Some notes concerning Norden-Walker 8-manifolds

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Abstract. The main purpose of the present paper is to study almost Norden structures on 8-dimensional Walker manifolds. We discuss the integrability and Kähler(holomorphic) conditions for these structures. Nonexistence of (non-Kähler)quasi-Kähler structures on almost Norden-Walker 8-manifolds is proved.

M.S.C. 2010: 53C50, 53B30.
Key words: Walker 8-manifolds; Proper almost complex structure; Norden metrics; Holomorphic metrics.

1 Introduction

Let M_{2n} be a pseudo-Riemannian manifold with neutral metric, i.e., with pseudo-Riemannian metric g of signature (n, n). We denote by $\Im_q^p(M_{2n})$ the set of all tensor fields of type (p, q) on M_{2n}. Manifolds, tensor fields and connections are always assumed to be differentiable and of class C^∞.

Let (M_{2n}, φ) be an almost complex manifold with almost complex structure φ. This structure is said to be integrable if the matrix $\varphi = (\varphi_j^i)$ is reduced to constant form in a certain holonomic natural frame in a neighborhood U_x of every point $x \in M_{2n}$. In order that an almost complex structure φ be integrable, it is necessary and sufficient that it be possible to introduce a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant, i.e., $\nabla\varphi = 0$. It is also know that the integrability of φ is equivalent to the vanishing of the Nijenhuis tensor $N_\varphi \in \Im_2^1(M_{2n})$. If φ is integrable, then φ is a complex structure and, moreover, M_{2n} is a C-holomorphic manifold $X_n(\mathrm{C})$ whose transition functions are holomorphic mappings.

A metric g is a Norden metric [2] if

$$g(\varphi X, Y) = g(X, \varphi Y),$$

for any $X, Y \in \Im_0^1(M_{2n})$. Metrics of this type have also been studied under the names: pure and B-metrics (see [1], [2], [4], [12], [17], [19]). If (M_{2n}, φ) is an almost complex manifold with Norden metric g, we say that (M_{2n}, φ, g) is an almost Norden manifold. If φ is integrable, we say that (M_{2n}, φ, g) is a Norden manifold.

Applied Sciences, Vol. 16, 2014, pp. 23-31.

In the present paper, we shall focus our attention to the Norden manifolds in dimension eight. Using the Walker metric we constructive a new Norden-Walker metrics together with so called proper almost complex structures. Note that an indefinite Kähler-Einstein metric on an eight-dimensional Walker manifolds has been recently investigated in [8]. Many authors have also been studied recently on Norden-Walker manifolds (see [13], [14], [15], [16]).

1.1 Holomorphic (almost holomorphic) tensor fields

Let $\overset{*}{t}$ be a complex tensor field on $X_n(\mathrm{C})$. The real model of such a tensor field is a tensor field t on M_{2n} of the same order such that the action of the structure affinor φ on t does not depend on which vector or covector argument of t φ acts. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [4], [9], [10], [17]-[19], [21]). In particular, being applied to a $(0,q)$-tensor field ω, the purity means that for any $X_1,...,X_q \in \Im_0^1(M_{2n})$, the following conditions should hold:

$$\omega(\varphi X_1, X_2, ..., X_q) = \omega(X_1, \varphi X_2, ..., X_q) = ... = \omega(X_1, X_2, ..., \varphi X_q).$$

We define an operator

$$\Phi_\varphi : \Im_q^0(M_{2n}) \to \Im_{q+1}^0(M_{2n})$$

applied to the pure tensor field ω by (see [21])

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, ..., Y_q) = (\varphi X)(\omega(Y_1, Y_2, ..., Y_q)) - X(\omega(\varphi Y_1, Y_2, ..., Y_q))$$

$$+\omega((L_{Y_1}\varphi)X, Y_2, ..., Y_q) + ... + \omega(Y_1, Y_2, ..., (L_{Y_q}\varphi)X),$$

where L_Y denotes the Lie differentiation with respect to Y.
When φ is a complex structure on M_{2n} and the tensor field $\Phi_\varphi\omega$ vanishes, the complex tensor field $\overset{*}{\omega}$ on $X_n(\mathrm{C})$ is said to be holomorphic (see [4], [17], [21]). Thus a holomorphic tensor field $\overset{*}{\omega}$ on $X_n(\mathrm{C})$ is realized on M_{2n} in the form of a pure tensor field ω, such that

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, ..., Y_q) = 0,$$

for any $X, Y_1, ..., Y_q \in \Im_0^1(M_{2n})$. Therefore such a tensor field ω on M_{2n} is also called holomorphic tensor field. When φ is an almost complex structure on M_{2n}, a tensor field ω satisfying $\Phi_\varphi\omega = 0$ is said to be almost holomorphic.

1.2 Holomorphic Norden(Kähler-Norden) metrics

In a Norden manifold a Norden metric g is called a *holomorphic* if

$$(\Phi_\varphi g)(X, Y, Z) = 0,$$

for any $X, Y, Z \in \Im_0^1(M_{2n})$, where

$$(\Phi_\varphi g)(X,Y,Z) = (\varphi X)(g(Y,Z)) - X(g(\varphi Y,Z)) + g((L_Y\varphi)X,Z)) + g(Y,(L_Z\varphi)X)). \tag{1.1}$$

By setting $X = \partial_k,\ Y = \partial_i,\ \mathrm{Z} = \partial_j$ in the equation (1.1), we see that the components $(\Phi_\varphi g)_{kij}$ of $\Phi_\varphi g$ with respect to a local coordinate system $x^1, ..., x^n$ may be expressed as follows:

$$(\Phi_\varphi g)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im}\partial_j \varphi_k^m.$$

If (M_{2n}, φ, g) is a Norden manifold with holomorphic Norden metric g, we say that (M_{2n}, φ, g) is a *holomorphic Norden manifold*.
In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

Theorem 1.1. *[3](For paracomplex version see [11]) For an almost complex manifold with Norden metric g, the condition $\Phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g.*

Kähler-Norden manifold can be defined as a triple (M_{2n}, φ, g) which consists of a manifold M_{2n} endowed with an almost complex structure φ and a pseudo-Riemannian metric g such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be Nordenian. Therefore, there exist a one-to-one correspondence between *Kähler-Norden* manifolds and Norden manifolds with a *holomorphic metric.* Recall that in such a manifold, the Riemannian curvature tensor is pure and holomorphic, also the curvature scalar is locally holomorphic function (see [3], [12]).

Remark 1.1. We know that the integrability of the almost complex structure φ is equivalent to the existing a torsion-free affine connection with respect to which the equation $\nabla\varphi = 0$ holds. Since the Levi-Civita connection ∇ of g is a torsion-free affine connection, we have: If $\Phi_\varphi g = 0$, then φ is integrable. Thus, almost Norden manifold with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$, i.e. *almost holomorphic Norden manifolds does not exist.*

2 Norden-Walker metrics

2.1 Walker metric g

A neutral metric g on a 8-manifold M_8 is said to be Walker metric if there exists a 4-dimensional null distribution D on M_8, which is parallel with respect to g. From Walker theorem [20], there is a system of coordinates $(x^1, ..., x^8)$ with respect to which g takes the local canonical form

$$g = (g_{ij}) = \begin{pmatrix} 0 & I_4 \\ I_4 & B \end{pmatrix}, \tag{2.1}$$

where I_4 is the unit 4×4 matrix and B is a 4×4 symmetric matrix whose entries are functions of the coordinates $(x^1,...,x^8)$. Note that g is of neutral signature $(++++----)$, and that the paralel null 4-plane D is spanned locally by $\{\partial_1,\partial_2,\partial_3,\partial_4\}$, where $\partial_i = \frac{\partial}{\partial x^i}$, $(i=1,...,8)$.
In this paper, we consider the specific Walker metrics on M_8 with B of the form

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where a, b are smooth functions of the coordinates $(x^1,...,x^8)$.

2.2 Almost Norden-Walker 8-manifolds

We can construct various almost complex structures φ on a Walker 8-manifold M_8 with the metric g as in (2.1), (2.2) so that (M_8,φ,g) is almost Nordenian. The following φ is one of the simplest examples of such an almost complex structure:

$$\begin{aligned} &\varphi\partial_1 = \partial_3, \quad \varphi\partial_2 = \partial_4, \quad \varphi\partial_3 = -\partial_1, \quad \varphi\partial_4 = -\partial_2, \\ &\varphi\partial_5 = \tfrac{1}{2}(a+b)\partial_3 - \partial_7, \quad \varphi\partial_6 = -\partial_8, \\ &\varphi\partial_7 = -\tfrac{1}{2}(a+b)\partial_1 + \partial_5, \quad \varphi\partial_8 = \partial_6. \end{aligned}$$

In conformity with the terminology of Matsushita (see, [6]-[8]) we call φ the proper almost complex structure. The proper almost complex structure φ has the local components

$$\varphi = (\varphi_j^i) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & -\frac{1}{2}(a+b) & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \frac{1}{2}(a+b) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{2.3}$$

with respect to the natural frame $\{\partial_i\}$, $i=1,...,8$.

Remark 2.1. From (2.3) we see that in the case $a=-b$, φ is integrable.

2.3 Integrability of the structure φ

We consider the general case for integrability.

The proper almost complex structure φ on almost Norden-Walker manifolds is integrable if and only if

$$(N_\varphi)_{jk}^i = \varphi_j^m\partial_m\varphi_k^i - \varphi_k^m\partial_m\varphi_j^i - \varphi_m^i\partial_j\varphi_k^m + \varphi_m^i\partial_k\varphi_j^m = 0. \tag{2.4}$$

Since $N^i_{jk} = -N^i_{kj}$, we need only consider N^i_{jk} $(j < k)$. By explicit calculation, the nonzero components of the Nijenhuis tensor are as follows:

$$
\begin{aligned}
&N^1_{15} = N^1_{37} = N^3_{17} = -N^3_{35} = \tfrac{1}{2}(a_1 + b_1),\\
&N^3_{57} = \tfrac{1}{4}(a + b)(a_1 + b_1),\\
&N^1_{25} = N^1_{47} = N^3_{27} = -N^3_{45} = \tfrac{1}{2}(a_2 + b_2),\\
&N^1_{17} = -N^1_{35} = -N^3_{15} = -N^3_{37} = -\tfrac{1}{2}(a_3 + b_3),\\
&N^1_{57} = -\tfrac{1}{4}(a + b)(a_3 + b_3),\\
&N^1_{27} = N^1_{45} = N^3_{25} = N^3_{47} = \tfrac{1}{2}(a_4 + b_4),\\
&N^1_{56} = -N^1_{78} = N^3_{58} = -N^3_{67} = -\tfrac{1}{2}(a_6 + b_6),\\
&N^1_{58} = -N^1_{67} = -N^3_{56} = N^3_{78} = -\tfrac{1}{2}(a_8 + b_8).
\end{aligned}
\tag{2.5}
$$

From (2.5) we have

Theorem 2.1. *The proper almost complex structure φ on almost Norden-Walker manifolds is integrable if and only if the following PDEs hold:*

$$
\begin{aligned}
&a_1 + b_1 = 0, \quad a_2 + b_2 = 0, \quad a_3 + b_3 = 0,\\
&a_4 + b_4 = 0, \quad a_6 + b_6 = 0, \quad a_8 + b_8 = 0.
\end{aligned}
\tag{2.6}
$$

Corollary 2.2. *The proper almost complex structure φ on almost Norden-Walker manifolds is integrable if and only if*

$$a = -b + \xi, \tag{2.7}$$

where ξ is any function of x^5 and x^7 alone.

3 Holomorphic Norden-Walker (Kähler-Norden-Walker) metrics on (M_8, φ, g)

Let (M_8, φ, g) be an almost Norden-Walker manifold. If

$$(\Phi_\varphi g)_{kij} = \varphi^m_k \partial_m g_{ij} - \varphi^m_i \partial_k g_{mj} + g_{mj}(\partial_i \varphi^m_k - \partial_k \varphi^m_i) + g_{im}\partial_j \varphi^m_k = 0, \tag{3.1}$$

then by virtue of Theorem 1 φ is integrable and the triple (M_8, φ, g) is called a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold. Taking account of Remark 1, we see that almost Kähler-Norden-Walker manifold with condition $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$ does not exist.

We will write (2.1) and (2.2) in (3.1). Since $(\Phi_\varphi g)_{ijk} = (\Phi_\varphi g)_{ikj}$, we need only consider $(\Phi_\varphi g)_{ijk}$ $(j < k)$. By explicit calculation, the nonzero components of the

$\Phi_\varphi g$ tensor are as follows:

$$(3.2)\quad \begin{cases} (\Phi_\varphi g)_{155} = a_3,\ (\Phi_\varphi g)_{157} = \frac{1}{2}(b_1 - a_1),\ (\Phi_\varphi g)_{177} = b_3, \\ (\Phi_\varphi g)_{255} = a_4,\ (\Phi_\varphi g)_{257} = \frac{1}{2}(b_2 - a_2),\ (\Phi_\varphi g)_{277} = b_4, \\ (\Phi_\varphi g)_{355} = -a_1,\ (\Phi_\varphi g)_{357} = \frac{1}{2}(b_3 - a_3),\ (\Phi_\varphi g)_{377} = -b_1, \\ (\Phi_\varphi g)_{455} = -a_2,\ (\Phi_\varphi g)_{457} = \frac{1}{2}(b_4 - a_4),\ (\Phi_\varphi g)_{477} = -b_2, \\ (\Phi_\varphi g)_{517} = -(\Phi_\varphi g)_{715} = \frac{1}{2}(a_1 + b_1),\ (\Phi_\varphi g)_{527} = -(\Phi_\varphi g)_{725} = \frac{1}{2}(a_2 + b_2), \\ (\Phi_\varphi g)_{537} = -(\Phi_\varphi g)_{735} = \frac{1}{2}(a_3 + b_3),\ (\Phi_\varphi g)_{547} = -(\Phi_\varphi g)_{745} = \frac{1}{2}(a_4 + b_4), \\ (\Phi_\varphi g)_{555} = \frac{1}{2}(a+b)a_3 - a_7,\ (\Phi_\varphi g)_{557} = -b_5, \\ (\Phi_\varphi g)_{567} = -(\Phi_\varphi g)_{756} = \frac{1}{2}(a_6 + b_6),\ (\Phi_\varphi g)_{577} = \frac{1}{2}(a+b)b_3 + a_7, \\ (\Phi_\varphi g)_{578} = -(\Phi_\varphi g)_{758} = \frac{1}{2}(a_8 + b_8),\ (\Phi_\varphi g)_{655} = -a_8, \\ (\Phi_\varphi g)_{657} = \frac{1}{2}(b_6 - a_6),\ (\Phi_\varphi g)_{677} = -b_8,\ (\Phi_\varphi g)_{755} = -\frac{1}{2}(a+b)a_1 - b_5, \\ (\Phi_\varphi g)_{757} = -a_7,\ (\Phi_\varphi g)_{777} = -\frac{1}{2}(a+b)b_1 + b_5, \\ (\Phi_\varphi g)_{855} = a_6,\ (\Phi_\varphi g)_{857} = \frac{1}{2}(b_8 - a_8),\ (\Phi_\varphi g)_{877} = b_6. \end{cases}$$

From (3.2) we have

Theorem 3.1. *The triple (M_8, φ, g) is Kähler-Norden-Walker if and only if the following PDEs hold:*

$$(3.3)\quad \begin{aligned} a_1 = a_2 = a_3 = a_4 = a_6 = a_7 = a_8 = 0, \\ b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = b_8 = 0. \end{aligned}$$

Corollary 3.2. *(M_8, φ, g) is Kähler-Norden-Walker if and only if the matrix B in (2.1) has components*

$$B = \begin{pmatrix} a(x^5) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b(x^7) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

4 Nonexistence of (non-Kähler) quasi-Kähler-Norden-Walker structures on (M_8, φ, g)

The basis class of almost complex manifolds with Norden metric is the class of the quasi-Kähler manifolds. An almost Norden manifold (M_{2n}, φ, g) is called a quasi-Kähler [2], [5], if

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \phi)Y, Z) = 0,$$

where σ is the cyclic sum by three arguments.

By setting

$$(L_Y \varphi)X = L_Y(\varphi X) - \varphi(L_Y X) = \nabla_Y(\varphi X) - \nabla_{\varphi X} Y - \varphi(\nabla_Y X) + \varphi(\nabla_X Y)$$

in (1.1), we see that $(\Phi_\varphi g)(X,Y,Z)$ may be expressed as

$$(\Phi_\varphi g)(X,Y,Z) = -g((\nabla_X\varphi)Y,Z) + g((\nabla_Y\varphi)Z,X) + g((\nabla_Z\varphi)X,Y).$$

If we add $(\Phi_\varphi g)(X,Y,Z)$ and $(\Phi_\varphi g)(Z,Y,X)$, then by virtue of $g(Z,(\nabla_Y\varphi)X) = g((\nabla_Y\varphi)Z,X)$, we find

$$(\Phi_\varphi g)(X,Y,Z) + (\Phi_\varphi g)(Z,Y,X) = 2g((\nabla_Y\varphi)Z,X).$$

Since $(\Phi_\varphi g)(X,Y,Z) = (\Phi_\varphi g)(X,Z,Y)$, from the last equation we have

$$(\Phi_\varphi g)(X,Y,Z) + (\Phi_\varphi g)(Y,Z,X) + (\Phi_\varphi g)(Z,X,Y) = \underset{X,Y,Z}{\sigma} g((\nabla_X\varphi)Y,Z).$$

Thus we have

Theorem 4.1. *Let (M_{2n},φ,g) be an almost Norden manifold. Then the Norden metric g is a quasi-Kähler-Norden if and only if*

$$(\Phi_\varphi g)(X,Y,Z) + (\Phi_\varphi g)(Y,Z,X) + (\Phi_\varphi g)(Z,X,Y) = 0 \tag{4.1}$$

for any $X,Y,Z \in \Im_0^1(M_{2n})$.

From (4.1) we easily see that a Kähler-Norden manifold is a quasi-Kähler-Norden. Conversely, quasi-Kähler-Norden manifold is a non-Kähler-Norden, in general. In particular, let (M_8,φ,g) be an almost Norden-Walker 8-manifold. Using (3.2) and (4.1) we have

$$\begin{aligned}
&(\Phi_\varphi g)_{155} + (\Phi_\varphi g)_{551} + (\Phi_\varphi g)_{515} = a_3 = 0,\\
&(\Phi_\varphi g)_{157} + (\Phi_\varphi g)_{571} + (\Phi_\varphi g)_{715} = \tfrac{1}{2}(b_1 - a_1) = 0,\\
&(\Phi_\varphi g)_{177} + (\Phi_\varphi g)_{771} + (\Phi_\varphi g)_{717} = b_3 = 0,\\
&(\Phi_\varphi g)_{255} + (\Phi_\varphi g)_{552} + (\Phi_\varphi g)_{525} = a_4 = 0,\\
&(\Phi_\varphi g)_{257} + (\Phi_\varphi g)_{572} + (\Phi_\varphi g)_{725} = \tfrac{1}{2}(b_2 - a_2) = 0,\\
&(\Phi_\varphi g)_{277} + (\Phi_\varphi g)_{772} + (\Phi_\varphi g)_{727} = b_4 = 0,
\end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&(\Phi_\varphi g)_{355} + (\Phi_\varphi g)_{553} + (\Phi_\varphi g)_{535} = -a_1 = 0,\\
&(\Phi_\varphi g)_{357} + (\Phi_\varphi g)_{573} + (\Phi_\varphi g)_{735} = \tfrac{1}{2}(b_3 - a_3) = 0,\\
&(\Phi_\varphi g)_{377} + (\Phi_\varphi g)_{773} + (\Phi_\varphi g)_{737} = -b_1 = 0,\\
&(\Phi_\varphi g)_{455} + (\Phi_\varphi g)_{554} + (\Phi_\varphi g)_{545} = -a_2 = 0,\\
&(\Phi_\varphi g)_{457} + (\Phi_\varphi g)_{574} + (\Phi_\varphi g)_{745} = \tfrac{1}{2}(b_4 - a_4) = 0,\\
&(\Phi_\varphi g)_{477} + (\Phi_\varphi g)_{774} + (\Phi_\varphi g)_{747} = -b_2 = 0,\\
&(\Phi_\varphi g)_{557} + (\Phi_\varphi g)_{575} + (\Phi_\varphi g)_{755} = -\tfrac{1}{2}(a+b)a_1 - 3b_5 = 0,\\
&(\Phi_\varphi g)_{567} + (\Phi_\varphi g)_{675} + (\Phi_\varphi g)_{756} = \tfrac{1}{2}(b_6 - a_6) = 0,\\
&(\Phi_\varphi g)_{577} + (\Phi_\varphi g)_{775} + (\Phi_\varphi g)_{757} = \tfrac{1}{2}(a+b)b_3 - a_7 = 0,\\
&(\Phi_\varphi g)_{578} + (\Phi_\varphi g)_{785} + (\Phi_\varphi g)_{857} = \tfrac{1}{2}(b_8 - a_8) = 0,\\
&(\Phi_\varphi g)_{655} + (\Phi_\varphi g)_{556} + (\Phi_\varphi g)_{565} = -a_8 = 0,\\
&(\Phi_\varphi g)_{677} + (\Phi_\varphi g)_{776} + (\Phi_\varphi g)_{767} = -b_8 = 0,\\
&(\Phi_\varphi g)_{855} + (\Phi_\varphi g)_{558} + (\Phi_\varphi g)_{585} = a_6 = 0,\\
&(\Phi_\varphi g)_{877} + (\Phi_\varphi g)_{778} + (\Phi_\varphi g)_{787} = b_6 = 0,\\
&(\Phi_\varphi g)_{555} = \tfrac{1}{2}(a+b)a_3 - a_7 = 0,\\
&(\Phi_\varphi g)_{777} = -\tfrac{1}{2}(a+b)b_1 + b_5 = 0.
\end{aligned}
$$

From (3.2) and (4.2) we see that the triple (M_8, φ, g) is quasi-Kähler-Norden-Walker if and only if the PDEs in the form (3.3) holds. On the other hand, the equation (3.3) is a Kähler condition of almost Norden-Walker manifolds. Thus we have

Theorem 4.2. *Let (M_8, φ, g) be an almost Norden-Walker Manifold. Then there does not exist a (non-Kähler)quasi-Kähler structure on this manifold.*

5 Conclusions

A Walker $n-$manifold is a semi-Riemannian manifold which admits a field of parallel null r planes with $r \leq \frac{n}{2}$. In this article, we study the almost Norden structures of a Walker 8-manifold (M, g) which admits a field of parallel null 4-palnes. The metric g is necessarily of neutral signature $(+ + + + - - - -)$. In [8], the authors consider Goldberg's conjecture but for the metrics with neutral signature. They initially display examples of almost Kahler-Einstein neutral structures on R^8 such that the almost complex structure is not integrable. Then, they obtain the structures of the same type on the torus T^8. Therefore, it is proved that the neutral version of Goldberg's conjecture fails. For such restricted Walker 8-manifolds, we study almost Norden structures on 8-dimensional Walker manifolds. We discuss the integrability and Kähler(holomorphic) conditions for these structures. Also, nonexistence of (non-Kähler)quasi-Kähler structures on almost Norden-Walker 8-manifolds is proved.

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