A relativistic viscoanelastic model

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Abstract. A relativistic model for viscoanelastic phenomena in continuous media is proposed its the development is based on the general methods of the non-equilibrium thermodynamic. It is assumed that in the rest frame several microscopic phenomena occur which give rise to inelastic strains and the contributions of these phenomena are introduced as internal degrees of freedom in Gibbs relation. In the adiabatic case we shall assume that the internal irreversible process produce a variation of the rest mass of the generic element and applying the methods of the relativistic dynamic in variable rest mass the four-dimensional equation of motion is derived and the explicit expression for energy-momentum tensor is obtained. We shall limit ourselves to the special relativity.

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1 Introduction

The consideration of relativistic effects in continuum mechanics is of fundamental importance when the speed of the macroscopic motion is comparable with the speed of light, like for example in the fluids consisting of particles which are accelerated in synchrotrons and betatrons, or in the giant and super-giant stars, etc..
Therefore it is necessary to formulate a relativistic theory of continuous media which conforms to the Einstein's principles and where an important role is played by the energy tensor which leads to the equations of motion.
Several authors ([1] - [4]) and see the bibliography in [5]) have been interested in this issue but usually the definition of the energy tensor is based on a-priori considerations and the proposed expressions are not univocally given.
Often, various irreversible processes are present in the continuous media, such as the viscosity and the inelastic behavior of the material, which produce relaxation phenomena.
In some previous papers ([6]-[9] and see the bibliography in [10]), from a classical (non-relativistic) point of view, it was proposed a model based on the non-equilibrium thermodynamics theory. In this theory, it is assumed that as a consequence of several microscopic phenomena there appears macroscopic inelastic strains (for instance, slip,

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dislocation, etc.) and a viscous flow phenomenon occurs analogously to the viscous flow of ordinary fluids. An explicit form for the entropy production was derived and the phenomenological equations were obtained. These phenomena affect the generic element of the continuous medium and as a consequence have a strong influence on its structure.
In this paper the generalization of the classical model based on the non-equilibrium thermodynamics theory to the relativistic case is proposed. This generalization is based on the assumption that the rest mass of the generic element undergoes a variation on time due to the presence of the aforementioned irreversible phenomena and this variation is due to the relativistic principle of equivalence between mass and energy [11, 12].
In section 2 some basic results of the non-relativistic theory are recalled and it is introduced the concept of relativistic thermodynamic potential that can be considered as a generalization of the enthalpy of the ideal fluids.
In section 3, from the principles of special relativity, is deduced the law of variation of the rest mass and, finally, in section 4 the four-dimensional equation of motion is derived and the explicit expression for energy-momentum tensor is obtained as well.

2 Non relativistic theory for viscous-anelastic media

Mechanical processes in continuous media usually are irreversible phenomena and therefore is necessary to investigate these phenomena with the help of the non-equilibrium thermodynamics theory.

A very important physical quantity, on which this theory is based, is the entropy. In the holonomic case [10] it was assumed that the specific entropy s depends on the internal energy u, on the symmetric strain tensor[1] ε_{ik} and on some variables Z_{ik}, so that

$$s = s(u, \varepsilon_{ik}, Z_{ik})\,. \tag{2.1}$$

The tensor variable Z_{ik} is a macroscopic variable whose physical nature can't be *a-priori* specified but it is necessary in order to give a complete description of the state of the medium. For this reason this tensor is called *hidden variable* and its components are hidden variables. In the following we shall only assume that Z_{ik} has some influence on the mechanical properties of the medium and that it is a symmetric tensor field, i.e.

$$Z_{ik} \,=\, Z_{ki}\,. \tag{2.2}$$

Theorem 2.1 (Gibbs relation). *By using (2.1) we can obtain the following Gibbs relation:*

$$T\,ds \,=\, du \,-\, \nu\,\tau^{ik}_{(eq)}\,d\varepsilon_{ik} \,+\, \nu\,G^{ik}\,dZ_{ik}\,, \tag{2.3}$$

[1] In the following it is assumed that the Latin indexes run from 1 to 3. The Einstein's convection of sum over repeated indexes is also used.

where ϱ is the mass density, $\tau^{ik}_{(eq)}$ is the equilibrium stress tensor of thermoelastic nature [10], T the absolute) temperature and G^{ik} is the affinity tensor, which is defined as the conjugate of the tensor Z_{ik} .

Proof. We define the following tensor fields

$$\text{(2.4)}\qquad \begin{cases} \tau^{ik}_{(eq)} \stackrel{\text{def}}{=} -\varrho T \dfrac{\partial}{\partial \varepsilon_{ik}} s(u, \varepsilon_{ik}, Z_{ik}) \,, \\ G^{ik} \stackrel{\text{def}}{=} \varrho T \dfrac{\partial}{\partial Z_{ik}} s(u, \varepsilon_{ik}, Z_{ik}), \end{cases}$$

and the *absolute temperature*

$$\text{(2.5)}\qquad \frac{1}{T} \stackrel{\text{def}}{=} \frac{\partial}{\partial u} s(u, \varepsilon_{ik}, Z_{ik}) \,,$$

By using equations (2.4) and (2.5) from (2.1) we obtain the differential ds of s *Gibbs relation*:

$$\text{(2.6)}\qquad T\, ds = du - \nu\, \tau^{ik}_{(eq)}\, d\varepsilon_{ik} + \nu\, G^{ik}\, dZ_{ik} \,,$$

being $\varrho \stackrel{\text{def}}{=} \nu^{-1}$. □

Of course, from (2.6) one has:

$$\tau^{ik}_{(eq)} = \tau^{ki}_{(eq)} \qquad G^{ik} = G^{ki} \,.$$

Theorem 2.2 (Decomposition of the total strain). *By using the generalized Gibbs thermodynamic potential:*

$$g = u - Ts - \nu\, \tau^{ik}_{(eq)}\, \varepsilon_{ik} \,,$$

the total strain ε_{ik} can be splitt in the thermoelastic, $\varepsilon^{(0)}_{ik}$, and anelastic part $\varepsilon^{(1)}_{ik}$, i.e.

$$\varepsilon_{ik} = \varepsilon^{(0)}_{ik} + \varepsilon^{(1)}_{ik} \,.$$

Proof. We consider the following generalized Gibbs thermodynamic potential

$$g = u - Ts - \nu\, \tau^{ik}_{(eq)}\, \varepsilon_{ik} \,.$$

From the equation (2.6) one has

$$\text{(2.7)}\qquad d\, g = -\, s\, dT - \varepsilon_{ik}\, d(\nu\, \tau^{ik}_{(eq)}) - \nu\, G^{ik}\, dZ_{ik} \,,$$

and we get

$$\text{(2.8)}\qquad \begin{cases} g = g(T, \nu\, \tau^{ik}_{(eq)}, Z_{ik}) \,, \\ \varepsilon_{ik} = -\dfrac{\partial}{\partial\, (\nu\, \tau^{ik}_{(eq)})}\, g(T, \nu\, \tau^{ik}_{(eq)}, Z_{ik}). \end{cases}$$

Thus we have the following tensor-valued function

$$\varepsilon_{ik} \;=\; \varphi_{ik}\,(T, \nu\,\tau^{ik}_{(eq)}, Z_{ik})\,. \tag{2.9}$$

By introducing a reference state of the medium $\Sigma_{(0)}$ in which we assume $T = T_0$, $\tau^{(0)ik}_{(eq)} = 0$ and $\nu = \nu_0$ we define the function $\varphi^{(1)}_{ik}$ of Z_{ik} by

$$\varphi^{(1)}_{ik}(Z_{ik}) \;=\; \varphi_{ik}\,(T_0, \nu_0\,\tau^{(0)ik}_{(eq)}, Z_{ik})\,. \tag{2.10}$$

Now, we can define the following strains:

$$\begin{cases} \varepsilon^{(1)}_{ik} \;=\; \varphi^{(1)}_{ik}(Z_{ik})\,, \\[2ex] \varepsilon^{(0)}_{ik} \;= \varepsilon_{ik} \;-\; \varphi^{(1)}_{ik}(Z_{ik})\,, \end{cases} \tag{2.11}$$

so that, by virtue of (2.9) and (2.10), from $(2.11)_2$ one has

$$\varepsilon^{(0)}_{ik} \;=\; \varphi_{ik}\,(T, \nu\,\tau^{ik}_{(eq)}, Z_{ik}) \;-\; \varphi_{ik}\,(T_0, \nu_0\,\tau^{(0)ik}_{(eq)}, Z_{ik})\,. \tag{2.12}$$

This equation shows that if $T = T_0$ and $\nu\,\tau^{ik}_{(eq)} \;=\; \nu_0\,\tau^{(0)ik}_{(eq)}$ then

$$\varepsilon^{(0)}_{ik} \;=\; 0 \quad \text{for all value of } \; Z_{ik}\,. \tag{2.13}$$

From (2.11) one has:

$$\varepsilon_{ik} \;=\; \varepsilon^{(0)}_{ik} \;+\; \varepsilon^{(1)}_{ik}\;.$$

and we can say that tensor Z_{ik} is a kind of operator which splits the strain into two parts: the strain $\varepsilon^{(0)}_{ik}$ which vanishes for all values of Z_{ik} if $T \;=\; T_0$ and $\nu\,\tau^{ik}_{(eq)} \;=\; \nu_0\,\tau^{(0)ik}_{(eq)}$ (see (2.12) and (2.13)) and the strain $\varepsilon^{(1)}_{ik}$ which depends only on Z_{ik} (see $(2.11)_1$ and (2.13)). Tensors $\varepsilon^{(0)}_{ik}$ and $\varepsilon^{(1)}_{ik}$ are, respectively, the *thermoelastic* part and the *anelastic* part of the strain.

□

Assuming that the tensor-valued function $\varphi^{(1)}_{ik}$ which occur in $(2.11)_1$ has an inverse $\psi^{(1)}_{ik}$, we have:

$$Z_{ik} \;=\; \psi^{(1)}_{ik}(\varepsilon^{(1)}_{ik})\,, \tag{2.14}$$

so that from (2.1) one obtains:

$$s \;=\; s(u, \varepsilon_{ik}, \varepsilon^{(1)}_{ik})\,. \tag{2.15}$$

By taking into account Equations (2.2)-(2.5) we can define the following fields:

$$\begin{cases} \dfrac{1}{T} \;=\; \dfrac{\partial}{\partial u}\, s(u, \varepsilon_{ik}, \varepsilon^{(1)}_{ik})\,, \\[3ex] \tau^{ik}_{(eq)} \;=\; -\,\varrho\, T\, \dfrac{\partial}{\partial\, \varepsilon_{ik}}\, s(u, \varepsilon_{ik}, \varepsilon^{(1)}_{ik})\,, \\[3ex] \tau^{ik}_{(1)} \;=\; \varrho\, T\, \dfrac{\partial}{\partial\, \varepsilon^{(1)}_{ik}}\, s(u, \varepsilon_{ik}, \varepsilon^{(1)}_{ik})\,. \end{cases} \tag{2.16}$$

From (2.15) and (2.16) one obtains the Gibbs relation in the form

$$T\,ds \,=\, du \,-\, \nu\,\tau^{ik}_{(eq)}\,d\varepsilon_{ik} \,+\, \nu\,\tau^{ik}_{(1)}\,d\varepsilon^{(1)}_{ik}\,. \tag{2.17}$$

Let us now consider the first law of thermodynamic which is:

$$\varrho\,\frac{du}{dt} \,=\, -\operatorname{div}\boldsymbol{J}^{(q)} \,+\, \tau^{ik}\,\frac{d\varepsilon_{ik}}{dt}\,, \tag{2.18}$$

$\boldsymbol{J}^{(q)}$ being the *heat flow* and τ^{ik} is the mechanical stress tensor which occurs in the equations of motion.
Similarly to the definition given for the ordinary fluid we define the *viscous stress tensor* $\tau^{ik}_{(vi)}$ as

$$\tau^{ik}_{(vi)} \,\stackrel{\text{def}}{=}\, \tau^{ik} \,-\, \tau^{ik}_{(eq)}\,, \tag{2.19}$$

and from the Equations (2.17), (2.18) we have

$$\varrho\,\frac{ds}{dt} \,=\, -\operatorname{div}\Big(\frac{\boldsymbol{J}^{(q)}}{T}\Big) \,+\, \sigma^{(s)}\,, \tag{2.20}$$

where

$$\sigma^{(s)} \,\stackrel{\text{def}}{=}\, \frac{1}{T}\Big[-\frac{1}{T}\,\boldsymbol{J}^{(q)}\,\cdot\,\operatorname{grad}T \,+\, \tau^{ik}_{(vi)}\,\frac{d\varepsilon_{ik}}{dt} \,+\, \tau^{ik}_{(1)}\,\frac{d\varepsilon^{(1)}_{ik}}{dt}\Big]\,, \tag{2.21}$$

is the *entropy production* per unit of volume and per unit of time.

According to the usual procedure of non-equilibrium thermodynamic and by virtue of the form (2.21), for the entropy production, we obtain the following *phenomenological equations*

$$j^{(q)}_i \,=\, -T^{-1}\,L^{k}_{(q)(q)i}\,\frac{\partial T}{\partial x^k} + L^{kj}_{(q)(0)i}\,\frac{d\varepsilon_{kj}}{dt} + L_{(q)(1)ikj}\,\tau^{kj}_{(1)}\,, \tag{2.22}$$

$$\tau^{ik}_{(vi)} \,=\, -T^{-1}\,L^{ikj}_{(0)(q)}\,\frac{\partial T}{\partial x^j} + L^{ikjn}_{(0)(0)}\,\frac{d\varepsilon_{jn}}{dt} + L^{ik}_{(0)(1)jn}\,\tau^{jn}_{(1)}\,, \tag{2.23}$$

$$\frac{d\varepsilon^{(1)}_{ik}}{dt} \,=\, -T^{-1}\,L^{j}_{(1)(q)ik}\,\frac{\partial T}{\partial x^j} + L^{jn}_{(1)(0)ik}\,\frac{d\varepsilon_{jn}}{dt} + L_{(1)(1)ikjn}\,\tau^{jn}_{(1)}\,. \tag{2.24}$$

The tensors L are called *phenomenological tensors.* The equation (2.22) may be regarded as a generalization of the Fourier's law for the heat conduction. The equations (2.23) and (2.24) describe, respectively, the viscous flow phenomenon and the irreversible process of anelastic flow. The second and third term on the right hand side of (2.22), the first and third term of (2.23) and the first and second term of (2.24) represent cross effects among the irreversible phenomena mentioned above.
If the cross effects can be neglected the equations (2.22)-(2.24) become

$$j^{(q)}_i \,=\, -T^{-1}\,L^{k}_{(q)(q)i}\,\frac{\partial T}{\partial x^k}\,, \tag{2.25}$$

$$\tau^{ik}_{(vi)} \,=\, L^{ik}_{(0)(0)jn}\,\frac{d\varepsilon_{jn}}{dt}\,, \tag{2.26}$$

$$\frac{d\varepsilon^{(1)}_{ik}}{dt} \,=\, L^{jn}_{(1)(1)ik}\,\tau^{(1)}_{jn}\,, \tag{2.27}$$

Tensors ε_{ik}, $\varepsilon^{(1)}_{ik}$ and $\tau^{ik}_{(1)}$ are symmetric functions. The time derivatives $d\varepsilon_{ik}/dt$ and $d\varepsilon^{(1)}_{ik}/dt$ and the heat flux are odd functions of the microscopic particle velocities while the stresses $\tau^{ik}_{(vi)}$ and $\tau^{ik}_{(1)}$ and the temperature gradient $\partial T/\partial x^k$ are even function of the these velocities so that the Onsager-Casimir reciprocity relations read

$$L^{ik}_{(0)(0)jn} = L^{ki}_{(0)(0)jn} = L^{ik}_{(0)(0)nj} = L^{ki}_{(0)(0)nj}\,, \tag{2.28}$$

$$L^{jn}_{(1)(1)ik} = L^{jn}_{(1)(1)ki} = L^{nj}_{(1)(1)ik} = L^{nj}_{(1)(1)ki}\,. \tag{2.29}$$

The equations (2.28)-(2.29) reduce the number of the independent components of the phenomenological tensors.
We call *a viscoanelastic medium of order one* the medium having the entropy in the form (2.15) .
The rheological equations obtained from (2.25)-(2.27) were applied to different materials (polymers: polyisobutylene [8], isomers: m-toluidine [9]) with confirmation of the experimental data.
The nonholonomic case is studied in [13].
Taking into account the irreversible processes, previously discussed, the following thermodynamic function can be introduced

$$h \stackrel{\text{def}}{=} u \,-\, \nu\,\tau^{ik}_{(eq)}\,\varepsilon_{ik} \,+\, \nu\,\tau^{ik}_{(1)}\,\varepsilon^{(1)}_{ik}\,. \tag{2.30}$$

ant it will be called *generalized enthalpy.*
By neglecting the external heat flux we have the *adiabatic process*, i.e. $\boldsymbol{J}^{(q)} = 0$ so that from the Equations (2.18),(2.30) we have

$$\frac{dh}{dt} \,=\, \nu\left(\tau^{ik}\,\frac{d\varepsilon_{ik}}{dt} \,+\, \varrho\,\frac{d\Lambda}{dt}\right),$$

where

$$\Lambda \stackrel{\text{def}}{=} \nu\Big(\tau^{ik}_{(1)}\,\varepsilon^{(1)}_{ik} \,-\, \tau^{ik}_{(eq)}\,\varepsilon_{ik}\Big)\,.$$

Hence, in adiabatic case, the substantial variation with respect to time of thermodynamic potential h depends on the substantial variations with respect to time of both the total strain and Λ .

3 Relativistic considerations

Let K an arbitrary given inertial frame in which a four-dimensional coordinate system $\{x^0 = c\,t, x^1, x^2, x^3\}$ are introduced where c is the velocity of light, x^i $(i = 1, 2, 3)$ are the Euclidean coordinates in the ordinary flat space and t is the time.
Let K' $(x'^0 = c\,t', x'^i)$ the rest frame of the generic element M and $d\mathring{m} = \mathring{\varrho}\,d\mathring{V}$ is its rest mass.
If in K' no external heat flux occurs, i.e. the motion of M in K' is *adiabatic*, we have

$$\frac{d}{dt'}\,d\mathring{m} \,=\, \frac{d}{dt'}\Big(\mathring{\varrho}\,d\mathring{V}\Big) \,=\, 0\,. \tag{3.1}$$

By virtue of the relation

$$(3.2)\qquad \frac{d}{dt'}\, d\overset{\circ}{V} \;=\; \operatorname{div}\mathbf{v}'\, d\overset{\circ}{V} \;\; (\mathbf{v}' \,=\, 0\,,\, \operatorname{div}\mathbf{v}' \;=\; \frac{\partial v'^i}{\partial x'^i} \neq 0)\,,$$

from (3.1) one obtains

$$(3.3)\qquad \frac{d\,\overset{\circ}{\varrho}}{dt'} + \overset{\circ}{\varrho}\operatorname{div}\mathbf{v}' \;=\; \frac{\partial\,\overset{\circ}{\varrho}}{\partial t'} + \operatorname{div}(\overset{\circ}{\varrho}\,\mathbf{v}') \;=\; 0\,.$$

The quantity $\overset{\circ}{\varrho}$ in (3.3) plays the role of the *invariant rest mass density* which was introduced by Fock [12] in the adiabatic case of ideal fluids.
We consider in K the metric

$$(3.4)\quad d\,s^2 \;=\; c^2(dt)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \;=\; e_\alpha \delta^\alpha_\beta\, dx^\alpha\, dx^\beta \quad (\alpha, \beta \,=\, 0, 1, 2, 3)\,,$$

where δ^α_β is the Kronecker's tensor and e_α is the Eisenberg's symbol ($e_0 \,=\, 1\,, e_1 \,=\, e_2 = e_3 = -1$) which is not participating into the sum.
The four-velocity of the particle M in K is

$$(3.5)\qquad W^\sigma \;=\; \frac{dx^\sigma}{ds} \;=\; \Big(\alpha\,;\, \frac{\alpha}{c}\,\mathbf{v}\Big)\;;\; \alpha \;=\; \Big(1 - \frac{v^2}{c^2}\Big)^{-\frac{1}{2}}\,.$$

and in K' the relation (3.5) becomes

$$(3.6)\qquad W'^\sigma \;=\; \Big(1\,;\, \frac{1}{c}\,\mathbf{v}'\Big)\,,\;,\; \operatorname{div}\mathbf{v}' \;=\; \frac{\partial v'^i}{\partial x'^i} \neq 0)\,,$$

so that the equation (3.3) can be written as

$$(3.7)\qquad \frac{\partial}{\partial x'^\sigma}\Big(\overset{\circ}{\varrho}\, W'^\sigma\Big) \;=\; 0,$$

which in K becomes

$$(3.8)\qquad \frac{\partial}{\partial x^\sigma}\Big(\overset{\circ}{\varrho}\, W^\sigma\Big) \;=\; 0\,.$$

4 The relativistic equations of motion

Let be $d\overset{\circ}{E} \;=\; \overset{\circ}{h}\, d\overset{\circ}{m}$ the total energy in K', rest frame of the generic element M, in which

$$(4.1)\qquad \begin{cases} \overset{\circ}{h} \;=\; \overset{\circ}{u} + \overset{\circ}{\Lambda}\,, \\ \dfrac{d\overset{\circ}{h}}{dt'} \;=\; \overset{\circ}{\nu}\Big(\overset{\circ}{\tau}{}^{ik}\, \dfrac{d\,\overset{\circ}{\varepsilon}_{ik}}{dt'} + \overset{\circ}{\varrho}\, \dfrac{d\overset{\circ}{\Lambda}}{dt'}\Big)\,, \end{cases}$$

where

$$(4.2)\qquad \overset{\circ}{\Lambda} \;=\; \overset{\circ}{\nu}\Big(\overset{\circ}{\tau}{}^{ik}_{(1)}\, \overset{\circ}{\varepsilon}{}^{(1)}_{ik} - \overset{\circ}{\tau}{}^{ik}_{(eq)}\, \overset{\circ}{\varepsilon}_{ik}\Big).$$

According to the theory of relativity, and in particular to the equivalence energy-mass, we have

$$d\bar{m}^\circ = \frac{d\overset{\circ}{E}}{c^2} = \frac{1}{c^2}\,\overset{\circ}{h}\,d\overset{\circ}{m}\,. \tag{4.3}$$

The equation (4.3) gives the variation of the rest mass $d\overset{\circ}{m}$ of the element M in K' due to relativistic effects of the mechanical energy $d\overset{\circ}{E}$.
By putting

$$d\bar{m}^\circ = \overset{\circ}{\mu}\,d\overset{\circ}{V}\,, \tag{4.4}$$

from (4.3) we obtain

$$\overset{\circ}{\mu} = \frac{\overset{\circ}{H}}{c^2}\,, \tag{4.5}$$

where $\overset{\circ}{H} \overset{\text{def}}{=} \overset{\circ}{\varrho}\,\overset{\circ}{h}$ is the generalized enthalpy per unit of volume of the element M in K'.
By using Equations (3.1),$(4.1)_2$, from (4.3) one obtains

$$\frac{d}{dt'}(d\bar{m}^\circ) = \frac{1}{c^2}\,\frac{d\overset{\circ}{h}}{dt'}\,d\overset{\circ}{m} = \frac{1}{c^2}\left(\overset{\circ}{\tau}{}^{ik}\,\frac{d\,\overset{\circ}{\varepsilon}_{ik}}{dt'} + \overset{\circ}{\varrho}\,\frac{d\overset{\circ}{\Lambda}}{dt'}\right)d\overset{\circ}{V}, \tag{4.6}$$

Like in K' one has $\dfrac{d}{ds'} = \dfrac{1}{c}\,\dfrac{d}{dt'}$ and $x'^0 = ct'$, so that the Equation (4.6) can be written as

$$\frac{d}{ds'}(d\bar{m}^\circ) = \frac{1}{c^2}\left(\overset{\circ}{\tau}{}^{ik}\,\frac{d\,\overset{\circ}{\varepsilon}_{ik}}{dx'^0} + \overset{\circ}{\varrho}\,\frac{d\overset{\circ}{\Lambda}}{dx'^0}\right)d\overset{\circ}{V}. \tag{4.7}$$

The equation of motion for $d\bar{m}^\circ$ is

$$\frac{d}{ds'}(d\bar{m}^\circ\,W'^i) = \frac{1}{c^2}\,\frac{\partial\,\overset{\circ}{\tau}{}^{ik}}{\partial x'^k}\,d\overset{\circ}{V}\,, \tag{4.8}$$

where

$$\overset{\circ}{\tau}{}^{ik} = \overset{\circ}{\tau}{}^{ik}_{(eq)} + \overset{\circ}{\tau}{}^{ik}_{(vi)}\,. \tag{4.9}$$

By putting [2]

$$\begin{cases} f'^0 = \dfrac{1}{c^2}\left[\overset{\circ}{\tau}{}^{ik}\,\dfrac{d\,\overset{\circ}{\varepsilon}_{ik}}{dx'^0} + \overset{\circ}{\varrho}\,\dfrac{d\overset{\circ}{\Lambda}}{dx'^0}\right] \overset{(4.1)_2}{=} \dfrac{\overset{\circ}{\varrho}}{c^2}\,\dfrac{\partial\overset{\circ}{h}}{\partial x'^0}\,, \\[2ex] f'^i = \dfrac{1}{c^2}\,\dfrac{\partial\,\overset{\circ}{\tau}{}^{ik}}{\partial x'^k}\,, \end{cases} \tag{4.10}$$

[2] in Σ' we have $\frac{d}{dt'} = \frac{\partial}{\partial t'}$

the equations (4.7) and (4.8), according to (3.6) can be written in the following form

$$\frac{d}{ds'}(d\bar{m}^\circ\, W'^\sigma) \;=\; f'^\sigma\, d\overset{\circ}{V}. \tag{4.11}$$

Since $d\bar{m}^\circ$ and $d\overset{\circ}{V}$ are invariants, the Equation (4.11) assumes the form

$$\frac{d}{ds}(d\bar{m}^\circ\, W^\sigma) \;=\; f^\sigma\, d\overset{\circ}{V}\,. \tag{4.12}$$

In order to determine the expression of f^σ we consider the following Lorentz transformation between K and K'

$$x^\sigma \;=\; a_\nu^{\cdot\,\sigma}\, x'^\nu \;=\; a_0^{\cdot\,\sigma}\, x'^0 \;+\; a_i^{\cdot\,\sigma}\, x'^i\,, \tag{4.13}$$

where $a_\nu^{\cdot\,\sigma} \;=\; e_\nu\, a^{\nu\sigma}$ and $W^\sigma \;=\; a^{0\,\sigma}$, with

$$\begin{cases} a^{00} \;=\; \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} \;=\; \alpha \;=\; W^0\,, \\ a^{0i} \;=\; -\,a^{i0} \;=\; \dfrac{\alpha}{c} v_i \;=\; W^i\,, \\ a^{ik} \;=\; -\delta_{ik} \;-\; (\alpha-1)\dfrac{v_i v_k}{v^2}. \end{cases} \tag{4.14}$$

We have

$$\begin{aligned} f^\sigma \;&=\; f'^\nu\,\frac{\partial x^\sigma}{\partial x'^\nu} \;=\; f'^0\,\frac{\partial x^\sigma}{\partial x'^0} \;+\; f'^i\,\frac{\partial x^\sigma}{\partial x'^i} \;= \\ &=\; \frac{1}{c^2}\Big(\overset{\circ}{\varrho}\,\frac{\partial \overset{\circ}{h}}{\partial x'^0}\,W^\sigma \;+\; \frac{\partial \overset{\circ}{\tau}{}^{ik}}{\partial x'^k}\,a_i^{\cdot\,\sigma}\Big) \overset{(3.8)}{=} \\ &=\; \frac{1}{c^2}\Big(\frac{\partial(\overset{\circ}{H}\,W^\nu)}{\partial x^\nu}\,W^\sigma \;+\; \frac{\partial(\overset{\circ}{\tau}{}^{ik}\;a_k^{\cdot\,\nu}\,a_i^{\cdot\,\sigma})}{\partial x^\nu}\Big)\,. \end{aligned} \tag{4.15}$$

By taking into account the well know relation

$$\frac{d}{ds}\,d\overset{\circ}{V} = \frac{\partial W^\nu}{\partial x^\nu}\,d\overset{\circ}{V}\,, \tag{4.16}$$

the first member of the equation (4.12) can be written as

$$\frac{d}{ds}(d\bar{m}^\circ\, W^\sigma) \;=\; \frac{d}{ds}(\overset{\circ}{\mu}\,d\overset{\circ}{V}\,W^\sigma) \;=\; \frac{\partial(\overset{\circ}{\mu} W^\sigma W^\nu)}{\partial x^\nu}\,d\overset{\circ}{V}\,, \tag{4.17}$$

and according to Equations (4.5),(4.17),(4.15) from (4.12) we have the following tensorial equation

$$\frac{\partial T^{\sigma\nu}}{\partial x^\nu} \;=\; \Theta^\sigma\,, \tag{4.18}$$

where

$$\begin{cases} T^{\sigma\nu} \;=\; \overset{\circ}{H}\,W^\sigma\,W^\nu \;-\; \overset{\circ}{\tau}{}^{ik}\;a_i^{\cdot\,\sigma}\,a_k^{\cdot\,\nu}\,, \\ \Theta^\sigma \;=\; \dfrac{\partial(\overset{\circ}{H}\,W^\nu)}{\partial x^\nu}\,W^\sigma\,. \end{cases} \tag{4.19}$$

The tensor $T^{\sigma\nu}$ is the energy momentum tensor for the relativistic visco-anelastic medium.

Conclusions

In this paper it is has been a relativistic model for the classical viscoanelastic media. In particular, the energy tensor has been explicitly computed in the hypothesis that irreversible processes are considered to be responsible for the variation of the rest mass of the generic element of the continuum. The relativistic equation of motion is derived.

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