Index of a generalized inverse of an endomorphism

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Abstract. Let f be an endomorphism of rank r over a finite dimensional vector space E, with index p. The main result of this paper is to study the relationship between the index of f and the index of f^{-}, a generalized inverse of f. First, we determinate the index of f^{-}. Second, we study this index related to p and quote some results for the index of a reflexive generalized inverse of f. Finally, we compare between the indexes. The original ideas include sequences of subspaces. The techniques to satisfy our objective are the direct sum decomposition of the space E and the image of a subspace of E under f^{-}.

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1 Introduction

The concept of generalized inverses becomes the most important in the theory of operators, especially in linear operators. Existence and construction of several kinds of generalized inverses of linear operators and their applications are presented in [1]. Some algebraic properties and structures of generalized inverses of matrices were given in [2]. Because of the relation between the existence of some kinds of generalized inverses of singular matrices and some spectral properties like the eigenvectors, eigenvalues, the radius, they were also studied in [1]. Note, however, that the index of a generalized inverse of an endomorphism (square matrix) is a one of the spectral properties which needs the study, because of the relation between this one and the multiplicity of the zero eigenvalue of a singular matrix. So, it is the aim of this paper.

Let f be an endomorphism of rank r over a finite dimensional vector space E over a field $\mathbb{K}$, with index p. The range and the null space of f are denoted by $\operatorname{Im} f$ and $\ker f$ respectively. We denote the vector space of endomorphisms over E by $\ell(E)$. Let f^{-} be in $\ell(E)$. We say that f^{-} is a g- inverse or a $\{1\}$-inverse of f, if it satisfies $f\,f^{-}\,f = f$. If, in addition, $f^{-}ff^{-} = f^{-}$, then f^{-} is said to be a $\{1,2\}$-inverse or a reflexive g-inverse of f.

The paper consists of the introduction and two sections. Section 2 is the main results which contains four subsections. subsection one underlines the relation between the index of f^{-} and the inclusion between some null spaces and range spaces. The second clears up the precedent relation related to the index of f. The third one

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analyzes the precedent results to compare between indexes. In the fourth, we quote some results for the index of a $\{1,2\}$-inverse of f. Section 3 is a short summary of the precedent study. We begin the paper by recalling some definitions and well-known results which are needed in the content.

Definition 1.1. Let $f \in \ell(E)$. We say that f is with index p if $\ker f^p = \ker f^{p+1}$. If $\operatorname{Im} f^p = \{0\}$, then, f is said to be nilpotent with index p.

The following lemma summarizes the most important results needed for the index.

Lemma 1.1. *Let $f \in \ell(E)$. The following assertions are equivalent:*
1- f is with index p.
2- $\operatorname{Im} f^p = \operatorname{Im} f^{p+1}$.
3- $E = \ker f^p \oplus \operatorname{Im} f^p$.

Proof. 1) f is with index $p \Leftrightarrow \ker f^p = \ker f^{p+1} \Leftrightarrow \dim \operatorname{Im} f^p = n - \dim \ker f^p = n - \dim \ker f^{p+1} = \dim \operatorname{Im} f^{p+1}$. The inclusion $\operatorname{Im} f^{p+1} \subset \operatorname{Im} f^p$ yields $\operatorname{Im} f^{p+1} = \operatorname{Im} f^p$.

2) For the same raison of dimensions and the inclusion $\ker f^p \subset \ker f^{p+1}$, if $\operatorname{Im} f^p = \operatorname{Im} f^{p+1}$ we have $\ker f^p = \ker f^{p+1}$. Thus, f is with index p.

3) Again, for the raison of dimensions, it is sufficient to prove that $\ker f^p \cap \operatorname{Im} f^p = \{0\}$. Let $x \in \ker f^p \cap \operatorname{Im} f^p$. Then, there exists $y \in E$, such that $x = f^p(y)$. Moreover $x \in \ker f^p \Rightarrow 0 = f^p(x) = f^{2p}(y) \Rightarrow y \in \ker f^{2p}$. From $\ker f^p = \ker f^{p+1}$, we have $\ker f^{2p} = \ker f^p$. Then, $y \in \ker f^p$. Consequently, $x = f^p(y) = 0$.

Inversely, using the inclusion $\ker f \subset \ker f^p$, then, if $\ker f^p \cap \operatorname{Im} f^p = \{0\}$, we have, for $x \in \operatorname{Im} f^p$ such that $f(x) = 0$, we get $x = 0$. Thus the restriction of f to $\operatorname{Im} f^p$ is an automorphism, which means that $\operatorname{Im} f^p = \operatorname{Im} f^{p+1}$. $\square$

The following lemmas summarize the well-known results concerning the null space and the range space when f^- is a $\{1\}$-inverse of f, and when f^- is a $\{1,2\}$-inverse of f.

Lemma 1.2. *Let $f \in \ell(E)$ and f^- a $\{1\}$-inverse of f. Then:*
1- $\operatorname{Im} f = \operatorname{Im} ff^-$, $\ker f^-f = \ker f$, $\ker f^- \subset \ker f\, f^-$, $\operatorname{Im} f^-f \subset \operatorname{Im} f^-$.
2- $E = \ker ff^- \oplus \operatorname{Im} f$.
3- $E = \ker f \oplus \operatorname{Im} f^-f$.

Lemma 1.3. *Let $f \in \ell(E)$. and f^- a $\{1,2\}$-inverse of f. Then:*
1- $\operatorname{Im} f = \operatorname{Im} ff^-$, $\ker f^-f = \ker f$, $\ker f^- = \ker f\, f^-$, $\operatorname{Im} f^-f = \operatorname{Im} f^-$, $rank(f) = rank(f^-)$.
2- $E = \ker f^- \oplus \operatorname{Im} f$.
3- $E = \ker f \oplus \operatorname{Im} f^-$.

For the proof of these lemmas and more results on generalized inverses of matrices see [3]

2 Main results

2.1 Study of the index of a $\{1\}$-inverse of f

Proposition 2.1. *Let E be a finite dimensional vector space over a field $\mathbb{K}$, $f \in \ell(E)$ and f^- be a $\{1\}$-inverse of f. The following assertions are equivalent:*

1) There exists an integer $p \geq 1$ *such that* $\forall k \geq p$, $\ker ff^- \subseteq \ker (f^-)^k$ *and* $\operatorname{Im} f \subseteq \ker (f^-)^k$ *or,* $\ker ff^- \subset \ker (f^-)^k$ *and* $(f^-)^{k+1} \operatorname{Im} f = (f^-)^k \operatorname{Im} f$ *or,* $(f^-)^{k+1} \ker ff^- = (f^-)^k \ker ff^-$ *and* $\operatorname{Im} f \subset \ker (f^-)^k$ *or,* $(f^-)^{k+1} \ker ff^- = (f^-)^k \ker ff^-$ *and* $(f^-)^{k+1} \operatorname{Im} f = (f^-)^k \operatorname{Im} f$.
2) *index* $(f^-) = p$.

Proof. Since $E = \ker ff^- \oplus \operatorname{Im} f$, we have

$$\forall k \geq 1, \operatorname{Im} \left(f^-\right)^k = \left(f^-\right)^k \ker ff^- + \left(f^-\right)^k \operatorname{Im} f. \tag{2.1}$$

First, we remark that $\left((f^-)^k \ker ff^-\right)_{k\geq 1}$, $\left((f^-)^k \operatorname{Im} f\right)_{k\geq 1}$ are decreasing sequences of sub-vector spaces of E. So, each of them either tends to the null space or will be stationary. We shall prove the existence of p in all the following cases:

1) Both sequences tend to the null space. There exist k_0 and k_1, such that

$$\ker ff^- \subset \ker \left(f^-\right)^{k_0} \text{ and } \operatorname{Im} f \subset \ker \left(f^-\right)^{k_1},$$

which implies that there exists $p = \max(k_0, k_1)$, such that

$$\forall k \geq p, \ker ff^- \subseteq \ker \left(f^-\right)^k \text{ and } \operatorname{Im} f \subseteq \ker \left(f^-\right)^k. \tag{2.2}$$

By replacing (2.2) in (2.1), we have $\operatorname{Im} (f^-)^p = \{0\}$. Then f^- is a nilpotent endomorphism with index $p = \max(k_0, k_1)$.

Conversely, if f^- is a nilpotent endomorphism with index p, then, from (2.1), we have

$$\{0\} = \left(f^-\right)^p \ker ff^- + \left(f^-\right)^p \operatorname{Im} f,$$

which obviously yields (2.2). **2)** $\left((f^-)^k \ker ff^-\right)_{k\geq 1}$ **tend to the null space and** $\left((f^-)^k \operatorname{Im} f\right)_{k\geq 1}$ **is stationary.** There exist k_0 and k_1 such that,

$$\ker ff^- \subset \ker \left(f^-\right)^{k_0} \text{ and } \left(f^-\right)^{k_1+1} \operatorname{Im} f = \left(f^-\right)^{k_1} \operatorname{Im} f.$$

Then,

$$\forall k \geq \max(k_0, k_1), \ker ff^- \subset \ker \left(f^-\right)^k \text{ and } \left(f^-\right)^{k+1} \operatorname{Im} f = \left(f^-\right)^k \operatorname{Im} f. \tag{2.3}$$

We replace (2.3) in (2.1), we have $\forall k \geq p = \max(k_0, k_1)$,

$$\operatorname{Im} \left(f^-\right)^k = \left(f^-\right)^k \operatorname{Im} f = \left(f^-\right)^p \operatorname{Im} f = \operatorname{Im} \left(f^-\right)^p,$$

which means that f^- is with index $p = \max(k_0, k_1)$.

3) $\left((f^-)^k \ker ff^-\right)_{k\geq 1}$ **is stationary and** $\left((f^-)^k \operatorname{Im} f\right)_{k\geq 1}$ **tend to the null space.** There exist k_0 and k_1 such that,

$$\operatorname{Im} f \subset \ker \left(f^-\right)^{k_1} \text{ and } \left(f^-\right)^{k_0+1} \ker ff^- = \left(f^-\right)^{k_0} \ker ff^-.$$

Then, $\forall k \geq \max(k_0, k_1)$, we have

$$\operatorname{Im} f \subset \ker \left(f^-\right)^k \text{ and } \left(f^-\right)^{k+1} \ker ff^- = \left(f^-\right)^k \ker ff^-. \tag{2.4}$$

We replace (2.4) in (2.1), we have $\forall k \geq p = \max(k_0, k_1)$,

$$\operatorname{Im}\left(f^-\right)^k = \left(f^-\right)^k \ker ff^- = \left(f^-\right)^p \ker ff^- = \operatorname{Im}\left(f^-\right)^p.$$

Then, f^- is with index $p = \max(k_0, k_1)$.

4) Both of sequences are stationary. There exist k_0 and k_1 such that,

$$\left(f^-\right)^{k_0+1} \ker ff^- = \left(f^-\right)^{k_0} \ker ff^- \text{ and } \left(f^-\right)^{k_1+1} \operatorname{Im} f = \left(f^-\right)^{k_1} \operatorname{Im} f.$$

Then, $\forall k \geq \max(k_0, k_1)$,

$$\left(f^-\right)^{k+1} \ker ff^- = \left(f^-\right)^k \ker ff^- \text{ and } \left(f^-\right)^{k+1} \operatorname{Im} f = \left(f^-\right)^k \operatorname{Im} f. \tag{2.5}$$

Replacing (2.5) in (2.1), we have $\forall k \geq p = \max(k_0, k_1)$,

$$\operatorname{Im}\left(f^-\right)^k = \left(f^-\right)^p \ker ff^- + \left(f^-\right)^p \operatorname{Im} f = \operatorname{Im}\left(f^-\right)^p,$$

then, f^- is with index $p = \max(k_0, k_1)$. Inversely, If f^- is with index p, we have $\forall k \succeq p$, $\operatorname{Im}\left(f^-\right)^k = \operatorname{Im}\left(f^-\right)^p$ holds if and only if one of the precedent cases holds. □

2.2 Study of the index of f^- related to a given index of f

Proposition 2.2. *Let $f \in \ell(E)$ with index p, and f^- be a $\{1\}$-inverse of f. The following assertions are equivalent:*
1) There exists $k_0 \geq 1$, such that,

$$\ker f^p \subseteq \ker \left(f^-\right)^{k_0} \textit{ or, } \forall k \geq k_0, \left(f^-\right)^k \ker f^p = \left(f^-\right)^{k_0} \ker f^p \textit{ or, } \left(f^-\right)^{k_0} \ker f^p \subseteq \operatorname{Im} f^p.$$

2) f^- is with index k_0.

Proof. According to lemma 1, we have

$$index\,(f) = p \Leftrightarrow E = \ker f^p \oplus \operatorname{Im} f^p.$$

Then, we have

$$\forall k \geq 1, \operatorname{Im}\left(f^-\right)^k = \left(f^-\right)^k \ker f^p + \left(f^-\right)^k \operatorname{Im} f^p. \tag{2.6}$$

Since the restriction of f to $\operatorname{Im} f^p$ is an automorphism, then, the restriction of f^- to $\operatorname{Im} f^p$ becomes the usual inverse of f. In fact, for every $y \in \operatorname{Im} f^{p,}$ there exists $x \in \operatorname{Im} f^p$, such that $f^{-1}(y) = x$, then

$$f^{-1}(y) = f^{-1}(f(x)) = f^{-1}\left(ff^- f(x)\right) = \left(f^{-1} f\right) f^-(y) = f^-(y).$$

This yields $\forall k \geq 1$,

$$\left(f^-\right)^k \operatorname{Im} f^p = \operatorname{Im} f^p.$$

Then, (2.6) becomes

$$\forall k \geq 1, \operatorname{Im}\left(f^{-}\right)^{k} = \left(f^{-}\right)^{k} \ker f^{p} + \operatorname{Im} f^{p}. \tag{2.7}$$

The sequence $\left(\operatorname{Im}(f^{-})^{k}\right)_{k\geq 1}$ depends only on the sequence $\left((f^{-})^{k} \ker f^{p}\right)_{k\geq 1}$, so for the same reason in subsection 1, we have these two cases:

case 1) There exists $k_0 \geq 1$, such that $\ker f^{p} \subset \ker(f^{-})^{k_0}$. Replacing this in (2.7), we have

$$\forall k \geq k_0, \operatorname{Im}\left(f^{-}\right)^{k} = \operatorname{Im} f^{p} = \operatorname{Im}\left(f^{-}\right)^{k_0}. \tag{2.8}$$

Then, f^{-} is with index k_0. We also remark that if there exists $k_0 \geq 1$, such that

$$\left(f^{-}\right)^{k_0} \ker f^{p} \subseteq \operatorname{Im} f^{p},$$

then

$$\forall k \geq k_{0,} \left(f^{-}\right)^{k} \ker f^{p} \subseteq \left(f^{-}\right)^{k_0} \ker f^{p} \subseteq \operatorname{Im} f^{p}.$$

Therefore, $\forall k \geq k_0$,

$$\operatorname{Im}\left(f^{-}\right)^{k} = \operatorname{Im} f^{p} = \operatorname{Im}\left(f^{-}\right)^{k_0}.$$

Then f^{-} is with index k_0.

case 2) There exists $k_0 \geq 1$, such that $\forall k \geq k_0$, $(f^{-})^{k} \ker f^{p} = (f^{-})^{k_0} \ker f^{p}$. Then, we replace this in (2.7), we have

$$\forall k \geq k_0, \operatorname{Im}\left(f^{-}\right)^{k} = \left(f^{-}\right)^{k_0} \ker f^{p} + \operatorname{Im} f^{p} = \operatorname{Im}\left(f^{-}\right)^{k_0},$$

which shows that

$$\operatorname{Im} f^{p} \subsetneq \operatorname{Im}\left(f^{-}\right)^{k_0}. \tag{2.9}$$

Then f^{-} is with index $k_0 \geq 1$.

Inversely, If f^{-} is with index k_0, then, $\forall k \geq k_0$, $\operatorname{Im}(f^{-})^{k} = \operatorname{Im}(f^{-})^{k_0}$ holds if and only if one of the precedent cases holds. $\square$

2.3 Comparison of Indexes

Corollary 2.3. *Let $f \in \ell(E)$ with index p and f^{-} be a $\{1\}$-inverse of f with index k_0. Then,*
If $\operatorname{Im} f^{p} = \operatorname{Im}(f^{-})^{k_0}$, *then,* $p \leq k_0$.

Proof. According to equations (2.8) and (2.9), index$f^{-} = k_0$ if and only if, either $\operatorname{Im}(f^{-})^{k_0} = \operatorname{Im} f^{p}$ or, $\operatorname{Im} f^{p} \subsetneq \operatorname{Im}(f^{-})^{k_0}$. If $\operatorname{Im}(f^{-})^{k_0} = \operatorname{Im} f^{p}$, then from (2.7), we have $\operatorname{Im}(f^{-})^{k_0} \subseteq \operatorname{Im}(f^{-})^{p}$. Therefore $p \leq k_0$ because the sequence $\left(\operatorname{Im}(f^{-})^{k}\right)_{k\geq 1}$ is decreasing. $\square$

2.4 Index of a $\{1,2\}$-inverse of f

Corollary 2.4. *Let $f \in \ell(E)$ with index p, and f^- is a $\{1,2\}$-inverse of f. Then,*
1) f^- is with index p if and only if $\operatorname{Im} f^p = \operatorname{Im}(f^-)^p$.
2) f^- is with index p if and only if $\ker f^p = \ker(f^-)^p$.
3) f^- is nilpotent if and only if f is nilpotent.

Proof. The only if parts: f^- is a $\{1,2\}$-inverse of f means that f is a $\{1\}$-inverse of f^-. So, similar proof applied to corollary 2.3, we get the first result. the remaining statements deduced from the direct sum decomposition in lemma 1.1 and the definition of nilpotent.
The if parts: According to the above mentioned results and previous subsections, if f is with index p, then the restrictions of f^- and f to $\operatorname{Im} f^p$are the inverses of each other. which means that

$$\left(f^- f\right) \operatorname{Im} f^p = \left(f f^-\right) \operatorname{Im} f^p = \operatorname{Im} f^p = \operatorname{Im} f^{p+1} \text{ and } E = \ker f^p \oplus \operatorname{Im} f^p.$$

Then,

$$\operatorname{Im} f = f(E) = f \ker f^p + \operatorname{Im} f^{p+1} = f \ker f^p + \operatorname{Im} f^p = f \ker f^p \oplus \operatorname{Im} f^p \tag{2.10}$$

We have the direct sum, because for $x \in f \ker f^p$, there exists $y \in \ker f^p$ such that $x = f(y)$ Then $f^p(x) = f^{p+1}(y) = 0$ because we have $y \in \ker f^p = \ker f^{p+1}$. Then $x \in \ker f^p$. Thus $f \ker f^p \subseteq \ker f^p$. direct sum. From (2.10) we have,

$$\operatorname{Im} f^- = \operatorname{Im} f^- f = f^- f \ker f^p \oplus f^- \operatorname{Im} f^p = f^- f \ker f^p \oplus \operatorname{Im} f^p.$$

We have the direct sum because, for $x \in f^- f \ker f^p \cap \operatorname{Im} f^p$, there exists $y \in \ker f^p$ such that $x = f^- f(y)$ and $x \in \operatorname{Im} f^p$. Then, $f^p(x) = f^{p-1}(f f^- f)(y) = f^p(y) = 0$. Then, $x \in \ker f^p \cap \operatorname{Im} f^p = \{0\}$. With the same manner, we have

$$\forall k \geq 1, \operatorname{Im}\left(f^-\right)^k = \left(f^-\right)^k f \ker f^p \oplus \operatorname{Im} f^p \tag{2.11}$$

Now, if f^- is with index p, by using (2.11), we get

$$\ker f^p \oplus \operatorname{Im} f^p = E = \ker\left(f^-\right)^p \oplus \operatorname{Im}\left(f^-\right)^p = \ker\left(f^-\right)^p \oplus \left(f^-\right)^p f \ker f^p \oplus \operatorname{Im} f^p. \tag{2.12}$$

Since f^- is the inverse of f on $\operatorname{Im} f$, we have for $x \in (f^-)^p f \ker f^p$ there exists $y \in \ker f^p$ such that $x = (f^-)^p f(y) \Leftrightarrow f^{2p}(x) = f^{p+1}(y) = 0 \Leftrightarrow x \in \ker f^{2p} = \ker f^p$.

Thus, $(f^-)^p f \ker f^p \subseteq \ker f^p$. By using (2.12), we get $\ker(f^-)^p \subseteq \ker f^p$. Now, As f and f^- are g-inverses of each other, by reciprocity we have $\ker f^p \subseteq \ker(f^-)^p$.

Finally, the remaining if parts are just a consequence of this one. □

3 Summary

Let E be a finite dimensional vector space over a field $\mathbb{K}$, $f \in \ell(E)$ with index p and f^- be a $\{1\}$ −inverse of f, with index k_0, such that $\operatorname{Im}(f^-)^{k_0} = \operatorname{Im} f^p$, then $k_0 \geq p$ and $k_0 = p$ if f^- is a $\{1,2\}$ −inverse of f. If f^- is a nilpotent $\{1\}$ −inverse of f, then f is nilpotent.

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