Conformal change of Finsler-Ehresmann connections

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Abstract. In [3] the author studied the conformal change of Finsler metrics and related geometrics objects by considering invariant Ehresmann connections on Finsler manifolds. In this paper, we investigate the behavior of the Finsler-Ehresmann connection under conformal change of Finsler metrics. Then we study, the conformal change of related geometrics objects like Chern connection and associated curvatures.

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1 Introduction

Finsler geometry is a generalization of the Riemannian geometry, in the sense that the geometrical data in Finsler geometry consists of a smoothly varying family of Minkowski norms (one on each tangent space), rather than a family of inner products as in Riemannian case [1]. Studying Finsler geometry, one encounters substantial difficulties by trying to seek analogues of classical global, or sometimes even local, results of Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments.

An Ehresmann connection on the slit tangent bundle of Finsler manifold plays an important role in Finsler geometry. By adopting an intrinsic approach of Finsler geometry via the Koszul methods used in [3], the goal of this paper is to study the behavior of an Ehresmann connection under a conformal change of the Finsler metric.

In [3], the conformal change of Chern connection with respect to an invariant Ehresmann connection was investigated, and some characterizations on geometric objects associated were established. In this paper, we consider the case where the Ehresmann connection is not invariant and we obtain some characterizations of the Chern connection, and associated curvature with respect to the conformal change of Finsler-Ehresmann connection. Moreover we give a necessary and sufficient condition for an Ehresmann connection to be invariant. Under this condition, we find the results gived in [3].

In the next section, we give a brief account of the basic concepts necessary for this work. And in the third section, we give the study of the conformal change of

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Ehresmann connection with respect to the change of metric. The main results of this part are the characterizations of the evolution of the Chern connection and the associated curvatures with respect to the conformal change of Ehresmann connection. It should be noted that, the obtained results of conformal change of Ricci and scalar curvatures yields to the formulation of the Yamabe-type problem in Finsler geometry, which we plan to solve in future paper.

2 Preliminaries

2.1 Finsler-Ehresmann connection and Chern connection

Let $\pi : TM \to M$ be a tangent bundle of connected smooth manifold M of dimension m. We denote by $v = (x, y)$ the points in TM if $y \in \pi^{-1}(x) = T_xM$. We denote by $O(M)$ the zero section of TM, and by TM_0 the slit tangent bundle $TM \setminus O(M)$. We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with local coordinate $(x^1, ..., x^n)$. By setting $v = y^i \frac{\partial}{\partial x^i}$ for every $v \in \pi^{-1}(U)$, we introduce a local coordinate $(x, y) = (x^1, ..., x^n, y^1, ..., y^n)$ on $\pi^{-1}(U)$.

Definition 2.1. A function $F : TM \to [0, +\infty[$ is called a Finsler structure or Finsler metric on M if:

(i) $F \in C^\infty(TM_0)$

(ii) $F(x, \lambda y) = \lambda F(x, y)$, for all $\lambda > 0$.

(iii) The $m \times m$ Hessian matrix (g_{ij}), where $g_{ij} := \frac{1}{2}(F^2)_{y^i y^j}$ is positive-definite at every point of TM_0.

The pair (M, F) is called *Finsler manifold.* For the differential π_* of the submersion $\pi : TM_0 \to M$, the vertical subbundle $\mathcal{V}$ of TTM_0 is defined by $\mathcal{V} = \ker \pi_*$, and $\mathcal{V}$ is locally spanned by $\{F\frac{\partial}{\partial y^1}, ..., F\frac{\partial}{\partial y^n}\}$ on each $\pi^{-1}(U)$. Then, it induces the exact sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{i} TTM_0 \xrightarrow{\pi_*} \pi^*TM \longrightarrow 0, \tag{2.1}$$

where $\pi^*TM := \{(x, y, v) \in TM_0 \times TM : v \in T_{\pi(x,y)}M\}$ is the pull-back bundle. So, the vertical subbundle is defined by $\mathcal{V} = \ker \pi_*$, while the horizontal subbundle $\mathcal{H}$ is defined by a subbundle $\mathcal{H} \subset TTM_0$, which is complementary to $\mathcal{V}$. These subbundles give a smooth splitting

$$TTM_0 = \mathcal{H} \oplus \mathcal{V}. \tag{2.2}$$

Although the vertical subbundle $\mathcal{V}$ is uniquely determined, the horizontal subbundle is not canonically determined. An Ehresmann connection of the submersion $\pi : TM_0 \to M$ is a selection of horizontal subbundles.

In this paper, we shall consider the choice of Ehresmann connection which arises from the Finsler structure F, constructed as follows. Recall that every Finslerian structure F induces a spray (see [6], [4])

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

which the spray coefficients G^i are defined by

$$G^i(x,y) := \frac{1}{4}g^{il}\left[2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\right]y^j y^k, \tag{2.3}$$

where the matrix (g^{ij}) means the inverse of (g_{ij}).

With the functions $N^i_j(x,y) := \frac{\partial G^i}{\partial y^j}(x,y)$, we define a π^*TM-valued smooth form on TM_0 by

$$\theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j). \tag{2.4}$$

This π^*TM-valued smooth form θ is globally well defined on TM_0 [3].

By the form θ, defined in (2.4) which is called Finsler-Ehresmann form, we can define a Finsler-Ehresmann connection as follow.

Definition 2.2. A Finsler-Ehresmann connection of the submersion $\pi : TM_0 \to M$ is the subbundle $\mathcal{H}$ of TTM_0 given by $\mathcal{H} = \ker\theta$, where $\theta : TTM_0 \to \pi^*TM$ is the bundle morphism defined in (2.4), and which is complementary to the vertical subbundle $\mathcal{V}$.

Note that π^*TM can be naturally identified with the horizontal subbundle $\mathcal{H}$ [5], and thus, any section ξ of π^*TM is considered as a section of $\mathcal{H}$. We denote by ξ^H the section of $\mathcal{H}$ corresponding to $\xi \in \Gamma(\pi^*TM)$:

$$\xi = \frac{\partial}{\partial x^i} \otimes \xi^i \in \pi^*TM \Longleftrightarrow \xi^H = \frac{\delta}{\delta x^i} \otimes \xi^i \in \Gamma(\mathcal{H}), \tag{2.5}$$

where

$$\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^i_j\frac{\partial}{\partial y^i} = (\frac{\partial}{\partial x^i})^H\}_{i=1,...,m} \tag{2.6}$$

denotes the horizontal lift of natural local frame field $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}\}$ with respect to the Finsler-Ehresmann connection $\mathcal{H}$. The set $\{dx^1, ..., dx^m\}$ is the dual basis of $\mathcal{H}^*$. For the two bundle morphisms π_* and θ from TTM_0 onto π^*TM, we have

Proposition 2.1. *[5] The bundle morphism π_* and θ satisfy*

$$\pi_*(\xi^H) = \xi, \qquad \pi_*(\xi^V) = 0, \tag{2.7}$$

and

$$\theta(\xi^H) = 0 \qquad \theta(\xi^V) = \xi, \tag{2.8}$$

for every $\xi \in \Gamma(\pi^*TM)$.

From (2.1) the existence of the Chern connection on the pullback bundle is given by the following result.

Theorem 2.2. *[3] Let (M,F) be a Finsler manifold, g a fundamental tensor of F and θ the vector form give by (2.4). There exist a unique linear connection ∇ on π^*TM such that, for all $X, Y \in \Gamma(TTM_0)$ and $\xi, \eta \in \Gamma(\pi^*TM)$, we have the following:*

(a) *Symmetry*

$$\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X, Y], \tag{2.9}$$

(b) *Almost g-compatibility*

$$(\nabla_X g)(\xi, \eta) = 2A(\theta(X), \xi, \eta), \tag{2.10}$$

where A is the Cartan tensor.

Remark 2.3. A local version of the above theorem is given in [1]

2.2 Tensor formalism on curvatures of Chern connection

2.2.1 Tensor formalism in Finsler geometry

If Ξ is vector bundle on TM_0 or SM, then we note $\Gamma^p(\Xi)$ the $C^\infty(TM_0)$-module of differentiable sections of Ξ^p, where $\Xi^p = \underbrace{\Xi \times \Xi \times ... \times \Xi}_{p\text{-times}}$. By convention $\Gamma^0(\Xi) = C^\infty(TM_0)$ or $(C^\infty(SM))$. The tensors that we will consider are defined as follows.

Definition 2.4. Let (M, F) be a Finsler manifold. A tensor field T of type $(p_1, p_2; q)$ on (M, F) is a map:

$$T : \Gamma^{p_1}(\pi^* TM) \times \Gamma^{p_2}(TTM_0) \longrightarrow \Gamma^q(\pi^* TM),$$

which is $C^\infty(TM_0)$ or $C^\infty(SM)$-linear in each arguments.

Note that, we only focus on tensors for which $q \in \{0, 1\}$.

Example 2.5.

1. The fundamental tensor g is of type $(2, 0; 0)$,
2. The Cartan tensor A is of type $(3, 0; 0)$,
3. The vector form θ is of type $(0, 1; 1)$,
4. The distinguished section l is of type $(0, 0; 1)$,

Remark 2.6.

(i) The function F and the sections of TTM_0 are not the Finslerian tensors in our case.

(ii) Any tensor of type (p_1, p_2, q) with $p_2 \geq 1$ admits a decomposition in tensors of the same type according to the horizontal and vertical components of its p_2 arguments. For example If the tensor T is of the type $(p, 1; q)$, then we have the following decomposition: $T = T^H + T^V$, where $T^H(\xi_1 ... \xi_p, X) = T(\xi_1 ... \xi_p, X^H)$ and $T^V(\xi_1 ... \xi_p, X) = T(\xi_1 ... \xi_p, X^V)$.

(iii) If T is $(1,p;q)$-tensor, then we can associate to T two tensors $\hat{T}$ and $\check{T}$ of type $(0,p+1;q)$ which are horizontal and vertical, respectively, according to the $(p+1)$-th variable added, i.e., if $T(\xi, X_1, ..., X_p) \in \Gamma^q(\pi^*TM)$ with $\xi \in \Gamma(\pi^*TM)$ then we we can define two $(0,p+1;q)$-tensors $\hat{T}$ and $\check{T}$ as follows.

$$\begin{cases} \hat{T}(X, X_1, ..., X_p) = T(\pi_* X, X_1, ..., X_p) \\ \check{T}(X, X_1, ..., X_p) = T(\theta(X), X_1, ..., X_p) \end{cases} \tag{2.11}$$

2.2.2 Associated curvatures of Chern connection

In this paragraph, we give an intrinsic formulation of the Chern curvatures necessary for this work.

Definition 2.7. The full curvature ϕ of Chern connection ∇ is defined by:

$$\phi(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi, \tag{2.12}$$

where $X, Y \in \Gamma(TTM_0)$ and $\xi \in \Gamma(\pi^*TM)$.

Using the decomposition $\nabla_X = \nabla_{X^H} + \nabla_{X^V}$, we can write the full curvature in the following way:

$$\phi(X,Y)\xi = \phi^{HH}(X,Y)\xi + \phi^{HV}(X,Y)\xi + \phi^{VH}(X,Y)\xi + \phi^{VV}(X,Y)\xi, \tag{2.13}$$

where, $\phi^{HH}(X,Y)\xi = \phi(X^H, Y^H)$, $\phi^{HV}(X,Y)\xi = \phi(X^H, Y^V)$, etc.

Let R, P and Q respectively the hh-, hv- and vv-curvature tensors of Chern connection, we have:

$$\begin{cases} R(X,Y)\xi = \phi^{HH}(X,Y)\xi, \\ P(X,Y)\xi = \phi^{HV}(X,Y)\xi + \phi^{VH}(X,Y)\xi, \\ Q(X,Y)\xi = \phi^{VV}(X,Y)\xi. \end{cases} \tag{2.14}$$

By integrabilty of $\mathcal{V}$, we have, for all X, $Y \in \Gamma(TTM_0)$ and $\xi \in \Gamma(\pi^*TM)$,

$$Q(X,Y)\xi = 0.$$

Hence, the surviving part of ϕ are the horizontal part R and the mixed part P and we have:

$$\phi(X,Y)\xi = R(X,Y)\xi + P(X,Y)\xi. \tag{2.15}$$

Remark 2.8. The full curvature ϕ is the tensor of type $(1,2;1)$ and as in the Riemannian case, we can define a $(2,2;0)$ version of this tensor by the following formula:

$$\begin{aligned} \Phi(\xi,\eta,X,Y) &= g(\phi(X,Y)\xi,\eta) \\ &= g(R(X,Y)\xi,\eta) + g(P(X,Y)\xi,\eta) \\ &= R(\xi,\eta,X,Y) + P(\xi,\eta,X,Y). \end{aligned} \tag{2.16}$$

Note that the hh-curvature tensor R is a generalization of the usual Riemannian curvature. Thus, by an appropriate contractions of the tensor R, we can obtain some avatars of Ricci tensor and scalar curvature which generalize the usual one.

Definition 2.9. Let (M, F) be a Finsler manifold, R the horizontal part of the full curvature tensor associated with the Chern connection and $\{e_a\}_{a=1,\dots,m}$ an g-orthonormal basis sections of π^*TM. We define

1. The horizontal Ricci tensor Ric^H of Finsler manifold (M, F) by:

$$Ric^H(\xi, X) := trace_g(\eta \mapsto R(X, \eta^H)\xi)), \tag{2.17}$$

   for every, $X \in \Gamma(TTM_0)$ and $\xi, \eta \in \Gamma(\pi^*TM)$. In g-orthonormal basis sections $\{e_a\}_{a=1,\dots,m}$ of π^*TM, we have

$$Ric^H(\xi, X) := \sum_{a=1}^{m} R(e_a, \xi, X, e_a^H). \tag{2.18}$$

2. The horizontal scalar curvature $Scal^H$ of the Finsler manifold (M, F) is the trace of horizontal Ricci curvature. We get a function on TM_0 or SM. In $g-$orthonormal basis sections $\{e_a\}_{a=1,\dots,m}$, we have

$$Scal^H \quad := \quad \sum_{a=1}^{m} Ric^H(e_a, e_a^H) = \sum_{a,b=1}^{m} R(e_b, e_a, e_a^H, e_b^H). \tag{2.19}$$

Remark 2.10. The trace notion (2.17) is meaningful due to the natural identification of π^*TM with $\mathcal{H}$ or $\mathcal{V}$.

In the same manner we can define a notion of vertical Ricci curvature and vertical scalar curvature, by the mixed part P of the full curvature. In this work, we will used essentially the horizontal one.

2.3 Fundamental differential operators on (M, F)

In this paragraph, we study several operators which will be used in the following. Let $\tau : \pi^*TM \to TM$ the canonical map defined by $\tau(x, y, v) = v \in T_xM$. By τ, we define a notion of gradient on (M, F).

Definition 2.11. [3] For a smooth function u on M, the gradient of u noted by $\triangledown u$, is the section of π^*TM, characterized by

$$g(x, y)(\triangledown u_{(x,y)}, \xi_{(x,y)}) = du_{\pi(x,y)}(\tau\xi),\ \xi \in \Gamma(\pi^*TM),\ (x, y) \in TM_0. \tag{2.20}$$

Locally, we have

$$\triangledown u_{(x,y)} = g^{ij}(x, y)\frac{\partial u}{\partial x^i}\frac{\partial}{\partial x^j}. \tag{2.21}$$

Definition 2.12. For a smooth section $\xi \in \Gamma(\pi^*TM)$, we define the horizontal divergence by

$$div^H\xi = trace_g(\eta \to \nabla_{\eta^H}\xi), \tag{2.22}$$

and the vertical divergence by

$$div^V\xi = trace_g(\eta \to \nabla_{\eta^V}\xi), \tag{2.23}$$

where ∇ is the Chern connection.

Remark 2.13. In the basis sections $\{\frac{\partial}{\partial x^i}\}_{i=1,\dots,m}$ of the bundle π^*TM, we have

$$div^H\xi = g^{ij}g(\nabla_{\frac{\delta}{\delta x^i}}\xi, \frac{\partial}{\partial x^j}), \tag{2.24}$$

and

$$div^V\xi = g^{ij}g(\nabla_{F\frac{\partial}{\partial y^i}}\xi, \frac{\partial}{\partial x^j}). \tag{2.25}$$

Proposition 2.3. *When the manifold (M,F) is Riemannian, the horizontal divergence div^H is reduce to the classical Riemannian common notion of divergence, while the vertical div^V vanishes.*

Proof. For the horizontal divergence div^H the result follows from (2.24) and for the vertical one, we have, by a straightforward computations,

$$div^V\xi = g^{ij}A(\theta(F\frac{\partial}{\partial y^i}), \frac{\partial}{\partial x^j}, \xi), \tag{2.26}$$

and the result follows, by Deicke theorem [1]. $\square$

Remark 2.14. In view of the above proposition, we remark that the notion of vertical divergence is essentially Finslerian, while the horizontal one generalizes the classical Riemannian case.

Now by the Chern connection ∇, we shall define a Hessian of the smooth function u noted $\mathbf{H}u$ on (M,F) as the $(1,1;0)$-tensor defines as follows.

Definition 2.15. For $u \in C^\infty(M)$, a hessian of u is the map

$$\mathbf{H}u : \Gamma(\pi^*TM) \times \Gamma(TTM_0) \to C^\infty(TM_0), \tag{2.27}$$

such that

$$\mathbf{H}u(\xi, X) = g(\xi, \nabla_X(\triangledown u)). \tag{2.28}$$

Remark 2.16.

1. The associated $(0,2;0)$-tensor $\hat{\mathbf{H}}u$ of $\mathbf{H}u$ verify the following symmetric property [3].

$$\hat{\mathbf{H}}u(X,Y) - \hat{\mathbf{H}}u(Y,X) = 2[A(\theta(Y), \triangledown u, \pi_* X) - A(\theta(X), \triangledown u, \pi_* Y)], \tag{2.29}$$

where $X, Y \in \Gamma(TTM_0)$.

2. Note that $\mathbf{H}u(\xi, X)$ can be split in horizontal part and vertical part as follows:

$$\begin{aligned} \mathbf{H}u(\xi,X) &= \mathbf{H}^H u(\xi,X) + \mathbf{H}^V u(\xi,X) \\ &= g(\xi, \nabla^H_X \triangledown u) + g(\xi, \nabla^V_X \triangledown u). \end{aligned} \tag{2.30}$$

Then, by (2.29) the horizontal part of $\hat{\mathbf{H}}u$ is symmetric. i.e

$$\hat{\mathbf{H}}^H u(X,Y) = \hat{\mathbf{H}}^H u(Y,X). \tag{2.31}$$

From (2.22) and (2.23), we can define the horizontal and vertical Laplacian as follows

Definition 2.17. Let $u \in C^\infty(M)$. The horizontal Laplacian $\triangle^H u$ of u is given by

$$\triangle^H u = -div^H \nabla u, \tag{2.32}$$

and the vertical Laplacian $\triangle^V u$ by

$$\triangle^V u = -div^V \nabla u. \tag{2.33}$$

Proposition 2.4. *The horizontal $\triangle^H u$ Laplacian and vertical $\triangle^V u$ Laplacian of smooth function u, can be expressed in terms of the hessian $\mathbf{H}u$ of u, respectively by:*

$$\triangle^H u = -trace_g((\xi, \eta) \mapsto \mathbf{H}u(\xi, \eta^H)) \quad \eta, \xi \in \Gamma(\pi^* TM), \tag{2.34}$$

and

$$\triangle^V u = -trace_g((\xi, \eta) \mapsto \mathbf{H}u(\xi, \eta^V)) \quad \eta, \xi \in \Gamma(\pi^* TM). \tag{2.35}$$

Furthermore, in orthonormal basis sections $\{e_a\}_{a=1,\dots,m}$, we have

$$\triangle^H u = -\sum_{a=1}^{m} \mathbf{H}u(e_a, e_a^H), \tag{2.36}$$

and

$$\triangle^V u = -\sum_{a=1}^{m} \mathbf{H}u(e_a, e_a^V). \tag{2.37}$$

Proof. By definition of horizontal Laplacian. □

3 Conformal change of Finsler-Ehresmann connection, and related objects

In this section, we give some characterizations of Chern connection and associated curvatures, under the conformal change of Finsler-Ehresmann connection. We also give a necessary and sufficient condition for the Finsler-Ehresmann connection to be invariant, and under this condition, we obtain the results give in [3].

Definition 3.1. Let (M, F) and $(M, \tilde{F})$ be two Finsler manifolds. The two associated fundamental tensors g and $\tilde{g}$ are said to be conformal if there exists a positive smooth function u on M such that $\tilde{g} = e^{2u} g$. Equivalently g and $\tilde{g}$ are conformal if and only if $\tilde{F} = e^u F$. In this case, the two Finsler manifolds (M, F) and $(M, \tilde{F})$ are said to be conformal or conformally related.

Lemma 3.1. *Let (M, F) and $(M, \tilde{F})$ be conformally related Finsler manifolds, with the fundamental tensors related by $\tilde{g} = e^{2u} g$. The associated Finsler-Ehresmann valued forms θ and $\tilde{\theta}$ are related by*

$$\tilde{\theta} = e^{-u}(\theta - \mathcal{B}), \tag{3.1}$$

where $\mathcal{B}$ is the $(0,1;1)$-tensor

$$\mathcal{B} := \frac{1}{F}\mathcal{B}^i_j \frac{\partial}{\partial x^i} \otimes dx^j, \tag{3.2}$$

with $\mathcal{B}^i_j = \frac{\partial \mathcal{B}^{ir}}{\partial y^j}\frac{\partial u}{\partial x^r}$ and $\mathcal{B}^{ir} = \frac{F^2}{2}(g^{ir} - 2l^i l^r)$, where l is the distinguished section of π^*TM given by $l := l^i \frac{\partial}{\partial x^i} = \frac{y^i}{F}\frac{\partial}{\partial x^i}$.

Proof. Let G^i and $\tilde{G}^i$ the sprays coefficients induced respectively by F and $\tilde{F}$. we have

$$\begin{aligned}
\tilde{G}^i &= \frac{1}{4}\tilde{g}^{il}\left[2\frac{\partial \tilde{g}_{jl}}{\partial x^k} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}\right] y^j y^k \\
&= \frac{1}{4}e^{-2u} g^{il}\left[4e^{2u}\frac{\partial u}{\partial x^k} g_{jl} + 2e^{2u}\frac{\partial g_{jl}}{\partial x^k} - 2e^{2u}\frac{\partial u}{\partial x^l} g_{jk} - e^{2u}\frac{\partial g_{jk}}{\partial x^l}\right] y^j y^k \\
&= G^i + \frac{F^2}{2}(2l^i l^j - g^{ij})\frac{\partial u}{\partial x^j},
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{N}^i_j &= \frac{\partial \tilde{G}^i}{\partial y^j} = \frac{\partial G^i}{\partial y^j} + \frac{\partial}{\partial y^j}\left[\frac{F^2}{2}(2l^i l^r - g^{ir})\frac{\partial u}{\partial x^r}\right] \\
&= N^i_j - \frac{\partial}{\partial y^j}\left(\mathcal{B}^{ir}\frac{\partial u}{\partial x^r}\right) \\
&= N^i_j - \mathcal{B}^i_j,
\end{aligned} \tag{3.3}$$

where $\mathcal{B}^i_j = \frac{\partial}{\partial y^j}\left(\mathcal{B}^{ir}\frac{\partial u}{\partial x^r}\right)$ and $\mathcal{B}^{ir} = \frac{F^2}{2}(g^{ir} - 2l^i l^r)$. It follows that,

$$\begin{aligned}
\tilde{\theta} &:= \frac{\partial}{\partial x^i} \otimes \frac{\tilde{\delta} y^i}{\tilde{F}} = e^{-u}\left(\frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + \tilde{N}^i_j dx^j)\right) \\
&= e^{-u}\left(\frac{\partial}{\partial x^i} \otimes \frac{1}{F}\delta y^i - \frac{1}{F}\mathcal{B}^i_j \frac{\partial}{\partial x^i} \otimes dx^j)\right) = e^{-u}\left(\theta - \mathcal{B}\right),
\end{aligned}$$

where $\mathcal{B} = \frac{1}{F}\mathcal{B}^i_j \frac{\partial}{\partial x^i} \otimes dx^j$. □

Lemma 3.2. *Let (M, F) be a Finsler manifold. On the horizontal subbundle $\tilde{\mathcal{H}} \subset TTM_0$ with respect to the conformal change of the Finsler metric ($\tilde{F} = e^u F$), we have:*

$$\theta \equiv \mathcal{B}, \tag{3.4}$$

where $\mathcal{B}$ is the $(0,1;1)$-tensor given by (3.2).

Proof. By definition of $\tilde{\theta}$. □

Lemma 3.3. *If (M, F) and $(M, \tilde{F})$ are conformally related Finsler manifolds, and the associated Finsler-Ehresmann connections $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are equal, then,*

$$\mathcal{B} \equiv 0 \tag{3.5}$$

The converse is true.

Proof. By the fact that $\mathcal{B} = \theta - \tilde{\theta}$. □

Remark 3.2. The lemma 3.3 means that, $\mathcal{H}$ is invariant for $\mathcal{B} \equiv 0$, thus give the condition on the tensor $\mathcal{B}$ for which, the results of [3] on the conformal change of Chern connection and related curvatures was obtained.

Lemma 3.4. *Let (M, F) and $(M, \tilde{F})$ be conformally related Finsler manifolds, with $\tilde{g} = e^{2u} g$. The associated Cartan tensors A and $\tilde{A}$ are related by*

$$\tilde{A} = e^{3u} A, \tag{3.6}$$

and we have

$$\tilde{A}(\tilde{\theta}(X), \xi, \eta) = e^{2u} \left[A(\theta(X), \xi, \eta) - A(\mathcal{B}(X), \xi, \eta) \right], \tag{3.7}$$

for any $X \in \Gamma(TTM_0)$ and $\xi, \eta \in \Gamma(\pi^* TM)$.

Proof. In fact

$$\tilde{A}_{ijk} := \frac{\tilde{F}}{2} \frac{\partial \tilde{g_{ij}}}{\partial y^k} = e^{3u} \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} = e^{3u} A_{ijk}.$$

□

Before the characterization of the conformal change of Chern connection and taking into account the conformal change of Ehresmann connection, we have the following result. As for the Riemannian case, we have:

Lemma 3.5. *Let (M, F) be a Finsler manifold, g the fundamental tensor of F and $\pi^* TM$ the pulled-back bundle on TM_0 . Let $\xi \in \Gamma(\pi^* TM)$. There exists precisely one section $\xi^\flat \in \Gamma(\pi^* T^* M)$ such that for all $\eta \in \Gamma(\pi^* TM)$, $\xi^\flat(\eta) = g(\xi, \eta)$. Conversely, let $\alpha \in \Gamma(\pi^* T^* M)$ there exists precisely one section $\alpha^\sharp \in \Gamma(\pi^* TM)$ such that for all $\eta \in \Gamma(\pi^* TM)$, $g(\alpha^\sharp, \eta) = \alpha(\eta)$.*

Proof. By the nondegeneracy of g. □

Theorem 3.6. *If (M, F) and $(M, \tilde{F})$ are conformally related Finsler manifolds, then the associated Chern connnections respectively ∇ and $\tilde{\nabla}$ are related by:*

$$\tilde{\nabla}_X \pi_* Y = \nabla_X \pi_* Y + \mathcal{D}(X, Y), \quad X, Y \in \Gamma(TTM_0), , \; u \in C^\infty(M), \tag{3.8}$$

where

$$\mathcal{D}(X, Y) := du(\pi_* X) \pi_* Y + du(\pi_* Y) \pi_* X - g(\pi_* X, \pi_* Y) \triangledown u + \Theta(X, Y). \tag{3.9}$$

Θ being the $(0, 2; 1)$-tensor defined by

$$\Theta(X, Y) = (A(\mathcal{B}(X), \pi_* Y, \bullet))^\sharp + (A(\mathcal{B}(Y), \pi_* X, \bullet))^\sharp - (A(\pi_* X, \pi_* Y, \bullet)) \circ \mathcal{B})^\sharp$$

and for any $\eta \in \Gamma(\pi^* TM)$, $X \in \Gamma(TTM_0)$

$$\begin{array}{rcl} du(\pi_* X) & := & g(\triangledown u, \pi_* X) = X(u) \\ (A(\pi_* X, \pi_* Y, \bullet)) \circ \mathcal{B}) (\eta) & := & A(\pi_* X, \pi_* Y, \mathcal{B}(\eta^H)). \end{array} \tag{3.10}$$

Proof. For the Chern connection $\tilde{\nabla}$ associated to $(M, \tilde{F})$, we have by (2.2) the following generalized Koszul formula:

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X \pi_* Y, \pi_* Z) &= X.\tilde{g}(\pi_* Y, \pi_* Z) + Y.\tilde{g}(\pi_* Z, \pi_* X) - Z.\tilde{g}(\pi_* X, \pi_* Y) \\ &+ \tilde{g}(\pi_*[X,Y], \pi_* Z) - \tilde{g}(\pi_*[Y,Z], \pi_* X) + \tilde{g}(\pi_*[Z,X], \pi_* Y) \\ &- 2\tilde{\mathcal{A}}(X,Y,Z), \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathcal{A}}(X,Y,Z) = \tilde{A}(\tilde{\theta}(X), \pi_* Y, \pi_* Z)) + \tilde{A}(\tilde{\theta}(Y), \pi_* Z, \pi_* X)) \\ - \tilde{A}(\tilde{\theta}(Z), \pi_* X, \pi_* Y)). \end{aligned}$$

Using (3.1), we obtain,

$$\begin{aligned} &2\tilde{g}(\nabla_X \pi_* Y, \pi_* Z) = X(e^{2u}) g(\pi_* Y, \pi_* Z) + Y(e^{2u}) g(\pi_* Z, \pi_* X) \\ &-Z(e^{2u}) g(\pi_* X, \pi_* Y) + e^{2u} g(\pi_*[X,Y], \pi_* Z) - e^{2u} g(\pi_*[Y,Z], \pi_* X) \\ &+e^{2u} g(\pi_*[Z,X], \pi_* Y) - 2e^{2u}\mathcal{A}(X,Y,Z) + 2e^{2u} B(X,Y,Z), \end{aligned}$$

where

$$\begin{aligned} B(X,Y,Z) = A(\mathcal{B}(X), \pi_* Y, \pi_* Z)) + A(\mathcal{B}(Y), \pi_* Z, \pi_* X)) \\ - A(\mathcal{B}(Z), \pi_* X, \pi_* Y)). \end{aligned}$$

It follows that

$$\begin{aligned} & 2\tilde{g}(\tilde{\nabla}_X \pi_* Y, \pi_* Z) = 2e^{2u} g(\nabla_X \pi_* Y, \pi_* Z) + 2e^{2u} du(\pi_* X) g(\pi_* Y, \pi_* Z) \\ + \;& 2e^{2u} du(\pi_* Y) g(\pi_* X, \pi_* Z) - 2e^{2u} g(g(\pi_* X, \pi_* Y)\nabla u, \pi_* Z) \\ + \;& 2e^{2u} B(X,Y,Z). \end{aligned} \tag{3.11}$$

Thus,

$$\begin{aligned} g(\tilde{\nabla}_X \pi_* Y - \nabla_X \pi_* Y, \pi_* Z) &= du(\pi_* X) g(\pi_* Y, \pi_* Z) + du(\pi_* Y) g(\pi_* X, \pi_* Z) \\ &- g(g(\pi_* X, \pi_* Y)\nabla u, \pi_* Z) + B(X,Y,Z), \end{aligned} \tag{3.12}$$

with $B(X,Y,Z) = g(\Theta(X,Y), \pi_* Z)$, where, the $(0,2;1)$-tensor Θ is defined by:

$$\Theta(X,Y) = (A(\mathcal{B}(X), \pi_* Y, \bullet))^\sharp + (A(\mathcal{B}(Y), \pi_* X, \bullet))^\sharp - (A(\pi_* X, \pi_* Y, \bullet)) \circ \mathcal{B})^\sharp .$$

The result follows, from the lemma 3.5 and the nondegenerance of g. □

Remark 3.3.
1. When the Ehresmann connection is invariant under the conformal change of the metric, then, by Lemma 3.3, $\mathcal{B} = 0$ and (3.8) is equivalent to the result on conformal change of Chern connection given in [3].
2. The classical result on conformal change of Chern connection, obtained in local coordinates [2], derives immediately of our global result, derives immediately of our global result, using local relationship between both Chern connection and Cartan connection [1].

Definition 3.4. The generalized Kulkarni-Nomizu product $t \odot T$ of the $(1,1;0)$-tensor t and the $(2,0;0)$-tensor T, is the $(2,2;0)$-tensor $t \odot T$ given by:

$$
\begin{aligned}
(t \odot T)(\xi, \eta, X, Y) &= t(\xi, X)T(\eta, \pi_* Y) - T(\xi, \pi_* Y)t(\eta, X) \\
&+ T(\xi, \pi_* X)t(\eta, Y) - t(\xi, Y)T(\eta, \pi_* X)
\end{aligned} \tag{3.13}
$$

Theorem 3.7. *If (M,F) and $(M,\tilde{F})$ are conformally related Finsler manifolds, then the associated respectively full curvatures of Chern connection Φ and $\tilde{\Phi}$ are related by:*

$$
\tilde{\Phi}(\xi, \eta, X, Y) = e^{2u} \left[\Phi + (b_u \odot g) + T_u \right] (\xi, \eta, X, Y), \tag{3.14}
$$

where

$$
b_u(\xi, X) = \mathbf{H}_u(\xi, X) - du(\pi_* X) du(\xi) + \frac{1}{2} g(\triangledown u, \triangledown u) g(\xi, \pi_* X), \tag{3.15}
$$

and

$$
\begin{aligned}
T_u(\xi, \eta, X, Y) &= 2A(\theta(X), \triangledown u, \xi) g(\eta, \pi_* Y) - 2A(\theta(Y), \triangledown u, \xi) g(\eta, \pi_* X) \\
&+ 2A(\theta(Y), \pi_* X, \xi) du(\eta) - 2A(\theta(X), \pi_* Y, \xi) du(\eta) \\
&+ du(\Theta(Y, \xi^H)) g(\eta, \pi_* X) - du(\Theta(X, \xi^H)) g(\eta, \pi_* Y) \\
&+ g(\Theta(X, \xi^H), \pi_* Y) du(\eta) - g(\Theta(Y, \xi^H), \pi_* X) du(\eta) \\
&+ g(\Theta(Y, (\triangledown u)^H), \eta) g(\pi_* X, \xi) - g(\Theta(X, (\triangledown u)^H), \eta) g(\pi_* Y, \xi) \\
&+ g\left(\Theta(X, \Theta^H(Y, \xi^H)), \eta\right) - g\left(\Theta(Y, \Theta^H(X, \xi^H)), \eta\right) \\
&+ g((\nabla_X \Theta)(Y, \xi^H), \eta) - g((\nabla_Y \Theta)(X, \xi^H), \eta).
\end{aligned} \tag{3.16}
$$

Proof. Let $\xi, \eta \in \Gamma(\pi^* TM)$ and $X, Y \in \Gamma(TTM_0)$, the full curvature tensor associated to the conformal change of Chern connection, is given by:

$$
\tilde{\phi}(X, Y)\xi = \tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_Y \tilde{\nabla}_X \xi - \tilde{\nabla}_{[X,Y]} \xi. \tag{3.17}
$$

If we denote $W := \tilde{\nabla}_Y \xi$, then from (3.8) we have

$$
\begin{aligned}
\tilde{\nabla}_X W &= \nabla_X W + du(\pi_* X) W + du(W) \pi_* X - g(\pi_* X, W) \triangledown u \\
&\quad - \Theta(X, W^H).
\end{aligned} \tag{3.18}
$$

Computing any terms of the right hand of the above equation, we have:

$$
\begin{aligned}
\nabla_X W &= \nabla_X \nabla_Y \xi + du(\pi_* Y) \nabla_X \xi + 2A(\theta(X), \triangledown u, \pi_* Y)\xi + \mathbf{H}_u(\pi_* Y, X)\xi \\
&+ du(\nabla_X \pi_* Y)\xi + du(\xi) \nabla_X \pi_* Y + 2A(\theta(X), \triangledown u, \xi) \pi_* Y + \mathbf{H}_u(\xi, X) \pi_* Y \\
&+ du(\nabla_X \xi) \pi_* Y - g(\xi, \pi_* Y) \nabla_X \triangledown u - 2A(\theta(X), \xi, \pi_* Y) \triangledown u - g(\nabla_X \xi, \pi_* Y) \triangledown u \\
&- g(\nabla_X \pi_* Y, \xi) \triangledown u + \nabla_X \Theta(Y, \xi^H)
\end{aligned} \tag{3.19}
$$

$$
\begin{aligned}
du(\pi_* X) W &= du(\pi_* X) \nabla_Y \xi + du(\pi_* X) du(\pi_* Y)\xi + du(\pi_* X) du(\xi) \pi_* Y \\
&- g(\xi, \pi_* Y) du(\pi_* X) \triangledown u + du(\pi_* X) \Theta(Y, \xi^H)
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
du(W) \pi_* X &= du(\nabla_Y \xi) \pi_* X \nabla_Y \xi + 2 du(\xi) du(\pi_* Y) \pi_* X - g(\xi, \pi_* Y) du(\triangledown u) \pi_* X \\
&+ du(\Theta(Y, \xi^H)) \pi_* X
\end{aligned} \tag{3.21}
$$

$$\begin{aligned} g(W,\pi_*X)\triangledown u &= g(\nabla_Y\xi,\pi_*X)\nabla u + du(\pi_*Y)g(\xi,\pi_*X)\triangledown u + du(\xi)g(\pi_*X,\pi_*Y)\triangledown u \\ &- g(\xi,\pi_*Y)g(\triangledown u,\pi_*X)\triangledown u + g(\Theta(Y,\xi^H),\pi_*X)\triangledown u \end{aligned} \tag{3.22}$$

$$\begin{aligned} \Theta(X,W^H) &= \Theta(X,(\nabla_Y\xi)^H) + \Theta(X,\xi^H)du(\pi_*Y) + du(\xi)\Theta(X,Y) \\ &- g(\xi,\pi_*Y)\Theta(X,(\triangledown u)^H) \end{aligned} \tag{3.23}$$

Replacing (3.19), (3.20), (3.21) and (3.22) in (3.18), we get the first term on the right of (3.17). In the same manner, we obtain the second and the third terms.
Using (2.29) and the covariant derivative of the $(0,2;1)$-tensor Θ, defined as the $(0,2;1)$-tensor

$$(\nabla_X\Theta)(Y,Z) := \nabla_X\Theta(Y,Z) - \Theta\left((\nabla_X\pi_*Y)^H,Z\right) - \Theta\left(X,(\nabla_X\pi_*Z)^H\right) \tag{3.24}$$

We obtain,

$$\begin{aligned} g(\tilde{\phi}(X,Y)\xi,\eta) &= g(\phi(X,Y)\xi,\eta + \mathbf{H}_u(\xi,X)g(\eta,\pi_*Y) - \mathbf{H}_u(\eta,X)g(\xi,\pi_*Y) \\ &- du(X)du(\xi)g(\eta,\pi_*Y) + du(Y)du(\xi)g(\eta,\pi_*X) \\ &- g(\xi,\pi_*Y)\|\triangledown u\|_g^2 g(\eta,\pi_*X) - du(Y)du(\eta)g(\xi,\pi_*X) \\ &- \mathbf{H}_u(\xi,Y)g(\eta,\pi_*X) + \mathbf{H}_u(\eta,Y)g(\xi,\pi_*X) \\ &+ g(\xi,\pi_*X)\|\triangledown u\|_g^2 g(\eta,\pi_*Y) + du(X)g(\xi,\pi_*Y)du(\eta) \\ &+ 2A(\theta(X),\triangledown u,\xi)g(\eta,\pi_*Y) - du(\Theta(X,\xi^H))g(\eta,\pi_*Y) \\ &+ du(\Theta(Y,\xi^H))g(\eta,\pi_*X) - 2A(\theta(Y),\triangledown u,\xi)g(\eta,\pi_*X) \\ &+ 2A(\theta(Y),\pi_*X,\xi)du(\eta) - 2A(\theta(X),\pi_*Y,\xi)du(\eta) \\ &+ g(\Theta(X,\xi^H),\pi_*Y)du(\eta) - g(\Theta(Y,\xi^H),\pi_*X)du(\eta) \\ &+ g(\Theta(Y,(\triangledown u)^H),\eta)g(\pi_*X,\xi) - g(\Theta(X,(\triangledown u)^H),\eta)g(\pi_*Y,\xi) \\ &+ g\left(\Theta(X,\Theta^H(Y,\xi^H)),\eta\right) - g\left(\Theta(Y,\Theta^H(X,\xi^H)),\eta\right) \\ &+ g((\nabla_X\Theta)(Y,\xi^H),\eta) - g((\nabla_Y\Theta)(X,\xi^H),\eta) \end{aligned} \tag{3.25}$$

Noting that $e^{2u}g(\tilde{\phi}(X,Y)\xi,\eta) = \tilde{g}(\tilde{\phi}(X,Y)\xi,\eta) = \tilde{\Phi}(\xi,\eta,X,Y)$, we obtain the result.
$\square$

Elsewere, we can observed that, The decomposition of the $(2,2;0)$-tensor T_u is gived by $T_u = T_u^{HH} + T_u^{HV} + T_u^{VH} + T_u^{VV}$ where

$$\begin{aligned} T_u^{HH}(\xi,\eta,X,Y) &= du(\Theta(Y,\xi^H))g(\eta,\pi_*X) - du(\Theta(X,\xi^H))g(\eta,\pi_*Y) \\ &+ g(\Theta(X,\xi^H),\pi_*Y)du(\eta) - g(\Theta(Y,\xi^H),\pi_*X)du(\eta) \\ &+ g(\Theta(Y,(\triangledown u)^H),\eta)g(\pi_*X,\xi) - g(\Theta(X,(\triangledown u)^H),\eta)g(\pi_*Y,\xi) \\ &+ g\left(\Theta(X,\Theta^H(Y,\xi^H)),\eta\right) - g\left(\Theta(Y,\Theta^H(X,\xi^H)),\eta\right) \\ &+ g((\nabla_X\Theta)(Y,\xi^H),\eta) - g((\nabla_Y\Theta)(X,\xi^H),\eta) \end{aligned} \tag{3.26}$$

$$\begin{aligned} T_u^{HV}(\xi,\eta,X,Y) &= 2A(\theta(Y),\pi_*X,\xi)du(\eta) - 2A(\theta(Y),\triangledown u,\xi)g(\eta,\pi_*X) \\ &+ g((\nabla_X\Theta)(Y,\xi^H),\eta) - g((\nabla_Y\Theta)(X,\xi^H),\eta) \end{aligned} \tag{3.27}$$

and

$$T_u^{VV} = 0 \tag{3.28}$$

It follows that, the hh-curvature R and hv-curvature P of F are related at those of $\tilde{F}$ respectively $\tilde{R}$ and $\tilde{P}$ by:

Corollary 3.8. *Under the conformal change $\tilde{F} = e^u F$, the hh and hv curvatures of Chern connection are related by*

$$\begin{aligned} \tilde{R}(\xi,\eta,X,Y) &= e^{2u}\left[R + (b_u \odot g)^{HH} + T_u^{HH}\right](\xi,\eta,X,Y) \\ \tilde{P}(\xi,\eta,X,Y) &= e^{2u}\left[P + (b_u \odot g)^{HV} + (b_u \odot g)^{VH} + T_u^{VH} + T_u^{VH}\right](\xi,\eta,X,Y), \end{aligned} \tag{3.29}$$

where

$$b_u(\xi,X) = \mathbf{H}_u(\xi,X) - du(\pi_* X)du(\xi) + \frac{1}{2}g(\nabla u,\nabla u)g(\xi,\pi_* X), \tag{3.30}$$

$\odot$ *is the generalized Kulkarni-Nomizu product (3.13) of the* $(1,1;0)$*-tensor* b_u *and the* $(2,0;0)$*-tensor* g.

Proof. By (3.26), (3.27) and the proof of the theorem 3.7, we obtain the proof. □

Remark 3.5. When the horizontal subbundle is choice to be invariant under the conformal change, T_u^{HH} vanish by the lemma 3.3 and we obtain the result of [3].

Corollary 3.9. *If* (M,F) *and* $(M,\tilde{F})$ *are conformally related Finsler manifolds, then the associated horizontal Ricci curvatures* Ric^H *and* $\tilde{R}ic^{\tilde{H}}$ *are related by:*

$$\begin{aligned} \tilde{R}ic^{\tilde{H}}(\xi,X) &= \left[Ric^H + (\triangle^H u - (m-2)\|\nabla u\|_g^2)g - (m-2)(\mathbf{H}_u - du\circ du)\right](\xi,X) \\ &+ (1-m)du\left(\Theta(X,\xi^H)\right) + g(\Theta(X,\xi^H),\nabla u) - g(\Theta(\nabla^H u,\xi^H),\pi_* X) \\ &+ \Omega_{aa}(\xi,X) \end{aligned} \tag{3.31}$$

where

$$\begin{aligned} \Omega_{aa}(\xi,X) &= \sum_a^m \left[A(B(\nabla^H u)),e_a,e_a)g(\pi_* X,\xi)\right] - g(\Theta(X,(\nabla u)^H),\xi) \\ &+ \sum_a^m \left[g\left(\Theta(X,\Theta^H(e_a^H,\xi^H)),e_a\right) - g\left(\Theta(e_a^H,\Theta^H(X,\xi^H)),e_a\right)\right] \\ &+ \sum_a^m \left[g((\nabla_X\Theta)(e_a^H,\xi^H),e_a) - g((\nabla_{e_a^H}\Theta)(X,\xi^H),e_a)\right] \end{aligned} \tag{3.32}$$

Proof. By (2.18),(3.29) and the fact that $(\tilde{e}_a = e^{-u}e_a)_{a=1\dots m}$ we have

$$\begin{aligned} \tilde{R}ic^{\tilde{H}}(\xi,X) &= \sum_a^m \tilde{R}(\xi,\tilde{e}_a,\tilde{e}_a^H,X) \\ &= \sum_a^m e^{2u}\left[R + (b_u\odot g)^{HH} + T_u^{HH}\right](\xi,\tilde{e}_a,\tilde{e}_a^H,X) \\ &= Ric^H(\xi,X) + \sum_a^m \left[(b_u\odot g)^{HH} + T_u^{HH}\right](\xi,e_a,e_a^H,X) \end{aligned} \tag{3.33}$$

But,

$$\sum_a^m \left(b_u\odot g)^{HH}\right)(\xi,e_a,e_a^H,X) = \sum_a^m \left[g(e_a,e_a)b_u^H(\xi,X) - b_u^H(e_a,X)g(\xi,e_a)\right]$$

$$
\begin{aligned}
&+ \sum_a^m \left[g(\xi, \pi_* X) b_u(e_a, e_a^H) - g(e_a, \pi_* X) b_u(\xi, e_a^H) \right] \\
&= m b_u^H(\xi, X) - b_u^H(\xi, X) + g(\xi, \pi_* X) \sum_a^m \left[b_u(e_a, e_a^H) \right] \\
&- b_u^H(\xi, X) \\
&= (m-2) b_u^H(\xi, X) + g(\xi, \pi_* X) \sum_a^m \left[b_u(e_a, e_a^H) \right] \qquad (3.34)
\end{aligned}
$$

Then noting that,

$$
b_u^H(\xi, X) = \mathbf{H}_u(\xi, X) - du(\pi_* X) du(\xi) + \frac{1}{2} \|\nabla u\|_g^2 g(\xi, \pi_* X), \qquad (3.35)
$$

then

$$
\sum_a^m b_u(e_a, e_a^H) = \left[-\triangle^H u + \frac{m-2}{2} \|\nabla u\|_g^2 \right] \qquad (3.36)
$$

we obtain,

$$
\begin{aligned}
\sum_a^m \left(b_u \odot g)^{HH} \right) (\xi, e_a, e_a^H, X) &= \left((\triangle^H u - (m-2) |\nabla u\|_g^2) \right) g(\xi, X) \\
&- (m-2) \left(\mathbf{H}_u - du \circ du \right) (\xi, X) \qquad (3.37)
\end{aligned}
$$

by replacing (3.35) and (3.36) in (3.34). Likewise, we have

$$
\begin{aligned}
\sum_a^m T_u^{HH}(\xi, e_a, e_a^H, X) &= \sum_a^m \left[du(\Theta(e_a^H, \xi^H)) g(e_a, \pi_* X) - du(\Theta(X, \xi^H)) g(e_a, e_a) \right] \\
&+ \sum_a^m \left[g(\Theta(X, \xi^H), e_a) du(e_a) - g(\Theta(e_a^H, \xi^H), \pi_* X) du(e_a) \right] \\
&+ \sum_a^m \left[g(\Theta(e_a^H, (\nabla u)^H), e_a) g(\pi_* X, \xi) - g(\Theta(X, (\nabla u)^H), e_a) g(e_a, \xi) \right] \\
&+ \sum_a^m \left[g\left(\Theta(X, \Theta^H(e_a^H, \xi^H)), e_a \right) - g\left(\Theta(e_a^H, \Theta^H(X, \xi^H)), e_a \right) \right] \\
&+ \sum_a^m \left[g((\nabla_X \Theta)(e_a^H, \xi^H), e_a) - g((\nabla_{e_a^H} \Theta)(X, \xi^H), e_a) \right] \\
&= (1-m) du(\Theta(X, \xi^H)) + g(\Theta(\xi^H, X), \nabla u)) - g(\Theta(\xi^H, (\nabla u)^H), \pi_* X) \\
&+ \sum_a^m \left[A(B(\nabla^H u)), e_a, e_a) g(\pi_* X, \xi) \right] - g(\Theta(X, (\nabla u)^H), \xi) \\
&+ \sum_a^m \left[g\left(\Theta(X, \Theta^H(e_a^H, \xi^H)), e_a \right) - g\left(\Theta(e_a^H, \Theta^H(X, \xi^H)), e_a \right) \right] \\
&+ \sum_a^m \left[g((\nabla_X \Theta)(e_a^H, \xi^H), e_a) - g((\nabla_{e_a^H} \Theta)(X, \xi^H), e_a) \right] \qquad (3.38)
\end{aligned}
$$

Replacing (3.37) and (3.38) in (3.33) we obtain the result. □

Remark 3.6. When the horizontal subbundle is invariant under the conformal change, the horizontal Ricci curvature behaves like in Riemannian case for Ricci curvature of Levi-Civita connection.

Corollary 3.10. *If (M, F) and $(M, \tilde{F})$ are conformally related Finsler manifolds, then the associated horizontal scalar curvatures $Scal^H$ and $\tilde{Scal}^{\tilde{H}}$ are related by:*

$$\begin{aligned}
\tilde{Scal}^{\tilde{H}} &= e^{-2u}\left[Scal^H + 2(m-1)\triangle^H u - (m-2)(m-1)\|\nabla u\|_g^2)\right] \\
&+ e^{-2u}\left[\sum_b^m (2-m)\left[du(\Theta(e_b^H, e_b^H)) + g(\Theta(e_b^H, \nabla^H u), e_b)\right]\right] \\
&+ e^{-2u}\left[\sum_{a,b=1}^m \left[g\left(\Theta(e_b^H, \Theta^H(e_a^H, e_b^H)), e_a\right) - g\left(\Theta(e_a^H, \Theta^H(e_b^H, e_b^H)), e_a\right)\right]\right] \\
&+ e^{-2u}\left[\sum_{a,b=1}^m \left[g((\nabla_{e_b^H}\Theta)(e_a^H, e_b^H), e_a) - g((\nabla_{e_a^H}\Theta)(e_b^H, e_b^H), e_a)\right]\right] \qquad (3.39)
\end{aligned}$$

Proof. The proof is the same manner as the proof of corollary 3.9. □

Remark 3.7. By the above Corollary, we can obtain a good formulation of the Yamabe-type problem in Finsler geometry.

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