On Randers change of exponential metric

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Abstract. In this paper, we study the properties of special (α, β)-metric $\alpha e^{\frac{\beta}{\alpha}} + \beta$, the Randers change of exponential metric. We find a necessary and sufficient condition for this metric to be locally projectively flat and we prove the conditions for this metric to be of Berwald and Douglas type.

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1 Introduction

The Matsumoto metric is an interesting (α, β)-metric introduced by using gradient of slope, speed and gravity in [5]. This metric formulates the model of a Finsler space. Many authors [1, 5, 10], etc) have studied this metric by different perspectives. Projectively flat Finsler spaces are regular distance functions with straight geodesics. An extensive study of projectively flat Finsler metrics was taken up by authors [6, 7, 9, 11, 12, 13, 14] and [15]. Another interesting and important class of Finsler spaces is the class of Berwald spaces. Berwald spaces are the Finsler spaces with linear connections. As a generalization of Berwald space S. Bácsó and M. Matsumoto [2] introduced the notion of a Douglas space. A Douglas space is a Finsler space where the projectively invariant Douglas tensor vanishes.

The purpose of the present paper is to investigate the special (α, β)-metric $\alpha e^{\frac{\beta}{\alpha}} + \beta$ which is considered to be Randers change of exponential metric. After preliminaries in section 2, we prove the following in section 3:

The (α, β)-metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta$ is locally projectively flat if and only if:

(i) β is parallel with respect to α,
(ii) α is locally projectively flat, i.e., of constant curvature.

In section 4, we prove the conditions that the Finsler space F^n with the metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta$ is a Berwald space and a Douglas space.

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2 Preliminaries

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric, $\beta = b_i y^j$ a 1-form and let $F = \alpha\phi(s),\ s = \frac{\beta}{\alpha}$, where $\phi = \phi(s)$ is a positive C^∞ function defined in a neighborhood of the origin $s = 0$. It is well known that $F = \alpha\phi(\frac{\beta}{\alpha})$ is a Finsler metric for any α and β with $b = \parallel \beta \parallel_\alpha < b_0$ if and only if

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (\mid s \mid \leq b < b_0).$$

By taking $b = s$, we obtain

$$\phi(s) - s\phi'(s) > 0, \quad (\mid s \mid < b_0).$$

Let G^i and G^i_α denote the spray coefficients of F and α respectively, given by

$$G^i = \frac{g^{il}}{4}\left\{[F^2]_{x^k y^l} y^k - [F^2]_{x^k}\right\}, \quad G^i_\alpha = \frac{a^{il}}{4}\left\{[\alpha^2]_{x^k y^l} y^k - [\alpha^2]_{x^k}\right\},$$

where $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ and

$$(a^{ij}) = (a_{ij})^{-1},\ F_{x^k} = \frac{\partial F}{\partial x^k}, F_{y^k} = \frac{\partial F}{\partial y^k}.$$

For an (α,β)-metric $L(\alpha,\beta)$ the space $\mathbb{R}^n = (M^n, \alpha)$ is called associated Riemannian space to the Finsler space $F^n = (M^n, L(\alpha,\beta))$. The covariant differentiation with respect to the Levi-Civita connection $\gamma^i_{jk}(x)$ of $\mathbb{R}^n$ is denoted by $(;)$. We have the following [3]:

Lemma 2.1. *The spray coefficients G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i_0 + J(-2\alpha Q s_0 + r_{00})\frac{y^i}{\alpha} + H(-2\alpha Q s_0 + r_{00})\{b^i - \frac{y^i}{\alpha}\}, \tag{2.1}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'},$$
$$J := \frac{(\phi - s\phi')\phi'}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$
$$H := \frac{\phi''}{2((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

where $s_{l0} = s_{li}y^i, s_0 = s_{l0}b^l, r_{00} = r_{ij}y^i y^j, r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), r^i_j = a^{ir}r_{rj}, s^i_j = a^{ir}s_{rj}, r_j = b_r r^r_j, s_j = b_r s^r_j, b^i = a^{ir}b_r$ *and* $b^2 = a^{rs}b_r b_s$.

It is well-known that [4] a Finsler metric $F = F(x,y)$ on an open subset $U \subset \mathbb{R}^n$ is projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{2.2}$$

By (2.2), we have the following lemma [13]:

Lemma 2.2. *An (α,β)- metric $F=\alpha\phi(s)$, where $s=\frac{\beta}{\alpha}$, is projectively flat on an open subset $U\subset\mathbb{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0. \tag{2.3}$$

The functions $G^i(x,y)$ of F^n with an (α,β)-metric are written in the form [8]

$$2G^i = \gamma^i_{00} + 2B^i, \tag{2.4}$$

$$B^i = \frac{\alpha L_\beta}{L_\alpha} s^i_0 + C^*\left\{\frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}\left(\frac{1}{\alpha} y^i - \frac{\alpha}{\beta} b^i\right)\right\}, \tag{2.5}$$

provided $\beta^2 + L_\alpha + \alpha\gamma^2 L_{\alpha\alpha} \neq 0$, where $\gamma^2 = b^2\alpha^2 - \beta^2, \alpha = \frac{\partial L}{\partial\alpha}, L_\beta = \frac{\partial L}{\partial\beta}, L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial\alpha}$, the subscript 0 means contraction by y^i and we put

$$C^* = \frac{\alpha\beta\,(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2\,(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}.$$

We shall denote the homogeneous polynomials in (y^i) of degree r by $hp(r)$ for brevity. For example, γ^i_{00} is $hp(2)$. From (2.4) the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of F^n with an (α,β)-metric is given by [8]

$$G^i_j = \dot\partial_j G^i = \gamma^i_{0j} + B^i_j,$$
$$G^i_{jk} = \dot\partial_k G^i_j = \gamma^i_{jk} + B^i_{jk},$$

where we put $B^i_j = \dot\partial_j B^i$ and $B^i_{jk} = \dot\partial_k B^i_j$. On account of [8], B^i_{jk} are determined by

$$L_\alpha B^k_{ji} y^j y_t + \alpha L_\beta (B^k_{ji} b_t - b_{j;i}) y^j = 0, \tag{2.6}$$

where $y_k = a_{ik}y^i$. A Finsler space F^n with an (α,β)-metric is a Douglas space if and only if $B^{ij} \equiv B^i y^j - b^j y^i$ is $hp(3)$ [2]. From (2.5) B^{ij} is written as follows:

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i). \tag{2.7}$$

3 Projectively flat (α,β)-metric

In this section, we consider the metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta$ which is obtained by the Randers change of the Exponential metric.

$$F = \alpha\phi(s), \quad \phi(s) = (e^s + s),\ s = \frac{\beta}{\alpha}, \tag{3.1}$$

Let $b_0 > 0$ be the largest number such that

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \le b < b_0),$$

that is,

$$e^s(1 - s + b^2 - s^2) > 0, \quad (|\, s\,| \le b < b_0)$$

Lemma 3.1. *$F = \alpha e^{\frac{\beta}{\alpha}} + \beta$ is a Finsler metric, iff $\|\beta\|_\alpha < 1$.*

Proof. If $F = \alpha e^{\frac{\beta}{\alpha}} + \beta$ is a Finsler metric, then

$$e^s(1 - s + b^2 - s^2) > 0, \quad |s| \le b < b_0.$$

Let $s = b$, then we get $b < 1, \forall\, b < b_0$. Let $b \to b_0$, then $b_0 < 1$. So $\|\beta\|_\alpha < 1$. Now, if

$$|s| \le b < 1,$$

then

$$e^s(1 - s + b^2 - s^2) \ge e^s(1 - s) > 0. \quad (\textit{because } b^2 - s^2 \ge 0)$$

Thus $F = \alpha e^{\frac{\beta}{\alpha}} + \beta$ is a Finsler metric. □

By Lemma (2.1), the spray coefficients G^i of F are given by (2.1) with

$$Q = \frac{\alpha(1 + e^{\frac{\beta}{\alpha}})}{(\alpha - \beta)e^{\frac{\beta}{\alpha}}},$$
$$J = \frac{\alpha^2(\alpha - \beta)(1 + e^{\frac{\beta}{\alpha}})}{2(\alpha e^{\frac{\beta}{\alpha}} + \beta)(\alpha^2 - \alpha\beta + \alpha^2 b^2 - \beta^2)}$$
$$H = \frac{\alpha^2}{2(\alpha^2 - \alpha\beta + \alpha^2 b^2 - \beta^2)}.$$

Equation (2.3) is reduced to the following form:

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \frac{\alpha^4(1 + e^{\frac{\beta}{\alpha}})}{(\alpha - \beta)e^{\frac{\beta}{\alpha}}} s_{l0} + \frac{\alpha^3}{2(\alpha^2 - \alpha\beta + \alpha^2 b^2 - \beta^2)}\left[\frac{-2\alpha^2 s_0(1 + e^{\frac{\beta}{\alpha}})}{(\alpha - \beta)e^{\frac{\beta}{\alpha}}} + r_{00}\right]\left(b_l\alpha - \frac{\beta}{\alpha} y_l\right) = 0. \tag{3.2}$$

We use the following result [15]:

Lemma 3.2. *If $(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0$, then α is projectively flat.*

Proof. If $(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0$, then

$$\alpha^2 a_{ml} G^m_\alpha = y_m G^m_\alpha y_l,$$

then there is a $\eta = \eta(x, y)$ such that $y_m G^m_\alpha = \alpha^2\eta$, we get

$$a_{ml} G^m_\alpha = \eta y_l. \tag{3.3}$$

Contracting Eq. (3.3) with a^{il} yields $G^i_\alpha = \eta y^i$, and thus α is projectively flat. □

Theorem 3.3. *The (α, β)-metric $F = \alpha e^{\frac{\beta}{\alpha}} + \beta$ is locally projectively flat iff*
(i) β is parallel with respect to α
(ii) α is locally projectively flat, i.e., of constant curvature.

Proof. Suppose that F is locally projectively flat. First, we rewrite (3.2) as a polynomial in y^i and α. This gives

$$\begin{aligned}
&\Big[(-4\alpha^2\beta+2\beta^3-2b^2\alpha^2\beta)e^{\frac{\beta}{\alpha}}(a_{ml}\alpha^2-y_my_l)G^m_\alpha\\
&\quad+(2\alpha^6+2b^2\alpha^6-2\alpha^4\beta^2)(1+e^{\frac{\beta}{\alpha}})s_{l0}\\
&\quad-2\alpha^4s_0(1+e^{\frac{\beta}{\alpha}})(b_l\alpha^2-\beta y_l)-\alpha^2\beta e^{\frac{\beta}{\alpha}}r_{00}(b_l\alpha^2-\beta y_l)\Big]\\
&\quad+\alpha\Big[(2\alpha^2+2b^2\alpha^2)e^{\frac{\beta}{\alpha}}(a_{ml}\alpha^2-y_my_l)G^m_\alpha\\
&\quad-2\alpha^4\beta(1+e^{\frac{\beta}{\alpha}})s_{l0}+\alpha^2e^{\frac{\beta}{\alpha}}r_{00}(b_l\alpha^2-\beta y_l)\Big]=0,
\end{aligned}$$

$$\text{(3.4)} \qquad or \quad U+\alpha V=0,$$

where

$$\begin{aligned}
U&=-4\alpha^2\beta+2\beta^3-2b^2\alpha^2\beta)e^{\frac{\beta}{\alpha}}(a_{ml}\alpha^2-y_my_l)G^m_\alpha\\
&\quad+(2\alpha^6+2b^2\alpha^6-2\alpha^4\beta^2)(1+e^{\frac{\beta}{\alpha}})s_{l0}\\
&\quad-2\alpha^4s_0(1+e^{\frac{\beta}{\alpha}})(b_l\alpha^2-\beta y_l)-\alpha^2\beta e^{\frac{\beta}{\alpha}}r_{00}(b_l\alpha^2-\beta y_l)
\end{aligned}$$

and

$$\begin{aligned}
V&=(2\alpha^2+2b^2\alpha^2)e^{\frac{\beta}{\alpha}}(a_{ml}\alpha^2-y_my_l)G^m_\alpha\\
&\quad-2\alpha^4\beta(1+e^{\frac{\beta}{\alpha}})s_{l0}+\alpha^2e^{\frac{\beta}{\alpha}}r_{00}(b_l\alpha^2-\beta y_l).
\end{aligned}$$

Now, (3.4) is a polynomial in y^i, such that U and V are rational in y^i and α is irrational. Therefore, we must have

$$U=0 \text{ and } V=0,$$

which implies that

$$\begin{aligned}
&[-2\alpha^2\beta(2+b^2)+2\beta^3]e^{\frac{\beta}{\alpha}}(a_{ml}\alpha^2-y_my_l)G^m_\alpha\\
&\quad=-2\alpha^4[\alpha^2(1+b^2)-\beta^2](1+e^{\frac{\beta}{\alpha}})s_{l0}\\
&\qquad+2\alpha^4s_0(1+e^{\frac{\beta}{\alpha}})(b_l\alpha^2-\beta y_l)+\alpha^2\beta e^{\frac{\beta}{\alpha}}r_{00}(b_l\alpha^2-\beta y_l)
\end{aligned}\tag{3.5}$$

and

$$2(1+b^2)e^{\frac{\beta}{\alpha}}(a_{ml}\alpha^2-y_my_l)G^m_\alpha=2\alpha^2\beta(1+e^{\frac{\beta}{\alpha}})s_{l0}-e^{\frac{\beta}{\alpha}}r_{00}(b_l\alpha^2-\beta y_l)\tag{3.6}$$

Contracting (3.5) and (3.6) with b^l, we get

$$\begin{aligned}
&[-2\alpha^2\beta(2+b^2)+2\beta^3]e^{\frac{\beta}{\alpha}}(b_m\alpha^2-y_m\beta)G^m_\alpha\\
&\quad=-2\alpha^4[\alpha^2(1+b^2)-\beta^2](1+e^{\frac{\beta}{\alpha}})s_0\\
&\qquad+2\alpha^4s_0(1+e^{\frac{\beta}{\alpha}})(b^2\alpha^2-\beta^2)+\alpha^2\beta e^{\frac{\beta}{\alpha}}r_{00}(b^2\alpha^2-\beta^2)
\end{aligned}\tag{3.7}$$

and

$$2(1+b^2)e^{\frac{\beta}{\alpha}}(b_m\alpha^2 - y_m\beta)G^m_\alpha = 2\alpha^2\beta(1+e^{\frac{\beta}{\alpha}})s_0 - e^{\frac{\beta}{\alpha}}r_{00}(b^2\alpha^2-\beta^2) \tag{3.8}$$

(3.7) and (3.8)$\times\alpha^2\beta$ yields

$$\beta(b_m\alpha^2 - y_m\beta)G^m_\alpha = \alpha^4(1+e^{-\frac{\beta}{\alpha}})s_0. \tag{3.9}$$

The polynomial $\alpha^4(1+e^{-\frac{\beta}{\alpha}})$ is not divisible by β and β is not divisible by $\alpha^4(1+e^{-\frac{\beta}{\alpha}})$. Thus s_0 is divisible by β and $(b_m\alpha^2-y_m\beta)G^m_\alpha$ is divisible by $\alpha^4(1+e^{-\frac{\beta}{\alpha}})$. Therefore, there exist scalar functions $\tau=\tau(x),\ \chi=\chi(x)$ such that

$$s_0 = \tau\beta, \tag{3.10}$$

$$(b_m\alpha^2 - y_m\beta)G^m_\alpha = \chi\alpha^4(1+e^{-\frac{\beta}{\alpha}}). \tag{3.11}$$

Then (3.9) becomes

$$\beta\chi\alpha^4(1+e^{-\frac{\beta}{\alpha}}) = \alpha^4(1+e^{-\frac{\beta}{\alpha}})\tau\beta.$$

Thus $\tau=\chi$. Then (3.7) becomes

$$\chi(1+e^{-\frac{\beta}{\alpha}})[2\alpha^4(1+b^2) - 2\alpha^2\beta^2] = -r_{00}(b^2\alpha^2-\beta^2). \tag{3.12}$$

Since $(b^2\alpha^2-\beta^2)$ is not divisible by $(1+e^{-\frac{\beta}{\alpha}})[2\alpha^4(1+b^2)-2\alpha^2\beta^2]$, it follows from (3.12) that $\chi=0$. By (3.10), (3.11) and (3.12), we get

$$s_0 = 0, \tag{3.13}$$

$$(b_m\alpha^2 - y_m\beta)G^m_\alpha = 0, \tag{3.14}$$

$$\textit{and } r_{00} = 0. \tag{3.15}$$

Then substituting (3.13) and (3.14) into (3.6), we get

$$s_{l0} = 0. \tag{3.16}$$

Then by (3.14) and Lemma (3.2), α is projectively flat. And by (3.15) and (3.16), $b_{i;j}=0$. i., β is parallel with respect to α.

Conversely, if β is parallel with respect to α and α is locally projectively flat, then by Lemma (2.2), we can easily see that F is locally projectively flat. □

4 Berwald and Douglas spaces

In this section, we find the condition for a Finsler space F^n with (α,β)-metric (3.1) to be a Berwald space. In the n-dimensional Finsler space F^n with an (α,β)-metric (3.1), we have

$$\begin{aligned} L_\alpha &= \frac{(\alpha-\beta)e^{\frac{\beta}{\alpha}}}{\alpha}, \quad L_\beta = 1+e^{\frac{\beta}{\alpha}}, \\ L_{\alpha\alpha} &= \frac{\beta^2}{\alpha^3}e^{\frac{\beta}{\alpha}}, \quad L_{\beta\beta} = \frac{e^{\frac{\beta}{\alpha}}}{\alpha}. \end{aligned} \tag{4.1}$$

Substituting (4.1) into (2.6), we have

$$\alpha e^{\frac{\beta}{\alpha}} B^t_{ji} y^j y_t - \beta e^{\frac{\beta}{\alpha}} B^t_{ji} y^j y_t + \alpha^2 (1 + e^{\frac{\beta}{\alpha}})(B^t_{ji} b_t - b_{j;i}) y^j = 0 \tag{4.2}$$

Assume that F^n is a Berwald space, that is, $G^i_{jk} = G^i_{jk}(x)$. Then we have $B^t_{ji} = B^t_{ji}(x)$. Since α is irrational in (y^i), from (4.2), we have

$$e^{\frac{\beta}{\alpha}} B^t_{ji} y^j y_t = 0, \tag{4.3}$$

$$-\beta e^{\frac{\beta}{\alpha}} B^t_{ji} y^j y_t + \alpha^2 (1 + e^{\frac{\beta}{\alpha}})(B^t_{ji} b_t - b_{j;i}) y^j = 0. \tag{4.4}$$

From (4.3) and (4.4), we obtain

$$B^t_{ji} y^j y_t = 0 \quad \textit{and} \quad (B^t_{ji} b_t - b_{j;i}) y^j = 0.$$

which yields

$$B^t_{ji} a_{th} + B^t_{hi} a_{tj} = 0 \qquad B^t_{ji} b_t - b_{j;i} = 0.$$

Thus by the well known Christoffel process we get $B^t_{ji} = 0$. Therefore we have

Theorem 4.1. *The Randers change of the Exponential metric* (3.1) *provides a Berwald metric if and only if $b_{j;i} = 0$, and then the Berwald connection is Riemannian $(\gamma^i_{jk}, \gamma^i_{0j}, 0)$.*

Now, we consider the condition for a Finsler space F^n with an (α, β)-metric (3.1) to be a Douglas space. Substituting (4.1) into (2.7), we obtain

$$\begin{aligned} &2(\alpha - \beta) e^{\frac{\beta}{\alpha}} (\alpha^2 - \alpha\beta + b^2\alpha^2 - \beta^2) B^{ij} \\ &- 2\alpha^2 (1 + e^{\frac{\beta}{\alpha}})(\alpha^2 - \alpha\beta + b^2\alpha^2 - \beta^2)(s^i_0 y^j - s^j_0 y^i) \\ &- \alpha^2 \left\{(\alpha - \beta) e^{\frac{\beta}{\alpha}} r_{00} - 2\alpha^2 s_0 (1 + e^{\frac{\beta}{\alpha}})\right\} (b^i y^j - b^j y^i) = 0. \end{aligned} \tag{4.5}$$

Suppose that F^n is a Douglas space, that is B^{ij} are $hp(3)$. Separating (4.5) in rational and irrational terms of y^i, because α is irrational in (y^i), we have

$$\begin{aligned} &\Big[(-4\alpha^2\beta + 2\beta^3 - 2b^2\alpha^2\beta) e^{\frac{\beta}{\alpha}} B^{ij} \\ &+ (1 + e^{\frac{\beta}{\alpha}})(-2\alpha^4 - 2b^2\alpha^4 + 2\alpha^2\beta^2)(s^i_0 y^j - s^j_0 y^i) \\ &+ \alpha^2\beta e^{\frac{\beta}{\alpha}} r_{00} (b^i y^j - b^j y^i) + 2\alpha^4 s_0 (1 + e^{\frac{\beta}{\alpha}})(b^i y^j - b^j y^i)\Big] \\ &+ \alpha \Big[e^{\frac{\beta}{\alpha}} B^{ij} (2\alpha^2 + 2b^2\alpha^2) + 2\alpha^2\beta (1 + e^{\frac{\beta}{\alpha}})(s^i_0 y^j - s^j_0 y^i) \\ &- \alpha^2 e^{\frac{\beta}{\alpha}} r_{00} (b^i y^j - b^j y^i)\Big] = 0. \end{aligned} \tag{4.6}$$

Hence the equation (4.6) is divided into two equations as follows:

$$\begin{aligned} &(-4\alpha^2\beta + 2\beta^3 - 2b^2\alpha^2\beta) e^{\frac{\beta}{\alpha}} B^{ij} \\ &+ (1 + e^{\frac{\beta}{\alpha}})(-2\alpha^4 - 2b^2\alpha^4 + 2\alpha^2\beta^2)(s^i_0 y^j - s^j_0 y^i) \\ &+ \alpha^2\beta e^{\frac{\beta}{\alpha}} r_{00} (b^i y^j - b^j y^i) + 2\alpha^4 s_0 (1 + e^{\frac{\beta}{\alpha}})(b^i y^j - b^j y^i) = 0 \end{aligned} \tag{4.7}$$

and

(4.8) $e^{\frac{\beta}{\alpha}}B^{ij}(2\alpha^2+2b^2\alpha^2)+2\alpha^2\beta(1+e^{\frac{\beta}{\alpha}})(s^i_0y^j-s^j_0y^i)-\alpha^2e^{\frac{\beta}{\alpha}}r_{00}(b^iy^j-b^jy^i)=0$

Eliminating B^{ij} from (4.7) and (4.8), we obtain

$$A(s^i_0y^j-s^j_0y^i)+B(b^iy^j-b^jy^i)=0 \tag{4.9}$$

where

$$A=\left(-2\alpha^4-4b^2\alpha^4+6\alpha^2\beta^2-2b^4\alpha^4+4b^2\alpha^2\beta^2-2\beta^4\right)(1+e^{\frac{\beta}{\alpha}}), \tag{4.10}$$

$$B=e^{\frac{\beta}{\alpha}}r_{00}(\beta^3-\alpha^2\beta)+2\alpha^4(1+b^2)(1+e^{\frac{\beta}{\alpha}})s_0. \tag{4.11}$$

Transvecting (4.9) by b_iy_j, we get

$$As_0\alpha^2+B(b^2\alpha^2-\beta^2)=0. \tag{4.12}$$

The term of (4.12) which does not contain α^2 is $-\beta^5r_{00}$. Hence there exists $hp(5):v_5$ such that

$$-\beta^5r_{00}=\alpha^2V_5 \tag{4.13}$$

Here we consider two cases:

(i) $V_5=0$,
(ii) $V_5\neq 0, \alpha^2\not\equiv 0\ (mod\ \beta)$.

Case (i). When $V_5=0$, this leads to $r_{00}=0$. Therefore, substituting $r_{00}=0$ into (4.12), we get

$$s_0(A+\gamma^2B_1)=0, \tag{4.14}$$

where

$$B_1=2\alpha^2(1+b^2)(1+e^{\frac{\beta}{\alpha}}).$$

If $A+\gamma^2B_1=0$, then the term of $A+\gamma^2B_1=0$ which does not contain α^2 is $-4\beta^4$. Thus there exists $hp(2):V_2$ such that

$$-4\beta^4=\alpha^2V_2,$$

Hence we have $V_2=0$, which leads to a contradiction. Therefore, we must have $A+\gamma^2B_1\neq 0$. Therefore, we have $s_0=0$ from (4.14). Substituting $s_0=0$ and $r_{00}=0$ into (4.9), we get

$$A(s^i_0y^j-s^j_0y^i)=0. \tag{4.15}$$

If $A=0$, then from (4.10), we have

$$A=\left(-2\alpha^4-4b^2\alpha^4+6\alpha^2\beta^2-2b^4\alpha^4+4b^2\alpha^2\beta^2-2\beta^4\right)(1+e^{\frac{\beta}{\alpha}})=0. \tag{4.16}$$

The term of (4.16) which does not contain α^2 is $-4\beta^4$. Thus there exists $hp(2) : V_2$ such that

$$-4\beta^4 = \alpha^2 V_2,$$

from which we have $V_2 = 0$. It is a contradiction, therefore we must have $A \neq 0$. Therefore, from (4.15) we obtain

$$s^i_0 y^j - s^j_0 y^i = 0.$$

Transvecting the above equation by y_j gives $s^i_0 = 0$, which imply $s_{ij} = 0$. Consequently, we have $r_{ij} = s_{ij} = 0$, that is, $b_{i;j} = 0$.

Case (ii). The equation (4.13) shows that there exists a function $k = k(x)$ satisfying

$$r_{00} = k(x)\alpha^2.$$

Thus we have the term of (4.12) does not contain α^2 is included in the term $-\beta^5 r_{00}$. Hence we get $r_{00} = 0$. From (4.15), we have $A(s^i_0 y^j - s^j_0 y^i) = 0$. If $A = 0$, then it is a contradiction. Hence $A \neq 0$. Therefore, we obtain $s^i_0 y^j - s^j_0 y^i = 0$. Transvecting this equation by y_j we get $s^i_0 = 0$. Hence both the cases (i) and (ii) lead to $r_{ij} = 0$ and $s_{ij} = 0$, that is, $b_{i;j} = 0$.

Conversely if $b_{i;j} = 0$, then F^n is a Berwald space, so F^n is a Douglas space.

Thus we have the following

Theorem 4.2. *The Randers change of exponential metric is of Douglas type if and only if* $\alpha^2 \not\equiv 0 \ (mod\ \beta)$ *and* $b_{i;j} = 0$.

From Theorem 4.1 and Theorem 4.2, we have

Theorem 4.3. *If the Randers change of exponential metric is of Douglas type, then it is Berwaldian.*

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