$ = max, 0 and 1 being the least element and the greatest element, respectively.

A fuzzy set λ of a universe X is a function from X into the unit closed interval [0, 1], that is $\lambda : X \rightarrow [0,1]$. For a fuzzy set λ of a BCH-algebra X and $t \in (0,1]$, the crisp set

$$\lambda_t = \{x \in \mathrm{X} |\ \lambda(x) \geq t\}$$

is called *the level subset* of λ.

Definition 2.4. [24] A fuzzy set λ of a BCH-algebra X is called a fuzzy ideal of X if it satisfies the conditions (F1) and (F2), where

(F1) $\lambda(0) \geq \lambda(x)$,
(F2) $\lambda(x) \geq \lambda(x * y) \wedge \lambda(y)$,
for all $x, y \in X$.

Definition 2.5. [24] A fuzzy set λ of a BCH-algebra X is called a fuzzy fantastic ideal of X if it satisfies the conditions (F1) and (F3), where

(F1) $\lambda(0) \geq \lambda(x)$,
(F3) $\lambda(x * (y * (y * x))) \geq \lambda((x * y) * z) \wedge \lambda(z)$,
for all $x, y, z \in X$.

Theorem 2.6. [24] *A fuzzy set λ of a BCH-algebra X is a fuzzy fantastic ideal of X if and only if, for every $t \in (0,1]$, λ_t is either empty or a fantastic ideal of X.*

A fuzzy set λ of a BCH-algebra X having the form

$$\lambda(y) = \begin{cases} t \in (0,1] & \text{if } y = x \\ 0, & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t, and is denoted by x_t [24]. For a fuzzy point x_t and a fuzzy set λ in a set X, Pu and Liu [15] gave meaning to the symbol $x_t\alpha\lambda$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belong to (resp., quasi-coincident with) a fuzzy set λ, written as $x_t \in \lambda$ (resp. $x_tq\lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). By $x_t \in \vee q\lambda$ ($x_t \in \wedge q\lambda$) we mean that $x_t \in \lambda$ or $x_tq\lambda$ ($x_t \in \lambda$ and $x_tq\lambda$). For all $t_1, t_2 \in [0,1]$, $\min\{t_1, t_2\}$ and $\max\{t_1, t_2\}$ will be denoted by $t_1 \wedge t_2$ and $t_1 \vee t_2$, respectively.

In what follows let α and β denote any one of $\in, q, \in \vee q$, $\in \wedge q$ and $\alpha \neq \in \wedge q$ unless otherwise specified. To say that $x_t\bar{\alpha}\lambda$ means that $x_t\alpha\lambda$ does not hold.

3 $(\bar{\alpha}, \bar{\beta})$-fuzzy fantastic ideals

Throughout this paper X will denote a BCH-algebra and $\bar{\alpha}$, $\bar{\beta}$ are any one of $\bar{\in}, \bar{q}_k, \bar{\in} \vee \bar{q}_k, \bar{\in} \wedge \bar{q}_k$ unless otherwise specified.

Definition 3.1. A fuzzy set λ of a BCH-algebra X is called an $(\bar{\alpha}, \bar{\beta})$-fuzzy subalgebra of X, where $\bar{\alpha} \neq \bar{\in} \wedge \bar{q}_k$, if it satisfies the condition

$$(x * y)_{t_1 \wedge t_2} \bar{\alpha}\ \lambda \Rightarrow x_{t_1} \bar{\beta} \lambda \text{ or } y_{t_2} \bar{\beta} \lambda,$$

for all $t_1, t_2 \in (0, 1]$ and $x, y \in X$.

Let λ be a fuzzy set of a BCH-algebra X such that $\lambda(x) \geq \frac{1-k}{2}$ for all $x \in X$. Let $x \in X$ and $t \in (0, 1]$ be such that

$$x_t \bar{\in} \wedge \bar{q}_k.$$

Then

$$\lambda(x) < t \text{ and } \lambda(x) + t + k \leq 1.$$

It follows that

$$2\lambda(x) + k = \lambda(x) + \lambda(x) + k < \lambda(x) + t + k \leq 1.$$

This implies that $\lambda(x) < \frac{1-k}{2}$. This means that

$$\{x_t \mid x_t \bar{\in} \wedge \bar{q}_k \lambda\} = \phi.$$

Therefore, the case $\bar{\alpha} = \bar{\in} \wedge \bar{q}_k$ in the above definition is omitted.

Definition 3.2. A fuzzy set λ of a BCH-algebra X is called an $(\bar{\alpha}, \bar{\beta})$-fuzzy ideal of X, where $\bar{\alpha} \neq \bar{\in} \wedge \bar{q}_k$, if it satisfies the conditions (A) and (B), where

(A) $0_t \bar{\alpha} \lambda \Rightarrow x_t \bar{\beta} \lambda$,
(B) $x_{t_1 \wedge t_2} \bar{\alpha}\ \lambda \Rightarrow (x * y)_{t_1} \bar{\beta} \lambda$ or $y_{t_2} \bar{\beta} \lambda$,
for all $t, t_1, t_2 \in (0, 1]$ and $x, y \in X$.

Definition 3.3. A fuzzy set λ of a BCH-algebra X is called an $(\bar{\alpha}, \bar{\beta})$-fuzzy fantastic ideal of X, where $\bar{\alpha} \neq \bar{\in} \wedge \bar{q}_k$, if it satisfies the conditions (A) and (C), where

(A) $0_t \bar{\alpha} \lambda \Rightarrow x_t \bar{\beta} \lambda$,
(C) $(x * (y * (y * x)))_{t_1 \wedge t_2} \bar{\alpha}\ \lambda \Rightarrow ((x * y) * z)_{t_1} \bar{\beta} \lambda$ or $z_{t_2} \bar{\beta} \lambda$,
for all $t, t_1, t_2 \in (0, 1]$ and $x, y, z \in X$.

Theorem 3.4. *A fuzzy set λ of a BCH-algebra X is a fuzzy fantastic ideal of X if and only if λ is an $(\bar{\in}, \bar{\in})$-fuzzy fantastic ideal of X.*

Proof. Suppose λ is a fuzzy fantastic ideal of X. Let $0_t \bar{\in} \lambda$ for $t \in (0, 1]$. Then $\lambda(0) < t$. By Definition 2.5, we have

$$t > \lambda(0) \geq \lambda(x),$$

which implies that $t > \lambda(x)$, that is, $x_t \bar{\in} \lambda$. Let $x, y \in X$ and $t, r \in (0, 1]$ be such that

$$(x * (y * (y * x)))_{t \wedge r} \bar{\in} \lambda.$$

Then

$$\lambda(x * (y * (y * x))) < t \wedge r.$$

Since λ is a fuzzy fantastic ideal of X. So

$$t \wedge r > \lambda(x * (y * (y * x))) \geq \lambda((x * y) * z) \wedge \lambda(z).$$

This implies that

$$t > \lambda((x * y) * z) \text{ or } r > \lambda(z),$$

that is,

$$((x * y) * z)_t \bar{\in} \lambda \text{ or } z_r \bar{\in} \lambda.$$

This shows that λ is an $(\bar{\in}, \bar{\in})$-fuzzy fantastic ideal of X.

Conversely, assume that λ is an $(\bar{\in}, \bar{\in})$-fuzzy fantastic ideal of X. To show that λ is a fuzzy fantastic ideal of X, suppose there exists $x \in X$ such that

$$\lambda(0) < \lambda(x).$$

Select $t \in (0, 1]$, such that

$$\lambda(0) < t \leq \lambda(x).$$

Then $0_t \bar{\in} \lambda$ but $x_t \in \lambda$, which is a contradiction. Hence

$$\lambda(0) \geq \lambda(x), \text{ for all } x \in X.$$

Now suppose there exist $x, y, z \in X$, such that

$$\lambda(x * (y * (y * x))) < \lambda((x * y) * z) \wedge \lambda(z).$$

Select $t \in (0, 1]$, such that

$$\lambda(x * (y * (y * x))) < t \leq \lambda((x * y) * z) \wedge \lambda(z).$$

Then

$$(x * (y * (y * x)))_t \bar{\in} \lambda,$$

but

$$((x * y) * z)_t \in \lambda \text{ and } z_t \in \lambda,$$

which is a contradiction. Hence

$$\lambda(x * (y * (y * x))) \geq \lambda((x * y) * z) \wedge \lambda(z).$$

This shows that λ is a fuzzy fantastic ideal of X. $\square$

4 $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy ideals

In this section, we study the concept of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy ideal in a BCH-algebra and investigate some of their properties.

Definition 4.1. Let λ be a fuzzy set of a BCH-algebra X. Then λ is called an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy ideal of X if it satisfies the conditions (D) and (E), where
(D) $0_t \bar{\in} \lambda \Rightarrow x_t \bar{\in} \vee \bar{q}_k$,
(E) $x_{t \wedge r} \bar{\in} \lambda \Rightarrow (x * y)_t \bar{\in} \vee \bar{q}_k$ or $y_r \bar{\in} \vee \bar{q}_k$,
for all $x, y \in X$ and $t, r \in (0, 1]$.

Example 4.2. Let $X = \{0, a, b, c\}$ be a BCH-algebra with the following Cayley table:

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let λ be a fuzzy set in X defined by $\lambda(0) = 0.50$, $\lambda(a) = \lambda(\mathrm{b}) = 0.34$ and $\lambda(\mathrm{c}) = 0.23$. Simple calculations show that λ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy ideal of X for k = 0.1.

Theorem 4.3. *The conditions (D) and (E) in Definition 4.1, are equivalent to the following conditions, respectively:*
(F) $\lambda(0) \vee \frac{1-k}{2} \geq \lambda(x)$,
(G) $\lambda(x) \vee \frac{1-k}{2} \geq \lambda(x * y) \wedge \lambda(y)$,
for all $x, y \in X$.

Proof. Straightforward. □

Corollary 4.4. *A fuzzy set λ of a BCH-algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy ideal of X if it satisfies the conditions (F) and (G).*

5 $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals

In this section, we study the notion of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal in a BCH-algebra and further discuss some of their related properties.

Definition 5.1. Let λ be a fuzzy set of a BCH-algebra X. Then λ is called an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X if it satisfies the conditions (D) and (H), where
(D) $0_t \bar{\in} \lambda \Rightarrow x_t \bar{\in} \vee \bar{q}_k$,
(H) $(x * (y * (y * x)))_{t \wedge r} \bar{\in} \lambda \Rightarrow ((x * y) * z)_t \bar{\in} \vee \bar{q}_k$ or $z_r \bar{\in} \vee \bar{q}_k$,
for all $x, y, z \in X$ and $t, r \in (0, 1]$.

Example 5.2. Let $X = \{0, a, b, c\}$ be a BCH-algebra with Cayley table described below.

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let λ be a fuzzy set in X defined by $\lambda(0) = \lambda(\mathrm{c}) = 0.50$ and $\lambda(a) = \lambda(\mathrm{b}) = 0.38$. Simple calculations show that λ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X for $k = 0.1$.

Theorem 5.3. *Let λ be a fuzzy set of a BCH-algebra X. Then the condition (H) is equivalent to the condition (I), where*
(I) $\lambda(x * (y * (y * x))) \vee \frac{1-k}{2} \geq \lambda((x * y) * z) \wedge \lambda(z)$,
for all $x, y, z \in X$.

Proof. (H) $\Rightarrow$ (I). Suppose there exist $x, y, z \in X$, such that

$$\lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2} < t = \lambda((x * y) * z) \wedge \lambda(z).$$

Then

$$t \in (\tfrac{1-k}{2}, 1],\ (x * (y * (y * x)))_t \bar{\in} \lambda$$

and

$$((x * y) * z)_t \in \lambda,\ z_t \in \lambda.$$

It follows that

$$((x * y) * z)_t \bar{q}_k \lambda \text{ or } z_t \bar{q}_k \lambda.$$

Then

$$\lambda((x * y) * z) + t + k \leq 1$$

or

$$\lambda(z) + t + k \leq 1.$$

As $t \leq \lambda((x * y) * z)$ and $t \leq \lambda(z)$, it follows that

$$t \leq \tfrac{1-k}{2}.$$

This is a contradiction. So

$$\lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2} \geq \lambda((x * y) * z) \wedge \lambda(z).$$

(I) $\Rightarrow$ (H)
Let $x, y \in X$ and $t, r \in (0, 1]$ be such that

$$(x * (y * (y * x)))_{t \wedge r} \bar{\in} \lambda.$$

Then

$$\lambda(x * (y * (y * x))) < t \wedge r.$$

(a) If $\lambda(x * (y * (y * x))) \geq \frac{1-k}{2}$, then by condition (I)

$$\lambda(x * (y * (y * x))) \geq \lambda((x * y) * z) \wedge \lambda(z).$$

Thus

$$\lambda((x * y) * z) \wedge \lambda(z) < t \wedge r,$$

and consequently

$$\lambda((x * y) * z) < t \text{ or } \lambda(z) < r.$$

It follows that

$$((x * y) * z)_t \bar{\in} \lambda \text{ or } z_r \bar{\in} \lambda,$$

and hence

$$((x * y) * z)_t \bar{\in} \vee \bar{q}_k \lambda \text{ or } z_r \bar{\in} \vee \bar{q}_k \lambda.$$

(b) If $\lambda(x * (y * (y * x))) < \frac{1-k}{2}$, then by condition (I)

$$\tfrac{1-k}{2} \geq \lambda((x * y) * z) \wedge \lambda(z).$$

Suppose

$$((x * y) * z)_t \in \lambda \text{ and } z_r \in \lambda.$$

Then

$$\lambda((x * y) * z) \geq t \text{ and } \lambda(z) \geq r.$$

Thus $\frac{1-k}{2} \geq t \wedge r$. Hence

$$\lambda((x * y) * z) \wedge \lambda(z) + t \wedge r + k \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} + k = 1,$$

that is

$$((x * y) * z)_t \bar{q}_k \lambda \text{ or } z_r \bar{q}_k \lambda.$$

This implies that $((x * y) * z)_t \bar{\in} \vee \bar{q}_k \lambda$ or $z_r \bar{\in} \vee \bar{q}_k \lambda$. □

Corollary 5.4. *A fuzzy set λ of a BCH-algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X if and only if it satisfies the conditions (F) and (I).*

The properties of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals in BCH-algebras are given by the following Theorem.

Theorem 5.5. *Let λ be an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy ideal of a BCH-algebra X. Then the following are equivalent:*
(i) λ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X,
(ii) $\lambda(x * (y * (y * x))) \vee \frac{1-k}{2} \geq \lambda(x * y)$,
for all $x, y \in X$.

Proof. (i) ⇒ (ii). Since λ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X, we have

$$\lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2} \geq \lambda((x * y) * z) \wedge \lambda(z).$$

Putting $z = 0$ in the above relation, we get

$$\lambda(x*(y*(y*x)))\vee \tfrac{1-k}{2}\geq \lambda((x*y)*0)\wedge\lambda(0)$$
$$\lambda(x*(y*(y*x)))\vee \tfrac{1-k}{2}\geq \lambda(x*y)\wedge\lambda(0) \text{ (by Proposition 2.1(iv))}$$
$$\lambda(x*(y*(y*x)))\vee \tfrac{1-k}{2}\geq \lambda(x*y).$$

(ii) $\Rightarrow$ (i). By (ii), we have

$$\lambda(x*(y*(y*x)))\vee \tfrac{1-k}{2}\geq \lambda(x*y)$$
$$\lambda(x*(y*(y*x)))\vee \tfrac{1-k}{2}\vee \tfrac{1-k}{2}\geq \lambda(x*y)\vee \tfrac{1-k}{2}$$
$$\lambda(x*(y*(y*x)))\vee \tfrac{1-k}{2}\geq \lambda(x*y)\vee \tfrac{1-k}{2}.$$

Since λ is an $(\bar{\in},\bar{\in}\vee\bar{q}_k)$-fuzzy ideal of X, we have

$$\lambda(x*(y*(y*x)))\vee \tfrac{1-k}{2}\geq \lambda((x*y)*z)\wedge\lambda(z).$$

This shows that λ satisfies the condition (I). Hence λ is an $(\bar{\in},\bar{\in}\vee\bar{q}_k)$-fuzzy fantastic ideal of X. □

Theorem 5.6. *A fuzzy set λ of a BCH-algebra X is an $(\bar{\in},\bar{\in}\vee\bar{q}_k)$-fuzzy fantastic ideal of X if and only if for any $t\in(\frac{1-k}{2},1]$, $\lambda_t=\{x\in X|\ \lambda(x)\geq t\}$ is a fantastic ideal of X.*

Proof. Let λ be an $(\bar{\in},\bar{\in}\vee\bar{q}_k)$-fuzzy fantastic ideal of X and $\frac{1-k}{2}<t\leq 1$. If $\lambda_t\neq\phi$, then $x\in\lambda_t$. This implies that $\lambda(x)\geq t$. By condition (F)

$$\lambda(0)\vee \tfrac{1-k}{2}\geq \lambda(x)\geq t.$$

Thus $\lambda(0)\geq t$. Hence $0\in\lambda_t$. Let $(x*y)*z\in\lambda_t$ and $z\in\lambda_t$. Then

$$\lambda((x*y)*z)\geq t \text{ and } \lambda(z)\geq t.$$

By condition (I),

$$\begin{aligned}\lambda(x*(y*(y*x)))\vee \tfrac{1-k}{2}&\geq \lambda((x*y)*z)\wedge\lambda(z)\\ &\geq t\wedge t\\ &= \text{t}.\end{aligned}$$

Thus

$$\lambda(x*(y*(y*x)))\geq t,$$

that is,

$$(x*(y*(y*x)))\in\lambda_t.$$

Therefore λ_t is a fantastic ideal of X. Conversely, assume that λ is a fuzzy set of X, such that $\lambda_t(\neq\phi)$ is a fantastic ideal of X for all $\frac{1-k}{2}<t\leq 1$. Let $x\in X$ be such that

$$\lambda(0)\vee \tfrac{1-k}{2}<\lambda(x).$$

Select $\frac{1-k}{2}<t\leq 1$ such that

$$\lambda(0)\vee \tfrac{1-k}{2}<t\leq\lambda(x).$$

Then $x\in\lambda_t$ but $0\in\lambda_t$, a contradiction. Hence

$$\lambda(0)\vee \tfrac{1-k}{2}\geq\lambda(x).$$

Now assume that $x, y, z \in X$ such that

$$\lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2} < \lambda((x * y) * z) \wedge \lambda(z).$$

Select $\frac{1-k}{2} < t \leq 1$ such that

$$\lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2} < t \leq \lambda((x * y) * z) \wedge \lambda(z).$$

Then $(x * y) * z$ and z are in λ_t but

$$x * (y * (y * x)) \notin \lambda_t,$$

a contradiction. Hence

$$\lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2} \geq \lambda((x * y) * z) \wedge \lambda(z).$$

This shows that λ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X. $\square$

Remark 5.7. Let λ be a fuzzy set of a BCH-algebra X and

$$I_t = \{t \mid t \in (0, 1] \text{ such that } \lambda_t \text{ is a fantastic ideal of X}\}.$$

In particular,

(1) If $I_t = (0, 1]$, then λ is a fuzzy fantastic ideal of X (Theorem 2.6).
(2) If $I_t = (\frac{1-k}{2}, 1]$, then λ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X (Theorem 5.6).

Corollary 5.8. *Every fuzzy fantastic ideal of a BCH-algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.*

Theorem 5.9. *Let I be a non-empty subset of a BCH-algebra X. Then I is a fantastic ideal of X if and only if the fuzzy set λ of X defined by*

$$\lambda(x) = \begin{cases} \leq \frac{1-k}{2}, & \text{if } x \in X - I \\ 0, & \text{if } x \in I, \end{cases}$$

is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.

Proof. Let I be a fantastic ideal of X. Then $0 \in I$. This implies that $\lambda(0) = 1$. Thus

$$\lambda(0) \vee \tfrac{1-k}{2} = 1 \geq \lambda(x).$$

It means that λ satisfies the condition (F). Now let $x, y, z \in X$. If $(x * y) * z$ and z are in I, then

$$x * (y * (y * x)) \in I.$$

This implies that

$$\lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2} = 1 = \lambda((x * y) * z) \wedge \lambda(z).$$

If one of $(x * y) * z$ and z is not in I, then

$$\lambda((x * y) * z) \wedge \lambda(z) \leq \tfrac{1-k}{2} \leq \lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2}.$$

Thus λ satisfies the condition (I). Hence λ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X. Conversely, assume that λ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X. Let $x \in I$. Then by condition (F),

$$\lambda(0) \vee \tfrac{1-k}{2} \geq \lambda(x) = 1.$$

This implies that $0 \in I$. Let $x, y, z \in X$ be such that $(x * y) * z$ and z are in I. Then by condition (I)

$$\lambda(x * (y * (y * x))) \vee \tfrac{1-k}{2} \geq \lambda((x * y) * z) \wedge \lambda(z) = 1.$$

This implies that

$$\lambda(x * (y * (y * x))) = 1,$$

that is,

$$(x * (y * (y * x))) \in I.$$

Hence I is a fantastic ideal of X. □

Theorem 5.10. *Let I be a non-empty subset of a BCH-algebra X. Then I is a fantastic ideal of X if and only if the fuzzy set λ of X defined by*

$$\lambda(x) = \begin{cases} \leq \tfrac{1-k}{2}, & \text{if } x \in X - I \\ 0, & \text{if } x \in I, \end{cases}$$

is a $(\bar{q}_k, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.

Proof. Let I be a fantastic ideal of X. Let $t \in (0, 1]$ be such that $0_t \bar{q}_k \lambda$. Then

$$\lambda(0) + t + k \leq 1,$$

so $0 \notin I$. This implies that $I = \phi$. Thus, if $t > \frac{1-k}{2}$, then

$$\lambda(x) \leq \tfrac{1-k}{2} < t,$$

so $x_t \bar{\in} \lambda$. If $t \leq \frac{1-k}{2}$, then

$$\lambda(x) + t + k \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} + k = 1.$$

This implies that $x_t \bar{q}_k \lambda$. Hence

$$x_t \bar{\in} \vee \bar{q}_k \lambda.$$

Now let $x, y \in X$ and $t, r \in (0, 1]$ be such that

$$(x * (y * (y * x)))_{t \wedge r} \bar{q}_k \lambda.$$

Then

$$\lambda(x * (y * (y * x))) + t \wedge r + k \leq 1,$$

so

$$x * (y * (y * x)) \notin I.$$

This implies that either

$$(x * y) * \text{z} \notin I \text{ or } z \notin I.$$

Suppose

$$(x * y) * z \notin I.$$

Thus, if $t > \frac{1-k}{2}$, then

$$\lambda((x * y) * z) \leq \frac{1-k}{2} < t,$$

and so

$$\lambda((x * y) * z) \leq t.$$

This implies that

$$((x * y) * z)_t \bar{\in} \lambda.$$

If $t < \frac{1-k}{2}$ and $((x * y) * z)_t \in \lambda$, then

$$\lambda((x * y) * z) \geq t.$$

As

$$\frac{1-k}{2} \geq \lambda((x * y) * z),$$

so $\frac{1-k}{2} \geq t$. Thus

$$\lambda((x * y) * z) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1,$$

that is

$$((x * y) * z)_t \bar{q}_k \lambda.$$

Hence

$$((x * y) * z)_t \bar{\in} \vee \bar{q}_k \lambda.$$

Similarly, if $z \notin$ I, then

$$z_r \bar{\in} \vee \bar{q}_k \lambda.$$

This shows that λ is a $(\bar{q}_k, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.

Conversely, assume that λ is a $(\bar{q}_k, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X. Let $x \in I$. If $0 \notin I$, then $\lambda(0) \leq \frac{1-k}{2}$. Now for any $t \in (0, \frac{1-k}{2}]$

$$\lambda(0) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1,$$

this implies that $0_t \bar{q}_k \lambda$. Thus

$$x_t \bar{\in} \vee \bar{q}_k \lambda.$$

But

$$\lambda(x) = 1 > t \text{ and } \lambda(x) + t + k > 1$$

implies that

$$x_t \in \wedge q_k \lambda,$$

which is a contradiction. Hence $0 \in I$. Now suppose $x, y, z \in X$, such that

$$(x * y) * z \text{ and } z \in \text{I}.$$

We have to show that

$$x * (y * (y * x)) \in I.$$

Contrarily, assume that

$$x * (y * (y * x)) \notin I.$$

Then

$$\lambda(x * (y * (y * x))) \leq \tfrac{1-k}{2}.$$

Now for $t \in (0, \frac{1-k}{2}]$, we have

$$\lambda(x * (y * (y * x))) + t + k \leq \tfrac{1-k}{2} + \tfrac{1-k}{2} + k = 1,$$

this is,

$$(x * (y * (y * x)))_t \bar{q}_k \lambda.$$

Thus

$$((x * y) * z)_t \bar{\in} \vee \bar{q}_k \lambda \text{ or } z_t \bar{\in} \vee \bar{q}_k \lambda.$$

But

$$(x * y) * z \text{ and } z \in I$$

implies

$$\lambda((x * y) * z) = \lambda(z) = 1.$$

This implies that

$$((x * y) * z)_t \in \wedge q_k \lambda \text{ and } z_t \in \wedge q_k \lambda,$$

which is a contradiction. Hence $x * (y * (y * x)) \in I$. □

Theorem 5.11. *Let I be a non-empty subset of a BCH-algebra X. Then I is a fantastic ideal of X if and only if the fuzzy set λ of X defined by*

$$\lambda(x) = \begin{cases} \leq \frac{1-k}{2} & \text{if } x \in X - I \\ 0 & \text{if } x \in I, \end{cases}$$

is an $(\bar{\in} \vee \bar{q}_k, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.

Proof. The proof follows from the proof of Theorem 5.9 and Theorem 5.10. □

Theorem 5.12. *The intersection of any family of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals of a BCH-algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.*

Proof. Let $\{\lambda_i\}_{i\in I}$ be a family of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals of a BCH-algebra X and $x \in X$. So

$$\lambda_i(0) \vee \frac{1-k}{2} \geq \lambda_i(x),$$

for all $i \in I$. Thus

$$\begin{aligned}(\underset{i\in I}{\wedge} \lambda_i)(0) \vee \tfrac{1-k}{2} &= \underset{i\in I}{\wedge}(\lambda_i(0) \vee \tfrac{1-k}{2}) \\ &\geq \underset{i\in I}{\wedge}(\lambda_i(x)) = (\underset{i\in I}{\wedge} \lambda_i)(x).\end{aligned}$$

Thus

$$(\underset{i\in I}{\wedge} \lambda_i)(0) \vee \tfrac{1-k}{2} \geq (\underset{i\in I}{\wedge} \lambda_i)(x).$$

Let $x, y, z \in X$. Since each λ_i is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X. So

$$\lambda_i(x * (y * (y * x))) \vee \tfrac{1-k}{2} \geq \lambda_i((x * y) * z) \wedge \lambda_i(z)$$

for all $i \in I$. Thus

$$\begin{aligned}(\underset{i\in I}{\wedge} \lambda_i)(x * (y * (y * x))) \vee \tfrac{1-k}{2} &= \underset{i\in I}{\wedge}(\lambda_i(x * (y * (y * x))) \vee \tfrac{1-k}{2}) \\ &\geq \underset{i\in I}{\wedge}(\lambda_i((x * y) * z) \wedge \lambda_i(z)) \\ &= (\underset{i\in I}{\wedge} \lambda_i)((x * y) * z) \wedge (\underset{i\in I}{\wedge} \lambda_i)(z).\end{aligned}$$

Therefore

$$(\underset{i\in I}{\wedge} \lambda_i)(x * (y * (y * x))) \vee \tfrac{1-k}{2} \geq (\underset{i\in I}{\wedge} \lambda_i)((x * y) * z) \wedge (\underset{i\in I}{\wedge} \lambda_i)(z).$$

Hence, $\underset{i\in I}{\wedge} \lambda_i$ is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X. □

Theorem 5.13. *The union of any family of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals of a BCH-algebra X is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X.*

Proof. Let $\{\lambda_i\}_{i\in I}$ be a family of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideals of a BCH-algebra X and $x \in X$. So

$$\lambda_i(0) \vee \tfrac{1-k}{2} \geq \lambda_i(x),$$

for all $i \in I$. Thus

$$\begin{aligned}(\underset{i\in I}{\vee} \lambda_i)\ (0) \vee \tfrac{1-k}{2} &= \underset{i\in I}{\vee}(\lambda_i(0) \vee \tfrac{1-k}{2}) \\ &\geq \underset{i\in I}{\vee}(\lambda_i(x)) \\ &= (\underset{i\in I}{\vee} \lambda_i)(x).\end{aligned}$$

Thus

$$(\underset{i\in I}{\vee} \lambda_i)(0) \vee \tfrac{1-k}{2} \geq (\underset{i\in I}{\vee} \lambda_i)(x).$$

Let $x, y, z \in X$. Since each λ_i is an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$-fuzzy fantastic ideal of X. So

$$\lambda_i(x * (y * (y * x))) \vee \tfrac{1-k}{2} \geq \lambda_i((x * y) * z) \wedge \lambda_i(z),$$

for all $i \in I$. Thus

$$\begin{aligned}(\bigvee_{i\in I}\lambda_i)(x*(y*(y*x)))\vee\tfrac{1-k}{2} &= \bigvee_{i\in I}(\lambda_i(x*(y*(y*x)))\vee\tfrac{1-k}{2})\\ &\geq \bigvee_{i\in I}(\lambda_i((x*y)*z)\wedge\lambda_i(z))\\ &= (\bigvee_{i\in I}\lambda_i)((x*y)*z)\wedge(\bigvee_{i\in I}\lambda_i)(z).\end{aligned}$$

Therefore

$$(\bigvee_{i\in I}\lambda_i)(x*(y*(y*x)))\vee\tfrac{1-k}{2}\geq(\bigvee_{i\in I}\lambda_i)((x*y)*z)\wedge(\bigvee_{i\in I}\lambda_i)(z).$$

Hence, $\bigvee_{i\in I}\lambda_i$ is an $(\bar{\in},\bar{\in}\vee\bar{q}_k)$-fuzzy fantastic ideal of X. □

6 Conclusions

In order to investigate the structure of an algebraic system, we see that the fuzzy ideals with special properties always play an essential role. The purpose of this paper is to initiated the notion of $(\bar{\alpha},\bar{\beta})$-fuzzy fantastic ideal in BCH-algebra and investigate some of their related properties. We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy BCH-algebras and their applications in other branches of algebra. In the future study of fuzzy BCH-algebras, perhaps the following topics are worth to be considered:

(1) To characterize other classes of BCH-algebras by using this notion;

(2) To apply this notion to some other algebraic structures;

(3) To consider these results to some possible applications in computer sciences and information systems in the future.

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