Approximation on unbounded subsets and the moment problem

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Abstract. We apply L^1 approximation to characterize existence of the solutions of the multidimensional moment problems in terms of quadratic mappings, similarly to the one-dimensional case. To this end, we approximate any nonnegative continuous compactly supported function by sums of tensor products of positive polynomials in each separate variable. On the other hand, an application of an earlier result concerning Markov moment problems related to distanced convex subsets is discussed. Finally, we deduce an application of an abstract moment problem to a concrete Markov moment problem. The Hahn-Banach principle and its generalizations play an important role along this work.

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1 Introduction

Applying polynomial decomposition and approximation results in the moment problem is a well-known technique [1] - [7], [9] - [22]. Using Hahn-Banach principle in existence of the solution is a powerful tool. Some of these extension results are contained in [18]. In solving existence of the solutions of moment problems, upper L^1 approximation is sufficient. On the contrary, uniqueness and construction of the solution involve L^2 norms [5], [14], [20]. As it is well known, in several dimensions there are positive polynomials on $\mathbb{R}^n$, $n \geq 2$ which are not sum of squares of some other polynomials. We solve this difficulty by approximating a positive continuous function vanishing at infinity with sums of tensor products of positive polynomials in one separate variable. Each of the factors of a term of this sum is represented as a sum of squares [1]. Thus, one can solve multidimensional moment problems in terms of quadratic forms. A similar approximation result is presented in [13], for a complex moment problem. The proofs are different with respect to those of the present work, the latter following the real analysis methods. Another aim of this work is to find new applications of an earlier result that involves a distanced convex set with respect to a subspace. For the background of this work, see [1], [8]. Uniqueness of the solutions

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is discussed in [5], [9], [10]. The paper is organized as follows. In Section 2, we recall some basic polynomial approximation results. Section 3 contains an application of one of these results to a Markov moment problem on an unbounded subset of $\mathbb{R}^3$. Section 4 contains an application of an earlier extension result involving a distanced vector subspace with respect to a convex bounded set. We deduce an application of an abstract moment problem to a concrete one. Section 5 concludes the paper.

2 Approximation results on unbounded subsets

Theorem 2.1. (Lemma 1.4 [17]) *Let $A \subset \mathbb{R}^n$ be a closed subset and ν a determinate positive regular Borel measure on A with finite moments of all orders. Then for any $\psi \in (C_0(A))_+$ there is a sequence $(p_m)_m$ of polynomials on A, $p_m \geq \psi$, $p_m \to \psi$ in $L^1_\nu(A)$. We have*

$$\lim \int_A p_m d\nu = \int_A \psi d\nu,$$

the cone P_+ of positive polynomials is dense in $(L^1_\nu(A))_+$ and P is dense in $L^1_\nu(A)$.

We recall that a determinate measure is, by definition, uniquely determined by its moments ([5], [9], [10]). We remind the next result on uniform approximation on compact subsets. The approximation in usual L^1 spaces holds too.

Theorem 2.2. (Lemma 1.3 (d) [17]) *If $x \in C_0([0,\infty) \times [0,\infty))$ is a nonnegative continuous compactly supported function, then there exists a sequence $(p_m)_m$ of positive polynomials on $[0,\infty) \times [0,\infty)$, such that*

$$p_m(t) > x(t),\ \forall t \geq 0,\ \forall m \in \mathbb{Z}_+,\ p_m \to x$$

uniformly on compact subsets of $[0,\infty) \times [0,\infty)$.

3 Solving Markov moment problems on unbounded subsets

Let $\nu = \nu_1 \times \nu_2 \times \nu_3$, where ν_j is a positive determinate regular Borel measure on $\mathbb{R}$, $j = 1, 2$, while ν_3 is a regular Borel measure on $[0, 1]$. Let $S_3 = \mathbb{R}^2 \times [0, 1]$, and Y be an order complete Banach lattice, with solid norm:

$$|y_1| \leq |y_2| \Rightarrow ||y_1|| \leq ||y_2||,\ y_j \in Y,\ j = 1, 2.$$

Denote $\varphi_{j,k,l}(t_1, t_2, t_3) = t_1^j t_2^k t_3^l$, $(j, k, l) \in \mathbb{N}^3$, $(t_1, t_2, t_3) \in S_3$ and let $\{y_{j,k,l}\}_{(j,k,l)} \subset Y$.

Theorem 3.1. *Let $F_2 : L^1_\nu(S_3) \to Y$ be a positive linear bounded operator. The following statements are equivalent:*

(a) there exists a unique linear operator $F : L^1_\nu(S_3) \to Y$, such that

$$F(\varphi_{j,k,l}) = y_{j,k,l}, \forall (j, k, l) \in \mathbb{N}^3, 0 \leq F(\psi) \leq F_2(\psi),\ \psi \in (L^1_\nu(S_3))_+,\ ||F|| \leq ||F_2||;$$

(b) for any finite subsets $J_1, J_2 \subset \mathbb{N}$, any $\{\alpha_j\}_{j\in J_1} \subset \mathbb{R}$, $\{\beta_k\}_{k\in J_2} \subset \mathbb{R}$, and all $p, q \in \mathbb{N}$, we have:

$$0 \leq \sum_{\substack{i,j \in J_1 \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l \left(\sum_{r=0}^{q} (-1)^r \begin{pmatrix} q \\ r \end{pmatrix} y_{(i+j,k+l,p+r)} \right)$$

$$\leq \sum_{\substack{i,j \in J_1 \\ k,l \in J_2}} \alpha_i \alpha_j \beta_k \beta_l \left(\sum_{r=0}^{q} (-1)^r \begin{pmatrix} q \\ r \end{pmatrix} F_2(\varphi_{(i+j,k+l,p+r)}) \right).$$

Proof. We define F_0 on the space of polynomials, such that the moment conditions are accomplished. Condition (b) says that

$$0 \leq F_0(p_1 \otimes p_2 \otimes p_3) \leq F_2(p_1 \otimes p_2 \otimes p_3),\ \forall p_1, p_2 \in (\mathbb{R}[X])_+,\ p_3(t_3) > 0,\ t_3 \in [0,1],$$

since p_j, $j = 1, 2$ are sums of squares of some other polynomials with real coefficients [1], while p_3 is a linear combination with positive coefficients of special polynomials

$$t_3^p (1 - t_3)^q,\ t_3 \in [0,1],$$

following [6]. Hence, the implication (a) $\Rightarrow$ (b) is obvious. For the converse, let ψ be a continuous nonnegative function with compact support contained in S_3. One considers a parallelepiped $K_3 = [a_1, b_1] \times [a_2, b_2] \times [0,1]$ containing

$$pr_1(\text{support}(\psi)) \times pr_2(\text{support}(\psi)) \times [0,1].$$

Extend ψ to K_3 with zero values outside its support and approximate this function by means of Luzin's Theorem and the corresponding Bernstein polynomials in three variables. Each term of such a polynomial is a tensor product of positive polynomials on the corresponding compact interval. Extend each of these factors with zero value outside the compact interval and apply Luzin's Theorem in each of the first two variables. Next one approximates these continuous functions with compact support by means of Theorem 2.1, applied to $n = 1$, $A = \mathbb{R}$. The conclusion is that ψ can be approximated by sums of tensor products of positive polynomials on $\mathbb{R}$, respectively on [0,1]:

$$\sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j} \otimes m_{m,3,j} \to \psi,\ m \to \infty,$$

in the space $L^1_\nu(S_3)$. On the other hand, the linear positive operator F_0 has a linear positive extension F to the space of all integrable functions with their absolute value dominated on S_3 by a polynomial (following [8, p. 160]). This space contains the space of continuous functions with compact support. Hence $h \circ F$ can be represented by a regular positive Radon measure, for any linear positive functional h on Y. Moreover, using (b) and applying Fatou's lemma, one obtains:

$$0 \leq h(F(\psi)) \leq \liminf_m (h \circ F) \left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j} \otimes m_{m,3,j} \right) \leq \quad (1)$$

$$\leq \lim_m (h \circ F_2) \left(\sum_{j=0}^{k(m)} p_{m,1,j} \otimes p_{m,2,j} \otimes m_{m,3,j} \right) = h(F_2(\psi)),\ \psi \in (C_c(S_3))_+,\ h \in Y_+^*.$$

Assume that

$$F_2(\psi) - F(\psi) \notin Y_+ .$$

Using a separation theorem, it should exist a positive linear continuous functional $h \in Y_+^*$ such that

$$h(F_2(\psi)) < h(F(\psi)).$$

This relation contradicts (1). Hence we must have

$$F(\psi) \le F_2(\psi),\ \psi \in (C_c(S_3))_+ .$$

Then for arbitrary $g \in C_c(S_3)$ one writes

$$|F(g)| \le F_2(g^+) + F_2(g^-) = F_2(|g|) \Rightarrow ||F(g)|| \le ||F_2|| \cdot ||g||_1 .$$

The conclusion is that the operator F is positive and continuous, of norm dominated by $||F_2||$, on a dense subspace of $L^1_\nu(S_3)$. It has a unique linear extension preserving these properties. This concludes the proof. □

4 Extension of linear operators and the moment problem

The next theorem has a significant geometric meaning and leads to interesting results concerning the extension of linear functionals and operators.

If V is a convex neighborhood of the origin in a locally convex space, we denote by p_V the gauge attached to V.

Theorem 4.1. *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset with the following properties:*

(a) *there exists a neighborhood V of the origin such that $(S+V) \cap A = \Phi$ (A and S are distanced);*

(b) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j\in J} \subset L(S,Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_j\}_{j\in J} \subset L(X,Y)$ such that

$$F_j|_S = f_j \quad \text{and} \quad F_j|_A \ge \tilde{y}, \quad \forall j \in J.$$

Moreover, if V is a neighborhood of the origin such that

$$f_j(V \cap S) \subset [-u_0, u_0], \quad (S+V) \cap A = \Phi,$$

$$0 < \alpha \in R \text{ s.t. } p_V|_A \le \alpha, \quad \alpha_1 > 0 \text{ s.t. } \tilde{y} \le \alpha_1 u_0,$$

then the following relations hold

$$F_j(x) \le (1 + \alpha + \alpha_1) p_V(x) \cdot u_0, \quad x \in X,\ j \in J.$$

We denote by X the space of all power series in n variables with real coefficients, centered at $(0,\ldots,0)$, that are absolutely convergent in $\mathbb{C}^n$. Let

$$\varphi_j(z_1,\ldots,z_n)=z_1^{j_1}\ldots z_n^{j_n},\quad j=(j_k)_{k=1}^n,\ |j|=\sum_{k=1}^n j_k\geq 1.$$

On the other hand, consider a complex Hilbert space H, $A_k\in A(H)$, $k=1,\ldots,n$ commuting positive selfadjoint operators acting on H. Endow X with the norm

$$||\varphi||_\infty=\sup\{|\varphi(z_1,\ldots,z_n)|;\ |z_k|\leq 1,\ k=1,\ldots,n\}.$$

Denote

$$Y_1=\{U\in A(H);UA_k=A_kU,k=1,\ldots,n\},$$
$$Y=\{U\in Y_1;UV=VU,\ \forall V\in Y_1\},$$
$$Y_+=\{U\in Y;\langle U(h),h\rangle\geq 0,\ \forall h\in H\}.$$

Here $A(H)$ is the real vector space of all selfadjoint operators acting on H. Obviously, Y is a commutative algebra of selfadjoint operators. Moreover, Y is an order complete vector lattice (see [8], [12]), and the operatorial norm is solid on Y:

$$|U|\leq|V|\Rightarrow||U||\leq||V||,\ U,V\in Y.$$

Theorem 4.2. *Let* $(B_j)_{j\in\mathbb{N}^n}$, $\sum_{k=1}^n j_k\geq 1$ *be a sequence in* Y, $0<\varepsilon<1$, *such that there exists a real constant* M *with the qualities*

$$|B_j|\leq M\cdot\frac{A_1^{j_1}}{j_1!}\ldots\frac{A_n^{j_n}}{j_n!}\ \forall j=(j_1,\ldots,j_n)\in\mathbb{N}^n,\ |j|\geq 1.$$

Let $\{\psi_k\}_{k\in\mathbb{N}^n}$ *be a sequence in* X, *such that* $\psi_k(0,\ldots,0)=1$, $||\psi_k||\leq 1$, $\forall k\in\mathbb{N}^n$. *Let* $\tilde{B}\in Y_+$. *Then there is a linear operator applying* X *into* Y *such that:*

$$F(\varphi_j)=B_j,\ j\in\mathbb{N}^n,\ \sum_{k=1}^n j_k\geq 1,\ F(\psi_k)\geq\tilde{B},$$

$$F(\varphi)\leq\left(2+||\tilde{B}||\cdot M^{-1}\exp\left(\sum_{k=1}^n||A_k||\right)\right)\cdot||\varphi||_\infty u_0,\quad u_0:=M\cdot\exp\left(\sum_{k=1}^n A_k\right).$$

Proof. Due to the behavior at $(0,\ldots,0)$ of the functions φ_j, $|j|:=\sum_{k=1}^n j_k\geq 1$ and ψ_k, $k\in\mathbb{N}^n$, we have

$$||s-a||_\infty\geq|s(0)-a(0)|\geq 1,\ \forall s\in S:=Sp\{\varphi_j;|j|\geq 1\},$$
$$\forall a\in A:=\text{ conv }\{\psi_k;k\in\mathbb{N}^n\}\Rightarrow(S+B(0,1))\cap A=\Phi.$$

Using also the hypothesis on the norms of the functions ψ_k, $k\in\mathbb{N}^n$, we can take in Theorem 4.1 $V=B(0,1)$, $||\cdot||_{|A}\leq 1:=\alpha$. Now let $s=\sum_{j\in J_0}\lambda_j\varphi_j\in S\cap B(0,1)$ and define the linear operator F_0 on the subspace S, such that the moment conditions $F_0(\varphi_j) = B_j$, $|j| \geq 1$ to be accomplished. In the above relations, $B(0,1)$ is the unit ball in X. Cauchy's inequalities yield

$$|\lambda_j| \leq ||s||_\infty \leq 1,\ j \in J_0 \Rightarrow f(s) = \sum_{j \in J_0} \lambda_j B_j \leq \sum_{j \in J_0} |\lambda_j| \cdot |B_j| \leq$$

$$\sum_{j \in J_0} |B_j| \leq M \cdot \left(\sum_{j_1 \in \mathbb{N}} \frac{A_1^{j_1}}{j_1!} \right) \dots \left(\sum_{j_n \in \mathbb{N}} \frac{A_n^{j_n}}{j_n!} \right) = M \cdot \exp\left(\sum_{k=1}^{n} A_k \right) = u_0.$$

It is easy to see that u_0 is strong order unit in Y. On the other hand, we have:

$$\tilde{B} \leq ||\tilde{B}|| \cdot I = ||\tilde{B}|| \cdot M^{-1} \exp\left(- \left(\sum_{k=1}^{n} A_k \right) \right) \cdot u_0 \leq$$

$$\leq ||\tilde{B}|| \cdot M^{-1} \exp\left(\sum_{k=1}^{n} ||A_k|| \right) \cdot u_0 = \alpha_1 u_0.$$

Application of Theorem 4.1 leads to the conclusion. □

We recall the following result [18] on the abstract Markov moment problem, as an extension with two constraints theorem for linear operators. It is a constrained interpolation problem.

Theorem 4.3. *Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ given families and $F_1, F_2 \in L(X,Y)$ two linear operators. The following statements are equivalent:*

(a) *there is a linear operator $F \in L(X,Y)$ such that*

$$F_1(x) \leq F(x) \leq F_2(x),\ \forall x \in X_+,\ F(x_j) = y_j,\ \forall j \in J;$$

(b) *for any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset R$, we have:*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \psi_2 - \psi_1,\ \psi_1, \psi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \leq F_2(\psi_2) - F_1(\psi_2).$$

From Theorem 4.3 we deduce the following result. Let Y be a commutative real Banach algebra, which is also an order complete Banach lattice, with solid norm. Let

$$a_k, b_k \in Y_+,\ ||a_k|| < 1,\ ||b_k|| < 1,\ k = 1, \dots, n.$$

Let $(y_j)_{j \in \mathbb{N}^n}$ be a sequence in Y_+. Consider the space X of all continuous functions in the unit closed polydisc, which can be represented by sums of absolutely convergent power series with real coefficients in the open polydisc. The order relation on X is given by the coefficients of the power series. Namely,

$$X_+ = \left\{ \sum_{j \in \mathbb{N}^n} c_j z^j; c_j \geq 0,\ \forall j \in \mathbb{N}^n \right\}.$$

Let

$$\varphi_j(z_1, \dots, z_n) = z_1^{j_1} \dots z_n^{j_n},\ j = (j_1, \dots, j_n) \in \mathbb{N}^n,\ |z_k| \leq 1,\ k = 1, \dots, n.$$

Theorem 4.4. *With these notations, the following statements are equivalent:*
(a) *there exists* $F \in B(X,Y)$ *such that*

$$F(\varphi_j) = y_j,\ j \in \mathbb{N}^n,\ \psi(a_1,\dots,a_n) - \varepsilon \cdot \psi(b_1,\dots,b_n) \le F(\psi) \le$$

$$\le \psi(a_1,\dots,a_n) + \varepsilon \cdot \psi(b_1,\dots,b_n),\ \psi \in X_+,\ ||F|| \le 1+\varepsilon;$$

(b) *we have*

$$a_1^{j_1}\dots a_n^{j_n} - \varepsilon \cdot b_1^{j_1}\dots b_n^{j_n} \le y_j \le a_1^{j_1}\dots a_n^{j_n} + \varepsilon \cdot b_1^{j_1}\dots b_n^{j_n},\ j=(j_1,\dots,j_n)\in\mathbb{N}^n.$$

Proof. The implication (a) $\Rightarrow$ (b) is obvious, because of the relations

$$\begin{aligned}&\varphi_j \in X_+ \Rightarrow y_j = F(\varphi_j) \in \\ &\in [\varphi_j(a_1,\dots,a_n) - \varepsilon\cdot\varphi_j(b_1,\dots,b_n),\ \varphi_j(a_1,\dots,a_n) + \varepsilon\cdot\varphi_j(b_1,\dots,b_n)] = \\ &= [a_1^{j_1}\dots a_n^{j_n} - \varepsilon\cdot b_1^{j_1}\dots b_n^{j_n},\ a_1^{j_1}\dots a_n^{j_n} + \varepsilon\cdot b_1^{j_1}\dots b_n^{j_n}],\ j\in\mathbb{N}^n.\end{aligned}$$

Conversely, assume that (b) holds. We verify the implication in (b), Theorem 4.3. Namely, we have:

$$\begin{aligned}&\sum_{j\in J_0}\lambda_j\varphi_j = \psi_2 - \psi_1 = \sum_{m\in\mathbb{N}^n}\alpha_m\varphi_m - \sum_{m\in\mathbb{N}^n}\beta_m\varphi_m,\ \alpha_m,\beta_m \ge 0,\ m\in\mathbb{N}^n \Rightarrow \\ &\sum_{j\in J_0}\lambda_j y_j = \sum_{j\in J_0^+}\lambda_j y_j + \sum_{j\in J_0^-}\lambda_j y_j \le \sum_{j\in J_0^+}\lambda_j(a_1^{j_1}\dots a_n^{j_n} + \varepsilon\cdot b_1^{j_1}\dots b_n^{j_n}) + \\ &+\sum_{j\in J_0^-}\lambda_j(a_1^{j_1}\dots a_n^{j_n} - \varepsilon\cdot b_1^{j_1}\dots b_n^{j_n}) \le \sum_{j\in J_0}\lambda_j a_1^{j_1}\dots a_n^{j_n} + \varepsilon\left(\sum_{m\in\mathbb{N}^n}\alpha_j b_1^{j_1}\dots b_n^{j_n}\right) + \\ &+\varepsilon\left(\sum_{m\in\mathbb{N}^n}\beta_j b_1^{j_1}\dots b_n^{j_n}\right) = (\psi_2-\psi_1)(a_1,\dots,a_n) + \varepsilon\psi_2(b_1,\dots,b_n) + \varepsilon\psi_1(b_1,\dots,b_n) = \\ &= \psi_2(a_1,\dots,a_n) + \varepsilon\cdot\psi_2(b_1,\dots,b_n) - [\psi_1(a_1,\dots,a_n) - \varepsilon\psi_1(b_1,\dots,b_n)] = \\ &= F_2(\psi_2) - F_1(\psi_1),\ J_0^+ = \{j\in J_0; \lambda_j \ge 0\},\ J_0^- = \{j\in J_0; \lambda_j < 0\}.\end{aligned}$$

A direct application of Theorem 4.3 leads to the existence of a linear operator $F \in L(X,Y)$ such that

$$\psi(a_1,\dots,a_n) - \varepsilon\cdot\psi(b_1,\dots,b_n) \le F(\psi) \le \psi(a_1,\dots,a_n) + \varepsilon\cdot\psi(b_1,\dots,b_n),\ \forall\psi\in X_+ \Rightarrow$$

$$|F(\psi)| \le \psi(a_1,\dots,a_n) + \varepsilon\cdot\psi(b_1,\dots,b_n),\ \forall\psi\in X_+\ .$$

For an arbitrary $\varphi \in X$, one obtains:

$$|F(\varphi)| \le |F(\varphi^+)| + |F(\varphi^-)| \le |\varphi|(a_1,\dots,a_n) + \varepsilon\cdot|\varphi|(b_1,\dots,b_n) \Rightarrow$$

$$||F(\varphi)|| \le (1+\varepsilon)\cdot||\varphi||_\infty,\ \forall\varphi\in X \Rightarrow ||F|| \le 1+\varepsilon.$$

This concludes the proof. □

5 Conclusions

The present work starts by recalling two polynomial approximation results on unbounded subsets. Next, we deduce an application to a Markov moment problem of one of these approximation theorems. One uses decomposition of positive polynomials on [0,1] into sums of special generating polynomials too. Next, we apply a general extension result involving a subspace distanced with respect to a convex subset, to an operator valued moment problem. The last result is an application of an abstract moment problem to a concrete Markov moment problem. The results of Sections 3 and 4 are interpolating theorems with two constraints.

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