On the stability of perturbed time scale systems using integral inequalities

B. Ben Nasser, K. Boukerrioua and M. A. Hammami

Abstract. This paper focuses on the problem of uniform asymptotic stability of certain classes of dynamic perturbed systems on time scales using time scale versions of some Gronwall type inequalities. We prove under certain conditions on the linear and nonlinear perturbations that the resulting perturbed nonlinear initial value problem still acquire uniformly asymptotically stable, if the associated system has already owned this property.

M.S.C. 2010: 74H55, 93D20, 26E70, 35A23.
Key words: Stability; time scales; perturbed systems; Gronwall inequalities.

1 Introduction

The paper will illustrate the power of a burgeoning field of mathematics called dynamic equations on time scales. A time scale is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. We assume that $\mathbb{T}$ is a time scale and it has the inherited topology from the standard topology on the real numbers $\mathbb{R}$. During the last decades, time scale methods have rapidly been developed, and have received a lot of attention by several authors, not only to unify continuous and discrete processes, but also help reveal diversities in the corresponding results. The analysis of nonlinear perturbations of linear systems is not only important for its own sake but also has a broad range of applications. Let the regressive non-autonomous linear differential or difference equations described by the following equations in vector form

$$x^{\Delta}(t) = A(t)x(t), \tag{1.1}$$

where $x \in \mathbb{R}^n,\ t \in \mathbb{T}$, and $A : \mathbb{T} \to \mathbb{R}^{n*n}$ a regressive matrix valued function, has been treated for a long time in discrete and continuous case and under various hypotheses to investigate the behavior of solutions. Here, we represent the perturbation as an additive term on the right hand of (1.1). So, the associated perturbed system on time scales is given by

$$x^{\Delta}(t) = A(t)x(t) + F(t, x(t)), \tag{1.2}$$

Applied Sciences, Vol. 16, 2014, pp. 56-71.

where $F(t,0) = 0$ on $\mathbb{T}$, and $F : \mathbb{T}\times\mathbb{R}^n \to \mathbb{R}^n$ is an rd-continuous function. The theory of perturbed systems may be viewed as part of dynamical systems theory: Differential equations with deterministic (time varying) perturbations. For the continuous type, the authors in [3], [10] and [12] are interested to study the problem of asymptotic stability by referring to Lyapunov techniques. A lot of work which has been done concerning the stability and the state estimation of such systems was developed using the Lyapunov theory and integral Gronwall-Bellman inequalities. Properties of exponential stability of a time-varying dynamic equation on a time scale have also been investigated recently by Bohner and Martynyuk [5], Du and Tien [11], Hoffacker and Tisdell [15] and Martynyuk [18]. In 1919, T. Gronwall gave the Gronwall Bellman inequality (see [14]). After that, many authors gave a number of generalizations of this inequality and these generalizations had significant applications in differential and integral equations. The recent work of Willett (1964), Gollwitzer (1969), Butler and Rogers (1971) and Lakshmikantham (1973) show that the interest in the Gronwall-Bellman inequality still continues. During the past decades, with the development of the theory of differential and integral equations, a lot of integral and difference inequalities in time scales version like [16] and [5] and the references therein, have been discovered. This new integral inequality approaches deliver a good process to estimate solutions and submit several results that relate the stability properties of the perturbed systems (1.2) to those of unperturbed linear systems (1.1). The study of uniform asymptotic stability of perturbed systems (1.2) is related to two essential points. First, the solutions of unperturbed equations (1.1) are supposed to have bounded growth when we assume that the trajectories have a uniform asymptotic behavior. Secondly, the perturbation term $F(t,x)$ could result from modeling errors, aging, or uncertainties and disturbances, which exist in any realistic scientific problem. In this paper we gives some time scale versions of well known differential Gronwall-Bellman inequalities. Based on this new approaches we investigate uniform asymptotic stability of trajectories, after moderating the perturbation function F by the adequate growth rate. We note that this work contains generalizations of some results concerning the stability of dynamic systems related to perturbed term contained in works of J. J. Dacunha [8] section 5 and [9], using the integral inequality approach. Some numerical examples are given to ensure the useless of results.

2 Preliminaries

2.1 Time-scale analysis

For notational purposes, we shall further assume that in $\mathbb{T}$, the interval $[a,b]_{\mathbb{T}}$ means the set $\{t \in \mathbb{T}; a \le t \le b\}$ for the points $a < b$ in $\mathbb{T}$. If $b = \infty$, we denote $\mathbb{T}_a^+ = [a,\infty[_{\mathbb{T}}$. We introduce now some basic notions connected to the theory of time scales, which summarize the material from the recent book by Bohner and Peterson [4].

Definition 2.1. Let $t \in \mathbb{T}$. We define *the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T}; s > t\},$$

while *the backward jump operator* $\rho := \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T}; s < t\}.$$

In this definition, we put $\inf \varnothing = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \varnothing = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m), where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Points that are right-dense and left-dense at the same time are called *dense*. If $\mathbb{T}$ has a left-scatted maximum M, then we define $\mathbb{T}^k = \mathbb{T}\backslash\{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Definition 2.2. The function $f : \mathbb{T} \to \mathbb{R}$ is called Δ-differentiable at a point $t \in \mathbb{T}^k$ if there exist $\gamma \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a W-neighborhood of $t \in \mathbb{T}^k$ satisfying

$$|[f(\sigma(t)) - f(s)] - \gamma[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \ for\ all\ s \in W.$$

In this case we shall write $f^{\Delta}(t) = \gamma$.

When $\mathbb{T} = \mathbb{R}$, $x(t) = x'(t)$. When $\mathbb{T} = \mathbb{Z}$, $x(t)$ is the standard forward difference operator $x(n+1) - x(n)$.

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and left-sided limits exists(finite) at left-dense points in $\mathbb{T}$ and denotes by $\mathcal{C}_{rd} = \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.4. If $F^{\Delta}(t) = f(t),\ t \in \mathbb{T}$, then it is said that F is a (delta) antiderivative of f and the Cauchy (delta) integral is given by

$$\int_{\tau}^{s} f(t)\Delta t = F(s) - F(\tau), \text{ for } s, \tau \in \mathbb{T}.$$

For a more general definition of the delta integral, see [4]. We will use the following result later.

Theorem 2.1. *Let $t_0 \in \mathbb{T}^k$ and $w : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ be continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > t_0$. Assume that $w^{\Delta}(t,.)$ is rd-continuous on $[t_0, \sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$|w(\sigma(t),\tau) - w(s,\tau) - w^{\Delta}(t,\tau)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \ \ for\ all\ s \in U,$$

where w^{Δ} denotes the derivative of w with respect to the first variable. Then

$$g(t) = \int_{t_0}^{t} w(t,\tau)\Delta\tau$$

implies

$$g^{\Delta}(t) = \int_{t_0}^{t} w^{\Delta}(t,\tau)\Delta\tau + w(\sigma(t), t).$$

Definition 2.5. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regressive* provided that

$$1 + \mu(t)p(t) \neq 0 \ \ for\ all\ \ t \in \mathbb{T}^k.$$

The set of all *regressive and rd-continuous* function is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. The set $\mathcal{R}^+$ of all positively regressive function is

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}^k\}$$

We use the cylinder transformation to define a generalized exponential function for an arbitrary time scale $\mathbb{T}$.

Definition 2.6. If $p \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, then we define the generalized exponential function $e_p(t, s)$ by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \text{for } s, t \in \mathbb{T},$$

where $\xi_h(z)$ is the cylinder transformation given by

$$\xi_h(z) = \begin{cases} \frac{1}{h}\log(1 + zh) & \text{if } h \neq 0 \\ z & \text{if } h = 0. \end{cases}$$

Here log is the principal logarithm function.

Remark 2.7. Consider the regressive dynamic initial value-problem

$$x^{\Delta}(t) = p(t)x(t),\ x(t_0) = x_0,\ t_0 \in \mathbb{T}. \tag{2.1}$$

The exponential function $x(t) = e_p(t, t_0)x_0$ is the unique solution of (2.1).

For what follows, we need several theorems presented below.

Theorem 2.2. *(Comparison Theorem). Let $t_0 \in \mathbb{T}$, $x,\ f \in \mathcal{C}_{rd}$ and $p \in \mathcal{R}^+$. Then*

$$x^{\Delta}(t) \leq p(t)x(t) + f(t),\ for\ all\ t \in \mathbb{T}^+_{t_0}$$

implies

$$x(t) \leq x(t_0)e_p(t, t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau,\ for\ all\ t \in \mathbb{T}^+_{t_0}.$$

Theorem 2.3. *(Gronwall's Inequality). Let $t_0 \in \mathbb{T}$, $x,\ f \in \mathcal{C}_{rd}$ and $p \in \mathcal{R}^+, p \geq 0$. Then*

$$x(t) \leq f(t) + \int_{t_0}^t x(\tau)p(\tau)\Delta\tau,\ for\ all\ t \in \mathbb{T}^+_{t_0}$$

implies

$$x(t) \leq f(t) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)p(\tau)\Delta\tau,\ for\ all\ t \in \mathbb{T}^+_{t_0}.$$

It is clear from the proofs of the last two results in Bohner and Peterson [4] that in each case, reversing the inequalities in the assumptions yields corresponding lower (instead of upper) estimates for the solution. The next theorem proven in [20] deals with the time scale version of the inequality due to Sansone and Conti, see [19, p.86] . By employing these previous two theorems, in particular, the generalized Gronwall inequality, we have the following generalized dynamic inequality.

Theorem 2.4. *Let $a \in \mathbb{T}$, χ, ϱ, $\vartheta \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ and ϱ be delta-differentiable on $\mathbb{T}$ with $\varrho^{\Delta}(t) \geq 0$. If*

$$\chi(t) \leq \varrho(t) + \int_a^t \vartheta(s)\chi(s)\Delta s$$

for all $t \in \mathbb{T}_a^+$, then

$$\chi(t) \leq \varrho(a)e_{\vartheta}(t,a) + \int_a^t \varrho^{\Delta}(s)e_{\vartheta}(t,\sigma(s))\Delta s$$

for all $t \in \mathbb{T}_a^+$.

Now, we find a comparison relationship in [6] between the general exponential function on a time scale and the classical exponential function.

Lemma 2.5. *For nonnegative p with $-p \in \mathcal{R}^+$ we have the inequalities*

$$1 - \int_a^t p(u)\Delta u \leq e_{-p}(t,a) \leq \exp\{-\int_a^t p(u)\Delta u\} \ \ for\ all\ \ t \in \mathbb{T}_a^+. \tag{2.2}$$

For the purpose of use we state this relationship already proven in [6, remark.2]

Remark 2.8. If p is a nonnegative and a rd-continuous function, then

$$1 + \int_a^t p(u)\Delta u \leq e_{p}(t,a) \leq \exp\{\int_a^t p(u)\Delta u\} \ \ for\ all\ \ t \in \mathbb{T}_a^+. \tag{2.3}$$

Definition 2.9. A mapping $A : \mathbb{T} \to \mathbf{M}_n(\mathbb{R})$ is called *regressive* if for each $t \in \mathbb{T}$ the $n \times n$ matrix $I + \mu(t)A$ is invertible, where I the identity matrix. The class of all regressive and rd-continuous functions A from $\mathbb{T}$ to $\mathbf{M}_n(\mathbb{R})$ is denoted by $\mathcal{C}_{rd}\mathcal{R}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$.

We shall always assume that the coefficient matrix in the model (1.1) is sufficiently well behaved for there to exist a unique solution to the state-space model for any specified initial condition $x(t_0)$.
For instance, if the initial value problem with rd-continuous and regressive right hand side accepts a well-known point of departure, consequently, the desired existence and uniqueness properties hold.

Definition 2.10. Let $t_0 \in \mathbb{T}$. The unique matrix-valued solution of the `IVP`

$$X^{\Delta}(t) = A(t)X(t),\ X(t_0) = I_n, \tag{2.4}$$

where $A \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$, is called *the matrix exponential function* and it denoted by $\Phi_A(t, t_0)$.

Accordingly, the matrix function $\Phi_A(t, t_0)$ possesses the following two properties:

$$\Phi_A^{\Delta}(t, t_0) = A(t)\Phi_A(t, t_0) \tag{2.5}$$

$$\Phi_A(t_0, t_0) = I_n. \tag{2.6}$$

This matrix function is referred to as the state transition matrix, and our assumption in the nature of $A(t)$ turns out that the state transition matrix exists and is unique.

Theorem 2.6. *Suppose $A, B \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$ are matrix-valued functions on $\mathbb{T}$, then*
(i) *The transition operator has the linear co-cycle property, i.e., $\Phi_A(t,r)\Phi_A(r,s) = \Phi_A(t,s)$ for $r,s,t \in \mathbb{T}$;*
(ii) *$\Phi_A(\sigma(t),s) = (I + \mu(t)A(t))\Phi_A(t,s)$;*
(iii) *If $\mathbb{T} = \mathbb{R}$ and A is constant, then $\Phi_A(t,s) = e_A(t,s) = e^{A(t-s)}$;*
(iv) *If $\mathbb{T} = h\mathbb{Z}$, with $h > 0$, and A is constant, then $\Phi_A(t,s) = (I + hA)^{\frac{(t-s)}{h}}$.*

2.2 Stability definitions

Now, the concepts of stability (uniform stability, asymptotic stability, exponential stability,...) are defined by various way. We note that the stability of any solution of (1.1) or (1.2) is closely related to the stability of the null solution of the corresponding variational equation. Therefore, we will discuss the stability of perturbed dynamic system. In this section we we illustrate some useful definitions of stability which involves the boundedness of solutions of the regressive time varying perturbed dynamic equation (1.2).

Definition 2.11. a) Equation (1.2) is said to be *stable* if, for every $t_0 \in \mathbb{T}$ and for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that,

$$\|x(t_0)\| < \delta \text{ implies } \|x(t,t_0,x_0)\| < \varepsilon, \text{ for all } t \in \mathbb{T}^+_{t_0}. \tag{2.7}$$

b) Equation (1.2) is said to be *uniformly stable* if it is stable and for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ independent on initial point t_0, such that relation (2.7) is satisfied.
c) Equation (1.2) is said to be *uniformly asymptotically stable* if it is uniformly stable and it is uniformly attractive, i.e., there exists a positive constant c, independent of t_0, such that

$$\|x(t_0)\| < c \;\; \textit{implies} \;\; \lim_{t\to+\infty} \|x(t,t_0,x_0)\| = 0, \tag{2.8}$$

uniformly in t_0.
d) Equation (1.2) is said to be *globally uniformly asymptotically stable* if it is uniformly stable, $\delta(\varepsilon)$ can be chosen to satisfy $\lim\limits_{t\to+\infty} \delta(\varepsilon) = +\infty$ and the relation (2.8) holds for every $x(t_0) \in \mathbb{R}^n$.
e) Equation (1.2) is said to be *uniformly exponentially stable* if there exist the constants $\lambda, \gamma > 0$, with $-\lambda \in \mathcal{R}^+$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$\|x(t)\| \le \gamma e_{-\lambda}(t,t_0)\|x_0\|, \text{ for all } t \in \mathbb{T}^+_{t_0}. \tag{2.9}$$

S.K.Choi and al. [7] proved that the stability of (1.1) is equivalent to the boundedness of all its solutions when $A \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$. Also, DaCunha [8] proved that the uniform stability of (1.1) is equivalent to the uniform boundedness of all its solutions with respect to the initial state (t_0, x_0), when $A \in \mathcal{C}_{rd}\mathcal{R}(\mathbb{T}, \mathbf{M}_n(\mathbb{R}))$. He obtained the following characterization of uniform stability by means of the operator norm. It is no difficult to prove this result by using the maximum norm.

Theorem 2.7. *The linear system (1.1) is uniformly stable if and only if there exists a positive constant γ, such that*

$$\|\Phi_A(t,t_0)\| \leq \gamma, \text{ for all } t \in \mathbb{T}^+_{t_0}.$$

Dacunha defined in [8] a stability property that not only concerns the boundedness of trajectories to (1.1), but also the asymptotic features of the solutions as well. If the trajectories of (1.1) possess the following stability aspect, then the solutions approach zero exponentially when $t \to \infty$. Also, the uniformly exponential stability of the linear dynamic equation is characterized in terms of its transition operator.

Theorem 2.8. *The linear system (1.1) is* uniformly exponentially stable *if and only if there exists a positive constant λ with $-\lambda \in \mathcal{R}^+$ and there is $\gamma \geq 1$ independent on any initial point t_0, such that*

$$\|\Phi_A(t,t_0)\| \leq \gamma e_{-\lambda}(t,t_0), \text{ holds for all } t \in \mathbb{T}^+_{t_0}. \tag{2.10}$$

By using properties of transition matrix associated to (1.1), Dacunha [8] illustrated the equivalence between *uniform exponential stability* and *globally uniform asymptotic stability.*

Theorem 2.9. *The linear state equation (1.1) is uniformly exponentially stable if and only if it is globally uniformly asymptotically stable.*

3 Statement of results

3.1 Some integral inequalities

Now, we are in position to state a time scale version of Gollwitzer Theorem [13].

Lemma 3.1. *Let $a \in \mathbb{T}$, $\phi, \varphi, \alpha, \psi \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ which satisfy*

$$\phi(t) \leq \varphi(t) + \psi(t)\int_a^t \alpha(s)\phi(s)\Delta s.$$

Then

$$\phi(t) \leq \varphi(t) + \psi(t)\int_a^t \alpha(s)\varphi(s)\exp(\int_{\sigma(s)}^t \alpha(\tau)\psi(\tau)\Delta\tau)\Delta s, \text{ for all } t \in \mathbb{T}^+_a. \tag{3.1}$$

Proof. Let $t \in \mathbb{T}^+_a$. Define a function $z(t)$ by $z(t) = \int_a^t \alpha(s)\phi(s)\Delta s$. Then,

$$\phi(t) \leq \varphi(t) + \psi(t)z(t), \tag{3.2}$$

and

$$z^\Delta(t) = \alpha(t)\phi(t) \leq [\alpha(t)\psi(t)]z(t) + \alpha(t)\varphi(t). \tag{3.3}$$

Applying Theorem 2.2, we get

$$z(t) \leq \int_a^t e_{\alpha\psi}(t,\sigma(s))\alpha(s)\varphi(s)\Delta s. \tag{3.4}$$

Using the exponential approximation cited in remark 2.8, we obtain

$$e_{\alpha\psi}(t,\sigma(s)) \leq \exp(\int_{\sigma(s)}^{t} \alpha(\tau)\psi(\tau)\Delta\tau). \tag{3.5}$$

Substituting (3.4) and (3.5) in (3.2), we arrive at the required inequality in (3.1). □

We need the following comparison Lemma to prove Theorem (3.6).

Lemma 3.2. *Suppose $\phi, \varphi, \alpha, \psi, \chi \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$. Then*

$$\phi(t) \leq \varphi(t) + \psi(t)\int_a^t \{\alpha(s)\phi(s) + \chi(s)\}\Delta s, \ \textit{for all } t \in \mathbb{T}_a^+,$$

implies

$$\phi(t) \leq \varphi(t) + \psi(t)\int_a^t \{\alpha(s)\varphi(s) + \chi(s)\} \exp[\int_{\sigma(s)}^{t} \alpha(\tau)\psi(\tau)\Delta\tau]\Delta s, \ \textit{for all } t \in \mathbb{T}_a^+. \tag{3.6}$$

Proof. Let $t \in \mathbb{T}_a^+$. Define a function $z(t)$ by

$$z(t) = \int_a^t \{\alpha(s)\phi(s) + \chi(s)\}\Delta s.$$

The remainder of the proof also follows the arguments of the proof of Lemma 3.1, and hence it is also omitted. □

We prove the following result that will be used to ensure uniform asymptotic behavior of solutions in the next subsection.

Lemma 3.3. *Assume that $\phi, \varphi, \psi \in \mathcal{C}_{rd}([a, b[_{\mathbb{T}}, \mathbb{R}_+)$, let $R(t, x) : [a, b[_{\mathbb{T}} \times \mathbb{R}_+ \to \mathbb{R}_+$ be an rd-continuous function in its first argument, continuous in its second argument and differentiable on the domain $[a, b[_{\mathbb{T}} \times]0, \infty[$. If its partial derivative $\frac{\partial R}{\partial x}(t, x)$ is non-negative on $]a, b[_{\mathbb{T}} \times]0, +\infty[$ and there exists a continuous function $S :]a, b[_{\mathbb{T}} \times \mathbb{R}_+ \to \mathbb{R}_+$, such that*

$$\frac{\partial R}{\partial x}(t, x) \leq S(t, y) \quad \forall\, t \in]a, b[_{\mathbb{T}} \textit{ and } x \geq y > 0. \tag{3.7}$$

Then

$$\phi(t) \leq \varphi(t) + \psi(t)\int_a^t R(u, \phi(u))\Delta u, \ t \in [a, b[_{\mathbb{T}}, \tag{3.8}$$

implies

$$\phi(t) \leq \varphi(t) + \psi(t)\int_a^t R(s, \varphi(u)) \exp[\int_{\sigma(s)}^{t} S(r, \varphi(r))\psi(r)\Delta r]\Delta s, \quad t \in [a, b[_{\mathbb{T}}\ . \tag{3.9}$$

Proof. Let us consider the mapping $y : [a, b[_{\mathbb{T}} \to \mathbb{R}_+$ given by

$$y(t) := \int_a^t R(s, \phi(s))\Delta s.$$

Then y is Δ-differentiable on $]a, b[_{\mathbb{T}}$, $y^{\Delta}(t) = R(t, \phi(t))$ if $t \in]a, b[_{\mathbb{T}}$ and $y(a) = 0$. By applying the mean value theorem for the function $R(t, .)$, then for every $x \geq y > 0$ and $t \in]a, b[_{\mathbb{T}}$, there exists $c \in]y, x[$ such that

$$R(t, x) - R(t, y) = \frac{\partial R(t, c)}{\partial x}(x - y),$$

and therefore

$$R(t, x) - R(t, y) \leq S(t, y)(x - y).$$

Now, it follows that for any $t \in]a, b[_{\mathbb{T}}$

$$\begin{aligned} & y^{\Delta}(t) = R(t, \phi(t)) \leq R(t, \varphi(t) + \psi(t)y(t)), \\ & R(t, \varphi(t) + \psi(t)y(t)) \leq R(t, \varphi(t)) + S(t, \varphi(t))\psi(t)y(t); \end{aligned}$$

$$y^{\Delta}(t) \leq R(t, \varphi(t)) + S(t, \varphi(t))\psi(t)y(t). \tag{3.10}$$

Thus, integrating both sides of (3.10) from a to $t \in \mathbb{T}$, we obtain

$$y(t) = \int_a^t y^{\Delta}(s)\Delta s \leq \int_a^t R(s, \varphi(s))\Delta s + \int_a^t S(s, \varphi(s))\psi(s)y(s)\Delta s. \tag{3.11}$$

We further define

$$\varrho(t) = \int_a^t R(s, \varphi(s))\Delta s \textit{ and } \vartheta(t) = S(s, \varphi(s))\psi(s). \tag{3.12}$$

Then (3.11) becomes

$$y(t) \leq \varrho(t) + \int_a^t \vartheta(s)y(s)\Delta s. \tag{3.13}$$

Now applying Theorem 2.4 to (3.13), we get

$$y(t) \leq \int_a^t R(s, \varphi(s))e_{\vartheta}(t, \sigma(s))\Delta s. \tag{3.14}$$

Taking into account the relation in (2.3), we have

$$e_{\vartheta}(t, \sigma(s)) \leq \exp\{\int_{\sigma(s)}^t S(r, \varphi(r))\psi(r)\Delta r\}. \tag{3.15}$$

By substituting (3.15) and (3.12) in (3.14) we obtain

$$y(t) \leq \int_a^t R(s, \varphi(s)) \exp\{\int_{\sigma(s)}^t S(r, \varphi(r))\psi(r)\Delta r\}\Delta s.$$

From this results the desired estimation is (3.9), which completes the proof. □

3.2 Stability using Gronwall type inequalities

Certainly, between solutions of a linear equation (1.1) and the corresponding perturbed equation (1.2), there is a relation expressed by the formula for variation of parameters for the discrete and continuous case, given by Aleksev's formula in 1961 (see [1], or some of its generalizations in [17]). Now, we are in a position to state a time scale version of this formula.

Theorem 3.4. *We consider the regressive time varying perturbed system*

$$(1.2) \text{ and } x(t_0) = x_0, \tag{3.16}$$

where $x_0 \in \mathbb{R}^n$. Then the solution of (3.16) is given by

$$x(t) = \Phi_A(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,x(s))\Delta s. \tag{3.17}$$

Proof. Since the matrix function $\Phi_A(t,t_0)$ satisfies (2.5) and (2.6), the solution to the regressive time-varying perturbed system (1.2) with the initial condition x_0, has the explicit form (3.17). We note again that the terms corresponding to the zero-input and zero state responses are evident in (3.17). In order to verify (3.17), using Theorem 2.1, we differentiate it with respect to t:

$$x^{\Delta}(t) = [\Phi_A(t,t_0)]^{\Delta_t}x_0 + \int_{t_0}^{t} [\Phi_A(t,\sigma(s)]^{\Delta_t}F(s,x(s))\Delta s + \Phi_A(\sigma(t),\sigma(t))F(t,x(t)).$$

By using (2.5) and (2.6), we obtain

$$x^{\Delta}(t) = A(t)\Phi_A(t,t_0) + \int_{t_0}^{t} A(t)\Phi_A(t,\sigma(s))F(s,x(s))\Delta s + F(t,x(t)).$$

Now, since the integration is taken with respect to s, $A(t)$ can be factored out:

$$\begin{aligned} x^{\Delta}(t) &= A(t)[\Phi_A(t,t_0)x_0 + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,x(s))\Delta s] + F(t,x(t)) \\ &= A(t)x(t) + F(t,x(t)). \end{aligned}$$

So, the expression in (3.17) does indeed satisfy the state evaluation equation. We note that it also matches the specified initial condition, $\Phi_A(t_0,t_0)x(t_0) = x(t_0)$. □

Finally, from formula (3.17), we can derive conclusions about the behavior of solutions of the perturbed equation (1.2), given the behavior of the solutions of equation (1.1), the fundamental matrix $\Phi_A(.,.)$, and the perturbation F.

The next theorem shows that the uniform exponential stability is also preserved under some integral perturbations.

Theorem 3.5. *If the following conditions are satisfied,*

(i) *Equation (1.1) is uniformly exponentially stable with positive constants λ and γ,*

(ii) $\|F(t,x)\| \leq d(t)\|x\|$, $d(.) \in \mathcal{C}_{rd}(\mathbb{T},\mathbb{R}_+)$,

(iii) $\int_a^{+\infty} \frac{d(s)}{1-\lambda\mu(s)}\Delta s \leq \tilde{d} < +\infty$,

then, the trivial solution $x(t) \equiv 0$ of equation (1.2) is uniformly exponentially stable.

Proof. Let $t_0 \in \mathbb{T},\ t \in \mathbb{T}_{t_0}^+$ *and* $x(t) = x(t, t_0, x_0)$ denote the solution of the dynamic initial value problem (1.2). Using the variation of constants formula (3.17) with conditions(i) and (ii), we obtain the following estimation for an arbitrary solution $x(t)$ of system (1.2)

$$\|x(t)\| \leq \|\Phi_A(t, t_0)\| \cdot \|x_0\| + \int_{t_0}^{t} \|\Phi_A(t, \sigma(s))\| \cdot \|F(s, x(s))\|\Delta s,$$

and

$$\|x(t)\| \leq \gamma e_{-\lambda}(t, t_0)\|x_0\| + \gamma e_{-\lambda}(t, t_0) \int_{t_0}^{t} e_{-\lambda}(t_0, \sigma(s))d(s)\|x(s)\|\Delta s.$$

By applying Lemma 3.1, to the above estimation, we get

$$\|x(t)\| \leq \gamma e_{-\lambda}(t, t_0)\|x_0\| + \gamma^2 e_{-\lambda}(t, t_0)\|x_0\| \int_{t_0}^{t} \frac{d(s)}{1 - \lambda\mu(s)} \times \exp[\gamma \int_{\sigma(s)}^{t} \frac{d(\tau)}{1 - \lambda\mu(\tau)}\Delta\tau]\Delta s.$$

Taking into consideration the condition (iii), we infer

$$\|x(t)\| \leq \gamma[1 + \gamma\tilde{d}\exp[\gamma\tilde{d}]]e_{-\lambda}(t, t_0)\|x_0\|.$$

Hence, this estimation holds for each $t \in \mathbb{T}_{t_0}^+$. This inequality shows that the zero equilibrium state of system (1.2) is uniformly exponentially stable, and the proof is complete. □

Remark 3.1. The discrete version ($\mathbb{T} = \mathbb{Z}$) can be found in [2, Theorem 5.6.1].

Theorem 3.6. *Assume that there exist $d, k \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ that satisfy the following conditions*

(i) $\|F(t, x)\| \leq d(t)\|x\| + k(t)$,

(ii) $\int_a^{+\infty} \frac{d(s)}{1-\lambda\mu(s)}\Delta s \leq \tilde{d} < +\infty,\ \int_a^{+\infty} k(s)e_{-\lambda}(a, \sigma(s))\Delta s \leq \bar{k} < +\infty$,

and the homogeneous system is uniformly asymptotically stable. Then the perturbed system is uniformly attractive.

Proof. Let $t_0 \in \mathbb{T},\ t \in \mathbb{T}_{t_0}^+$ *and* $x(t) := x(t, t_0, x_0)$ a solution of system (1.2) that satisfies the variation of constant formula (3.17). Since the homogeneous system is uniformly asymptotically stable, we can estimate $x(t)$ by using the growth condition of the perturbed term in hypothesis (i)

$$\|x(t)\| \leq \gamma e_{-\lambda}(t, t_0)\|x_0\| + \gamma e_{-\lambda}(t, t_0) \int_{t_0}^{t} e_{-\lambda}(t_0, \sigma(s))(d(s)\|x(s)\| + k(s))\Delta s.$$

So referring to Lemma 3.2, we have this estimation

$$\|x(t)\| \leq \gamma e_{-\lambda}(t, t_0)[\|x_0\| + \gamma\{\int_{t_0}^{t} e_{-\lambda}(t_0, \sigma(s))(\gamma d(s)e_{-\lambda}(s, t_0)\|x_0\| + k(s)), \tag{3.18}$$

$$\times \exp(\gamma \int_{\sigma(s)}^{t} d(r) e_{-\lambda}(r, t_0) e_{-\lambda}(t_0, \sigma(r)) \Delta r) \Delta s\}.$$

By using the integral upper bound in the condition (ii) to inequality (3.18), we have

$$\|x(t)\| \leq \gamma e_{-\lambda}(t, t_0) \|x_0\| + \gamma^2 \tilde{d} \exp[\gamma \tilde{d}] e_{-\lambda}(t, t_0) \|x_0\| + \gamma \exp[\gamma \tilde{d}] \tilde{k} e_{-\lambda}(t, t_0). \tag{3.19}$$

Now we are in a position to state the uniform attractiveness of the trivial solution. We fix an arbitrary $\varepsilon > 0$ and $c > 0$ such that $\|x_0\| < c$. By using Lemma 2.5, we obtain the following relation

$$e_{-\lambda}(t, t_0) \leq e^{-\lambda(t-t_0)}, \ \textit{for all}\ t \in \mathbb{T}^{+}_{t_0}.$$

Afterwards, we can estimate the solution in this way

$$\|x(t)\| \leq \gamma_1 e^{-\lambda(t-t_0)} c + \gamma_2 e^{-\lambda(t-t_0)}, \ \textit{for all}\ t \in \mathbb{T}^{+}_{t_0}.$$

However, it is easy to see that

$$\gamma_1 e^{-\lambda(t-t_0)} c \leq \frac{\varepsilon}{2} \ \ \textit{if}\ \ t \geq t_0 + \frac{1}{\lambda} \log\left(\frac{2\gamma_1 c}{\varepsilon}\right).$$

By putting

$$T(\varepsilon, c) = \max\left\{\frac{1}{\lambda} \log\left(\frac{2\gamma_2}{\varepsilon}\right), \frac{1}{\lambda} \log\left(\frac{2\gamma_1 c}{\varepsilon}\right)\right\},$$

with $\varepsilon > 0$ sufficiently small, we obtain that

$$t \in \mathbb{T}^{+}_{t_0+T(\varepsilon,c)} \text{ implies } \|x(t, t_0, x_0)\| \leq \varepsilon,$$

which means that the trivial solution of the system (1.2) is uniformly attractive. □

Theorem 3.7. *Assume that the following conditions are satisfied:*

(i) *equation (1.1) is uniformly exponentially stable with growth constants γ and λ;*

(ii) *the perturbed term with growth bound*

$$\|F(t, x)\| \leq R(t, \|x\|), \ t \in \mathbb{T}, \ x \in \mathbb{R}^n, \tag{3.20}$$

with the function R with the same characteristics as in Lemma 3.3 with $R(t, 0) = 0$ on $\mathbb{T}$;

(iii) *there exists a $\theta_0 > 0$ such that functions R, S as defined in (3.7) verify*

$$\int_a^{+\infty} \frac{S(s, \theta)}{1 - \lambda\mu(s)} \Delta s \leq \tilde{S} < +\infty, \tag{3.21}$$

and

$$\int_a^{+\infty} e_{-\lambda}(t_0, \sigma(s)) R(s, \theta) \Delta s \leq \tilde{R} < +\infty, \ \textit{for all}\ \theta \in]0, \theta_0]. \tag{3.22}$$

Then the trivial solution is uniformly asymptotically stable.

Proof. For any initial condition $x_0 = x(t_0)$, the solution of (1.2) satisfies

$$x(t) = \Phi_A(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi_A(t,\sigma(s))F(s,x(s))\Delta s,$$

By the uniform exponential stability of (1.1) with growth constants γ and λ, the transition matrix verifies $\|\Phi_A(t,t_0)\| \le \gamma e_{-\lambda}(t,t_0)$, for all $t \in \mathbb{T}_{t_0}^+$. Taking the norms of both sides and utilizing Lemma 3.3, we get

$$\begin{aligned}\|x(t)\| &\le \gamma e_{-\lambda}(t,t_0)\|x_0\| + \gamma e_{-\lambda}(t,t_0) \textstyle\int_{t_0}^{t} e_{-\lambda}(t_0,\sigma(s))R(s,\gamma e_{-\lambda}(s,t_0)\|x_0\|)\\ &\times \exp[\textstyle\int_{\sigma(s)}^{t} e_{-\lambda}(t_0,\sigma(r))\, S(r,\gamma e_{-\lambda}(r,t_0)\|x_0\|)\gamma e_{-\lambda}(r,t_0)\Delta r]\Delta s, \text{ for all } t \in \mathbb{T}_{t_0}^+.\end{aligned}$$

Let us consider an arbitrary $\varepsilon > 0$. Since $z \mapsto \int_{t_0}^{+\infty} e_{-\lambda}(t_0,\sigma(s))R(s,z)\Delta s$ is continuous at $x = 0$ and $R(t,0) = 0$ on $\mathbb{T}$, then there exists $\theta_1(\varepsilon) > 0$, such that

$$\|x_0\| \le \frac{\theta_1(\varepsilon)}{\gamma} \text{ implies } \int_{t_0}^{+\infty} e_{-\lambda}(t_0,\sigma(s))R(s,\gamma e_{-\lambda}(s,t_0)\|x_0\|)\Delta s \le \frac{\varepsilon}{2\gamma\exp(\gamma\tilde{S})}.$$

Let us denote the constant $\theta(\varepsilon) := \min\{\frac{\theta_0}{\gamma}, \frac{\varepsilon}{2\gamma}, \frac{\theta_1(\varepsilon)}{\gamma}\}$. If the initial condition satisfies $\|x_0\| \le \theta(\varepsilon)$, then $\|x(t)\| \le \varepsilon$, $\forall t \in \mathbb{T}_{t_0}^+$.

Consequently, the trivial solution of the perturbed system (1.2) is uniformly stable. In addition to that, combining (3.21) satisfied by function S and (3.22), we obtain the estimation

$$\|x(t)\| \le \gamma e_{-\lambda}(t,t_0)\|x_0\| + \gamma\tilde{R}\exp(\gamma\tilde{S})e_{-\lambda}(t,t_0),\ \forall t \in \mathbb{T}_{t_0}^+.$$

Now, if $\|x_0\| \le c := \frac{\theta_0}{\gamma}$ then, $\|x(t)\| \le [\gamma c + \gamma\tilde{R}\exp(\gamma\tilde{S})]e_{-\lambda}(t,t_0)$, $\quad \forall t \in \mathbb{T}_{t_0}^+$. By putting

$$T(\varepsilon,c) = \frac{1}{\lambda}\ln\left(\frac{\gamma\tilde{R}\exp(\tilde{S}\gamma) + c}{\varepsilon}\right),$$

with $\varepsilon > 0$ sufficiently small, we obtain that $t \in \mathbb{T}_{t_0+T(\varepsilon,c)}^+$ implies $||x(t)|| \le \varepsilon$, which means that the trivial solution of equation (1.2) is uniformly asymptotically stable. □

4 Numerical examples

To illustrate the usefulness of Theorem 3.7, we state the corresponding examples in the previous section for the special case $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \bigcup_0^{+\infty}[2k, 2k+1]$.

Example 4.1. Let us consider the first order problem

$$x^{\Delta} = -px + q(t)\ln(x+1),$$

where $p > 0$ with $-p \in \mathcal{R}^+$ and $q \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}_+)$ such that

$$\int_a^{+\infty} q(s)e_{-p}(a, \sigma(s))\Delta s < +\infty, \ \textit{for all } a \in \mathbb{T}.$$

It is easy to see that all statements of the last theorem are validated and conclude that this system is uniformly asymptotically stable.

Example 4.2. We consider the nonlinear perturbed initial value problem:

$$x^{\Delta}(t) = \begin{pmatrix} \frac{-1}{4} & 0 \\ 0 & \frac{-1}{4} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + q(t)\arctan(\sqrt{x_1^2 + x_2^2}) \begin{pmatrix} \frac{-x_1^2}{x_1^2+x_2^2} \\ \frac{x_2^2}{x_1^2+x_2^2} \end{pmatrix}, \tag{4.1}$$

where $t \in \mathbb{T} = \bigcup_0^{+\infty}[2k, 2k+1]$.

We observe that $A = \begin{pmatrix} \frac{-1}{4} & 0 \\ 0 & \frac{-1}{4} \end{pmatrix}$ is regressive ($\mu(t) \neq 4$ for all $t \in \mathbb{T}$). In this case the linear system has the unique solution $x(t, t_0, x_0) = \Phi_A(t, t_0)x_0$, where $\Phi_A(t, t_0)$ is the matrix exponential function. It is given by

$$\Phi_A(t, t_0) = \begin{pmatrix} e_{\frac{-1}{4}}(t, t_0) & 0 \\ 0 & e_{\frac{-1}{4}}(t, t_0) \end{pmatrix}, \ \ t \in \mathbb{T}.$$

We see that the generalized exponential function $e_{\frac{-1}{4}}(t, t_0)$ is given by

$$e_{\frac{-1}{4}}(t, t_0) = e^{-\frac{1}{4}(k_0-k)}e^{\frac{-1}{4}(t-t_0)}, \ k_0 = [t_0/2], \ t \in \bigcup_0^{+\infty}[2k, 2k+1].$$

If the rd-continuous nonnegative function q satisfies this relation

$$\int_a^{+\infty} q(s)e_{-p}(a, \sigma(s))\Delta s < +\infty, \ \textit{for all } a \in \mathbb{T}.$$

It is no difficult to verify all the hypotheses of Theorem 3.7. So, we conclude the claimed behavior of the system (4.1).

5 Conclusions

The aim of this paper solves retention stability properties of the perturbed systems (1.2) to those of unperturbed linear systems (1.1), resorting to certain Gronwall inequalities based on the choice of the upper bound perturbation.

The classical approach for studying the stability of nonlinear systems is the use of Lyapunov theory. The present work illustrates the proficiency and diversity of the stability of dynamic systems using Lyapunov techniques.

References

[1] V. M. Alekseev, *An estimate for the perturbations of the solutions of ordinary differential equations* (in Russian), Vestnik Moskov. Univ. Ser. I. Mat. Meh. 2 (1961), 28-36.

[2] R. P. Agarwal, *Difference and Inequalities: Theory, Methods, and Applications*, Second ed., Monographs and Textbooks in Pure and Applied Mathematics 228, Marcel Dekker, Inc., New York 2000.

[3] A. BenAbdallah, M. Dlala and M. A. Hammami, *A new Lyapunov function for stability of time-varying nonlinear perturbed systems*, Syst. Control Lett. 56 (2007), 179-187 .

[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston 2001.

[5] M. Bohner and A. A. Martynyuk, *Elements of stability theory of A. M. Lyapunov for dynamic equations on time scales*, Nonlinear Dynamics and Systems Theory, 7, 3 (2007), 225-257.

[6] M. Bohner, *Some oscillation criteria for first order delay dynamic equations*, Far East J. Appl. Math. 18, 3 (2005), 289-304.

[7] S. K. Choi, D. M. Im. and N. Koo, *Stability of linear dynamic systems on time scales*, Advances in Difference Equations, Article ID 670203 (2008), 1-12.

[8] J. J. DaCunha, *Stability for time varying linear dynamic systems on time scales*, J. Comput. Appl. Math. 176 (2005), 381-410.

[9] J. J. Dacunha, *Dynamic inequalities, bounds and stability of systems with linear and nonlinear perturbations*, Nonlinear Dynamics and Systems Theory 9, 3 (2009), 239-248.

[10] M. Dlala and M. A. Hammami, *Uniform exponential practical stability of impulsive perturbed systems*, Journal of Dynamical and Control Systems 13 (2007), 373-386.

[11] N. H. Du and L. H. Tien, *On the exponential stability of dynamic equations on time scales*, J. Math. Anal. Appl. 331 (2007), 1159-1174.

[12] B. Ghanmi, N. Hadj Taieb and M. A. Hammami, *Growth conditions for exponential stability of time-varying perturbed systems*, International Journal of Control 86, 6 (2013), 1086-1097.

[13] H. E. Gollwitzer, *A note on a functional inequality*, Proceedings of the American Mathematical Society 23, 3 (1969) 642-647.

[14] T. H. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math. 20 (1919), 292-296.

[15] J. Hoffacker and C. C. Tisdell, *Stability and instability for dynamic equations on time scales*, Comput. Math. Appl. 49, 9-10 (2005), 1327-1334.

[16] W. N. Li, *Some Pachpatte type inequalities on time scales*, Computers and Mathematics with Applications, 57 (2009), 275-282.

[17] M. E. Lord and A. R. Mitchell, *A new approach to the method of nonlinear variation of parameters*, J. Appl. Math. Com. 4 (1978), 95-105.

[18] A. A. Martynyuk, *On the exponential stability of a dynamical system on a time scale* (in Russian), Dokl. Akad. Nauk. 421 (2008), 312-317.

[19] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, Mathematics in Science and Engineering Series 197, Academic Press, New York 1998.

[20] D. B. Pachpatte, *Explicit estimates on integral inequalities with time scale*, J. Inequal. Pure Appl. Math. 7, 4 (2006), 8 pages.

Authors' addresses:

Bacem Ben Nasser *and* Mohamed Ali Hammami
Department of Mathematics, Faculty of Sciences of Sfax,
Road of Soukra, B.P 1171, Sfax, 3000, Tunisia.
E-mail: bacem.bennasser@yahoo.fr, MohamedAli.Hammami@fss.rnu.tn

Khaled Boukerrioua
Department of Mathematics, University 08 mai 1945 of Guelma,
Avenue 19 May 1956, B.P 401, Guelma, Algeria.
E-mail: khaledv2004@yahoo.fr