Exact solution of the prey-predator model with diffusion using an expansion method

Mitul Islam, Bipul Islam and Nurul Islam

Abstract. In this paper the (G'/G)-expansion method is used to get more general exact solutions of the coupled non-linear prey-predator model equations incorporating a diffusion term. The importance of the approach is that the solutions of the given system get classified into three sub-classes - trigonometric, hyperbolic and rational. The exact solutions are further explored through numerical simulation. Their biological implications are accordingly discussed.

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Key words: Travelling wave solutions; (G'/G)-expansion method; diffusion; prey-predator model.

1 Introduction

In various areas of science and engineering, especially in fluid mechanics, solid-state physics, biophysics, biomathematics and so on, arise non-linear evolution equations. The exact solutions of non-liner partial differential equations may provide much physical informations to understand the mechanism of the physical models. Various powerful and exact methods have been developed, such as Hirota's bilinear method [5], Darboux transform method [9], inverse scattering transform [1], truncated Painleve' expansion [3, 10], variational iteration method [4], the sine-cosine method [2], the Jacobi elliptic function expansion method [8], the F-expansion method [14], tanh-function method [12] and so on. Recently, the (G'/G) -expansion method has been proposed by Wang et al. [11] to find exact solutions of non-liner partial differential equations.

The prey-predator model incorporating diffusion is of profound interest as it takes into account the heterogeneity of both the environment and the populations involved. The spatial pattern formation even in absence of environmental heterogeneity is another interesting phenomenon associated with the diffusion models [7]. Existence of exact solutions is pivotal to better the understanding about the processes involved. In this paper, we apply (G'/G) -expansion method to the prey-predator model with diffusion terms to obtain travelling wave solutions. Section 2 outlines this method stepwise. Section 3 involves the calculations which finally yield the solution set. This

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method produces exact solutions for the system, with the solutions classified into three sub classes depending upon the value of $\lambda^2 - 4\mu$. Section 4 tries to explain the biological aspects of the analytical results obtained. The simulation results clearly establishes $\lambda^2 - 4\mu$ as the driving factor in determining the long term behavior of the given system. The simulation also shows spatial pattern formation due to non-uniform distribution of the population densities along the axis of diffusion.

2 The (G'/G)-expansion method

In this section we present the (G'/G)-expansion method as described by Wang et al. [11] to get exact travelling wave solutions of non-liner partial differential equations. A partial differential equation in two independent variables x and t is taken in the form

$$F(u, u_t, u_x, u_{tx}, u_{tt}, ...) = 0, \tag{2.1}$$

where $u = u(x,t)$ and F is a polynomial of u and its derivatives involve non-linear terms. The next steps are as follows:

Step I: With the substitution of travelling wave variable $\xi = x + ct$, the equation (2.1) reduces to an ordinary differential equation(ODE) for $u = u(\xi)$ as

$$F(u, u', u'', u''', ...) = 0 \tag{2.2}$$

Step II: Solution of (2.2) is taken in the form

$$u(\xi) = \sum_{i=0}^{m} f_i (G'/G)^i, \tag{2.3}$$

where $G = G(\xi)$ satisfies the second order differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{2.4}$$

and $f_0, f_1, f_2, \dots, f_m, \lambda, \mu$ are constants to be determined. Considering the homogeneous balance between the highest order derivative and the non-linear terms appearing in the equation (2.3), the positive integer m is determined.

Step III: Using (2.3) and (2.4) in equation (2.2), it reduces to a polynomial in (G'/G) and finally yields a set of algebraic equations for $f_0, f_1, f_2, \dots, f_m, \lambda$, and μ by equating each coefficient to zero of the polynomial.

Step IV: Finding out $f_0, f_1, f_2, \dots, f_m$ from the system of equations obtained in step III and solving equation (2.4) one gets travelling wave solution of non-linear differential equation (2.2) by using them in equation (2.3).

3 Application of the (G'/G) -expansion method to the prey-predator model with diffusion

The (G'/G) -expansion method is applied for getting travelling wave solution of the prey-predator model equations with diffusion terms[6] given by

$$\begin{aligned} u_t &= u(a_1 - b_{11}u - b_{12}v) + D_1 u_{xx} \\ v_t &= v(-a_2 + b_{21}u - b_{22}v) + D_2 v_{xx} \ , \end{aligned} \tag{3.1}$$

where $u(x,t)$ and $v(x,t)$ represent respectively the prey and predator population densities at time t. Following step I, let us introduce the new independent variable

$$\xi = x + ct \tag{3.2}$$

and consequently u and v both become the functions of one variable ξ. With the help of equation (3.2), system of equations (3.1) takes the form

$$\begin{aligned} cu' + u(a_1 - b_{11}u - b_{12}v) + D_1u'' = 0 \\ cv' + v(-a_2 + b_{21}u - b_{22}v) + D_2v'' = 0. \end{aligned} \tag{3.3}$$

In terms of (G'/G), the solutions of the system (3.3) are taken as

$$\begin{aligned} u(\xi) &= \sum_{i=0}^{m} f_i(G'/G)^i \\ \textit{and} \quad v(\xi) &= \sum_{i=0}^{n} h_i(G'/G)^i \end{aligned} \tag{3.4}$$

where f_i and h_i are constants and m and n are integers. $G = G(\xi)$ satisfies the second order ordinary linear differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{3.5}$$

where λ and μ are constants. Considering the homogeneous balance between the highest order derivative and the non-linear terms appearing in the equations (3.3), the positive integers m and n are determined from the equations

$$m + 2 = 2m = m + n = n + 2 = 2n$$

which on simplification yield $m = n = 2$. Thus

$$\begin{aligned} u(\xi) = f_0 + f_1(G'/G) + f_2(G'/G)^2 \\ v(\xi) = h_0 + h_1(G'/G) + h_2(G'/G)^2 \end{aligned} \tag{3.6}$$

with $f_2 \neq 0$, $h_2 \neq 0$. Substituting (3.6) into (3.3) and comparing coefficients of like powers of (G'/G) one obtains the system of algebraic equations as follows:

$$\begin{aligned}
(G'/G)^0 &: \mu[f_1c - D_1(2\mu f_2 + \lambda f_1)] \\
&= a_1f_0 - b_{11}f_0^2 - b_{12}f_0h_0 \\
(G'/G)^1 &: 2\mu[f_2c - D_1(2\lambda f_2 + f_1)] + \lambda[f_1c - D_1(2\mu f_2 + \lambda f_1)] \\
&= a_1f_1 - 2b_{11}f_0f_1 - b_{12}(f_0h_1 + f_1h_0) \\
(G'/G)^2 &: f_1C - D_1(2\mu f_2 + \lambda f_1) - 6\mu D_1f_2 + \lambda[2cf_2 - 2D_1(f_1 + 2\lambda f_2)] \\
&= a_1f_2 - b_{11}[f_1^2 + 2f_0f_2] - b_{12}[f_0h_2 + f_2h_0 + f_1h_1] \\
(G'/G)^3 &: 6\lambda D_1f_2 - 2Cf_2 + 2D_1(f_1 + 2\lambda f_2) \\
&= 2b_{11}f_1f_2 + b_{12}[f_2h_1 + f_1h_2] \\
(G'/G)^4 &: 6D_1 = b_{11}f_2 + b_{12}h_2
\end{aligned}$$

and

$$
\begin{aligned}
(G'/G)^0 &: \mu h_1 C = -a_2 h_0 + b_{21} f_0 h_0 - b_{22} h_0^2 + \mu D_2(2\mu h_2 + \lambda h_1)\\
(G'/G)^1 &: C(\lambda h_1 + 2\mu h_2) = -a_2 h_1 + b_{21}(f_0 h_1 + f_1 h_0) - 2b_{22} h_0 h_1\\
&\quad + D_2[\lambda(2\mu h_2 + \lambda h_1) + 2\mu(h_1 + 2\lambda h_2)]\\
(G'/G)^2 &: C(h_1 + 2\lambda h_2) = -a_2 h_2 + b_{21}(f_0 h_2 + f_2 h_0 + f_1 h_1)\\
&\quad - b_{22}(h_1^2 + 2h_0 h_2) + D_2[2\mu h_2 + \lambda h_1 + 6\mu h_2 + 2\lambda(h_1 + 2\lambda h_2)]\\
(G'/G)^3 &: 2Ch_2 = b_{21}(f_1 h_2 + f_2 h_1) - 2b_{22} h_2 h_1 + D_2[2(h_1 + 2\lambda h_2) + 6\lambda h_2]\\
(G'/G)^4 &: 6D_2 = b_{22} h_2 - b_{21} f_2.
\end{aligned}
$$

This system of equations yields

$$
f_2 = \frac{6(D_1 b_{22} + D_2 b_{12})}{b_{11} b_{22} + b_{21} b_{12}} \quad \textit{and} \quad h_2 = \frac{6(D_1 b_{21} + D_2 b_{11})}{b_{11} b_{22} + b_{21} b_{12}} \tag{3.7}
$$

$$
f_1 = \frac{\Delta_1}{\Delta_0}, \quad \textit{and} \quad h_1 = \frac{\Delta_2}{\Delta_0}, \ \textit{provided}\ \Delta_0 \ \neq \ 0, \tag{3.8}
$$

where

$$
\Delta_0 = (2b_{11} f_2 + b_{12} h_2 - 2D_1)(b_{21} f_2 - 2b_{22} h_2 + 2D_2) - b_{12} b_{21} f_2 h_2 \neq 0
$$

$$
\Delta_1 = 2f_2(5\lambda D_1 - C)(b_{21} f_2 - 2b_{22} h_2 + 2D_2) - 2b_{12} f_2 h_2 (C - 5\lambda D_2)
$$

and

$$
\Delta_2 = 2h_2(C - 5\lambda D_2)(2b_{11} f_2 + b_{12} h_2 - 2D_1) + 2b_{21} h_2 f_2 (C - 5\lambda D_1)
$$

with Δ_0, Δ_1 and Δ_2 obtained by using the values of f_2, h_2 from (3.7), and

$$
f_0 = \frac{\Delta_1'}{\Delta_0'}, \quad \textit{and} \quad h_0 = \frac{\Delta_2'}{\Delta_0'}, \ \textit{provided}\ \Delta_0' \ \neq \ 0, \tag{3.9}
$$

where

$$
\begin{aligned}
\Delta_0' &= 2(b_{12} b_{22} h_2^2 + 2b_{11} b_{22} f_2 h_2 - b_{11} b_{21} f_2^2) \neq 0\\
\Delta_1' &= (b_{21} f_2 - 2b_{22} h_2)\{C(f_1 + 2\lambda f_2) + b_{11} f_1^2 + b_{12} f_1 h_1 - a_1 f_2\\
&\quad - D_1(8\mu f_2 + 4\lambda^2 f_2 + 3\lambda f_1)\} + b_{12} f_2 \{C(h_1 + 2\lambda h_2)\\
&\quad + a_2 h_2 - b_{21} f_1 h_1 + b_{22} h_1^2 - D_2(8\mu f_2 + 4\lambda^2 f_2 + 3\lambda h_1)\}\\
\Delta_2' &= -(2b_{11} f_2 + b_{12} h_2)\{C(h_1 + 2\lambda h_2) + a_2 h_2 - b_{21} f_1 h_1 + b_{22} h_1^2\\
&\quad - D_2(8\mu h_2 + 4\lambda^2 h_2 + 3\lambda h_1)\} + b_{21} h_2 \{a_1 f_2 - b_{12} f_1 h_1 - b_{11} f_1^2\\
&\quad - C(f_1 + 2\lambda f_2) + D_1(8\mu f_2 + 4\lambda^2 f_2 + 3\lambda f_1)\}
\end{aligned}
$$

with Δ_0', Δ_1', Δ_2' found by substituting the values of f_2,h_2,f_1,h_1 from (3.7) and (3.8). Again C is obtained from either of the relations,

$$
\begin{aligned}
&\mu h_1 C = -a_2 h_0 + b_{21} f_0 h_0 - b_{22} h_0^2 + \mu D_2(2\mu h_2 + \lambda h_1)\\
\textit{and,}\quad &\mu f_1 C = a_1 f_0 + b_{12} f_0 h_0 - b_{11} f_0^2 + \mu D_1(2\mu f_2 + \lambda f_1)
\end{aligned}
\tag{3.10}
$$

by using the values of f_0, f_1, f_2 and h_0, h_1, h_2 found earlier in (3.7), (3.8) and (3.9).

By substituting the solutions of (3.5) and the values of f_0, f_1, f_2 and h_0, h_1, h_2 into (3.6), one gets the following travelling wave solutions of the prey-predator model with diffusion terms:

Case I : $\lambda^2 - 4\mu < 0$. The trigonometric function travelling wave solutions are

$$\begin{aligned} u(\xi) &= f_0 + f_1(G'/G) + f_2(G'/G)^2 \\ v(\xi) &= h_0 + h_1(G'/G) + h_2(G'/G)^2, \end{aligned} \tag{3.11}$$

where

$$(G'/G) = -\frac{\lambda}{2} + \frac{\sqrt{4\mu - \lambda^2}}{2}\left\{\frac{d_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + d_1 sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{d_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi) + d_2 sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right\},$$

and d_1, d_2 are arbitrary constants.

Case II : $\lambda^2 - 4\mu > 0$. The hyperbolic function travelling wave solutions are

$$\begin{aligned} u(\xi) &= f_0 + f_1(G'/G) + f_2(G'/G)^2 \\ v(\xi) &= h_0 + h_1(G'/G) + h_2(G'/G)^2, \end{aligned} \tag{3.12}$$

where

$$(G'/G) = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left\{\frac{c_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{c_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi) + c_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right\},$$

and c_1, c_2 are arbitrary constants.

Case III : $\lambda^2 - 4\mu = 0$. The rational function travelling wave solutions are

$$\begin{aligned} u(\xi) &= f_0 + f_1(G'/G) + f_2(G'/G)^2 \\ v(\xi) &= h_0 + h_1(G'/G) + h_2(G'/G)^2, \end{aligned} \tag{3.13}$$

where $(G'/G) = -\frac{\lambda}{2} + \dfrac{p_1}{p_1 + p_2\xi}$, and p_1, p_2 are arbitrary constants.

4 Results and discussion

Numerical simulations are carried out with a fixed set of parameters for all the three above cases in order to explain the behavior of the obtained solutions qualitatively. The chosen parameter values are: $a_1 = 0.9, b_{11} = 0.05, b_{12} = 0.4, a_2 = 0.2, b_{21} = 0.17, b_{22} = 0.12, D_1 = 0.1$ and $D_2 = 0.01$. The parameter c is set 0.1. The different constants in $(3.11), (3.12)$ and (3.13) are chosen as: $c_1 = 1.5, c_2 = 1.7, d_1 = 0.4, d_2 = 0.6, p_1 = 0.2, p_2 = 0.4$.

Simulation results indicate that $\lambda^2 - 4\mu$ is the deciding factor for the long term behavior of the system. $\lambda^2 - 4\mu < 0$ produces sustained oscillations in the prey and predator populations, that is, stable coexistence of both species. On the other hand, $\lambda^2 - 4\mu > 0$ can lead to different situations such as excess of predators, extinction of both species, etc. depending upon the choice of constants c_1 and c_2. $\lambda^2 - 4\mu = 0$ leads to drastic loss of both the species.

4.1 The case $\lambda^2 - 4\mu < 0$

The graphs are plotted taking $\lambda = 0.4$ and $\mu = 0.4$. Fig. 1 and 2 shows $u(x,t)$ and $v(x,t)$ for different values of x. Oscillations are observed in both the prey and predator populations. Figs. 3 and 4 are plotted with $\lambda = 0.2$ and $\mu = 0.4$ at $x = 1$ and $x = 14$ respectively. Fig. 3 shows oscillation in the maxima of both the prey and predator populations. Also, it is evident that at $x = 1$, predators dominate over the preys for the given set of parameters. Fig. 4 shows monotonic increase in the maxima of both the populations. Moreover, at $x = 14$, the preys dominate over the predators. Thus, the preys and predators are distributed along the axis of diffusion with preys dominant at specific points and predators at others. This is the aspect of 'spatial pattern' formation. The population densities at all these points keep on oscillating periodically with the maximum size of both populations evolving depending upon its position coordinate x.

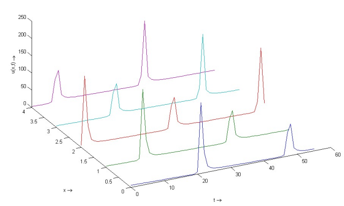

Fig. 1.

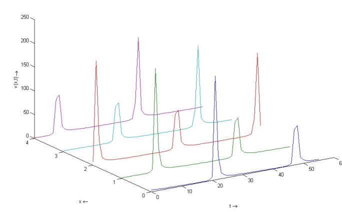

Fig. 2.

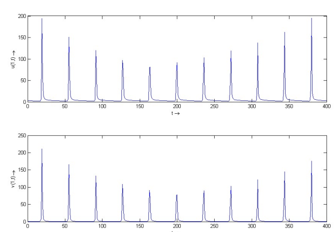

Fig. 3.

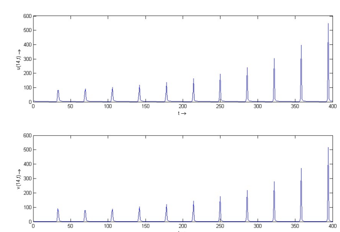

Fig. 4.

4.2 The case $\lambda^2 - 4\mu > 0$

Taking $\lambda = 2$ and $\mu = 0.009$, Figs. 5 and 6 are plotted for different values of x. The prey population decays rapidly to settle down at a much smaller value. The predators become dominant in regions close to the origin. However at places away from the origin, both the population densities attain a constant value, as seen in Fig. 7 with $x = 14$. This is a situation where overgrazing by the predators lead to scarcity of preys at certain points. However, by changing the constants c_1 and c_2, it is possible to attain a situation where both the species co-exist in areas close to the origin. The population remains constant over time at places farther away from the origin. Thus, when $\lambda^2 - 4\mu > 0$, the constants in the solution are the determining factors for long term behavior of the system close to the origin. For points farther away, the system always attains constant values irrespective of the choice of constants.

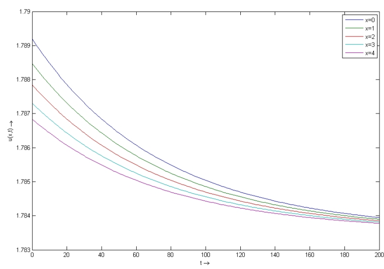

Fig. 5.

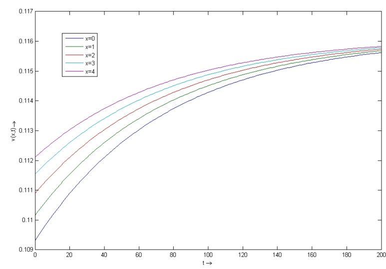

Fig. 6.

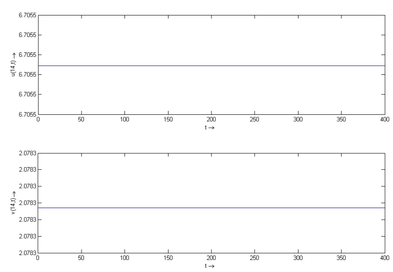

Fig. 7.

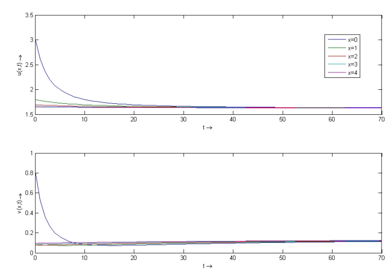

Fig. 8.

4.3 The case $\lambda^2 - 4\mu = 0$

Taking $\lambda = 0.6$ and $\mu = 0.09$, Fig. 8 is plotted for x close to the origin. At all these points, both the prey and predators decay rapidly and settle down to extremely small values. This portrays the 'near-extinction' situation for the entire prey-predator system. By altering the values of constants p_1 and p_2, it is possible to attain a situation where both prey and predators coexist. However, both $u(x,t)$ and $v(x,t)$ always remain constant at points away from origin.

4.4 Influence of diffusion

The parameters D_1 and D_2 are the diffusion parameters. When both of them are zero, the resulting solution can be deemed as the explicit solution of the classical prey-predator model. However the conditions imposed in (3.8) imply $D_1^2 + D_2^2 \neq 0$. Hence, we discuss two cases separately: (1) when $D_2 = 0, D_1 \neq 0$, and (2) when $D_1 = 0, D_2 \neq 0$.

4.4.1 The case $D_2 = 0$ - in absence of diffusion among predators

During the spawning season, salmon moves upstream from the seas. In North America, this period of 'salmon run' is a major preying event for the grizzly bears living in areas close to the river banks [13]. The grizzly bears, forming the predator class, is native of the area. The existing prey population of bass and trout is enhanced manifold by the diffusing salmon population. A classical prey-predator system with diffusion in prey population acts as a model for such a situation. Fig. 9 displays the time evolution prey and predator populations at the point $x = 1$ for this situation.

Fig. 10 displays the same at $x = 14$. The profile of the graph remains same as those obtained in Figs 3 and 4. However, the loss due to diffusion among predators causes the maximum size of both populations to fall.

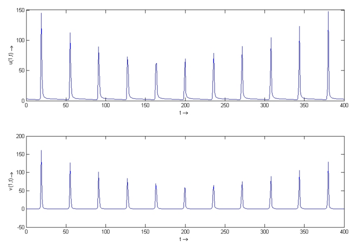

Fig. 9.

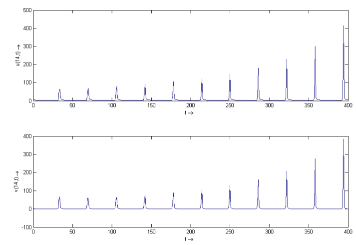

Fig. 10.

4.4.2 The case $D_1 = 0$ - in absence of diffusion among preys

Comparing Figs. 11 and 12 with Figs. 3 and 4, it is observed that the sizes of both the populations fall considerably in absence of diffusion. The maximum size of the prey population monotonically decays at $x = 1$ while it increases at $x = 14$. The diffusion among predators is instrumental in producing this non-uniform distribution of the populations. Interestingly, in Fig. 11 it is seen that though preys dominate at $x = 1$ initially, it is the predators who become more numerous there in the long run. This change over- of dominance was not observed in Figs 3 and 4.

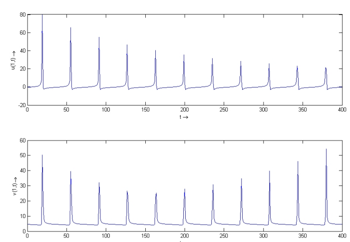

Fig. 11.

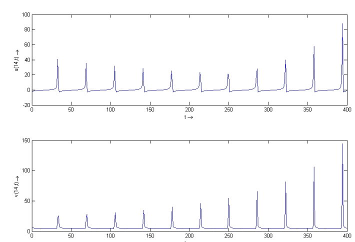

Fig. 12.

4.5 Conclusions

The expressions (3.11), (3.12) and (3.13) provide explicit solutions to the prey-predator system (3.1) incorporating diffusion. The above discussion qualitatively explains the results obtained through numerical simulation. The conclusion is that the prey and predator populations remain non-uniformly distributed along the axis of diffusion at points close to the origin. Whenever $\lambda^2 - 4\mu \geq 0$, all points far away from the origin have constant values of $u(x,t)$ and $v(x,t)$. The solution obtained by our method also classifies all the solutions of the system into two broad classes: (1) Periodically oscillating solutions characterized by $\lambda^2 - 4\mu < 0$, and (2) Monotonic solutions characterized by $\lambda^2 - 4\mu \geq 0$, thus simplifying the analysis of the solutions. Moreover, the influence of diffusion terms in shaping the dynamics of the obtained solutions is also investigated. The idea of introducing $D_1 = 0$ and $D_2 = 0$ to obtain explicit solutions of the classical prey-predator model, provides scope for future work.

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Author's address:

Mitul Islam
Department of Mathematics, Jadavpur University, Kolkata-700032, India.
E-mail: mitul.islam@gmail.com

Bipul Islam
Indian Statistical Institute, 203 B.T.Road, Barrackpore, Kolkata-700108, India.
E-mail: islam.bipul@gmail.com

Nurul Islam
Ramakrishna Mission Residential College (Autonomous),
Narendrapur, Kolkata-700103, India.
E-mail: dr.i.nurul@gmail.com