On ϕ-pseudo symmetric Kenmotsu manifolds with respect to quarter-symmetric metric connection

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Abstract. The object of the present paper is to study ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric Kenmotsu manifolds with respect to quarter-symmetric metric connection and obtain a necessary and sufficient condition of a ϕ-pseudo symmetric Kenmotsu manifold with respect to quarter symmetric metric connection to be ϕ-pseudo symmetric Kenmotsu manifold with respect to Levi-Civita connection.

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1 Introduction

In [48] Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He proved that they could be divided into three classes: (i) homogeneous normal contact Riemannian manifolds with $c > 0$, (ii) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$ and (iii) a warped product space $\mathbb{R} \times_f \mathbb{C}^n$ if $c < 0$. It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure. The manifolds of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu [29] characterized the differential geometric properties of the manifolds of class (iii) which are nowadays called Kenmotsu manifolds and later studied by several authors.

As a generalization of both Sasakian and Kenmotsu manifolds, Oubiña [34] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type (0, 0), $(\alpha, 0)$ and $(0, \beta)$ are called the cosympletic, α-Sasakian and β-Kenmotsu manifolds respectively, α, β being scalar functions. In particular, if $\alpha = 0, \beta = 1$; and $\alpha = 1, \beta = 0$ then a trans-Sasakian manifold will be a Kenmotsu and Sasakian manifold respectively.

The study of Riemann symmetric manifolds began with the work of Cartan [6]. A Riemannian manifold (M^n, g) is said to be locally symmetric due to Cartan [6] if

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its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by Walker [55], semisymmetric manifold by Szabó [47], pseudosymmetric manifold in the sense of Deszcz [20], pseudosymmetric manifold in the sense of Chaki [7].

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be pseudosymmetric in the sense of Chaki [7] if it satisfies the relation

$$\begin{aligned} (1.1) \qquad (\nabla_W R)(X, Y, Z, U) &= 2A(W)R(X, Y, Z, U) + A(X)R(W, Y, Z, U) \\ &+ A(Y)R(X, W, Z, U) + A(Z)R(X, Y, W, U) \\ &+ A(U)R(X, Y, Z, W), \end{aligned}$$

i.e.,

$$\begin{aligned} (1.2) \qquad (\nabla_W R)(X, Y)Z &= 2A(W)R(X, Y)Z + A(X)R(W, Y)Z \\ &+ A(Y)R(X, W)Z + A(Z)R(X, Y)W \\ &+ g(R(X, Y)Z, W)\rho \end{aligned}$$

for any vector field X, Y, Z, U and W, where R is the Riemannian curvature tensor of the manifold, A is a non-zero 1-form such that $g(X, \rho) = A(X)$ for every vector field X. Such an n-dimensional manifold is denoted by $(PS)_n$.

Every recurrent manifold is pseudosymmetric in the sense of Chaki [7] but not conversely. Also the pseudosymmetry in the sense of Chaki is not equivalent to that in the sense of Deszcz [20]. However, pseudosymmetry by Chaki will be the pseudosymmetry by Deszcz if and only if the non-zero 1-form associated with $(PS)_n$, is closed. Pseudosymmetric manifolds in the sense of Chaki have been studied by Chaki and Chaki [9], Chaki and De [10], De [12], De and Biswas [14], De, Murathan and Özgür [17], Özen and Altay ([36], [37]), Tarafder ([51], [52]), Tarafder and De [53] and others.

A Riemannian manifold is said to be Ricci symmetric if its Ricci tensor S of type (0,2) satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. During the last five decades, the notion of Ricci symmetry has been weakened by many authors in several ways to a different extent such as Ricci recurrent manifold [38], Ricci semisymmetric manifold [47], pseudo Ricci symmetric manifold by Deszcz [21], pseudo Ricci symmetric manifold by Chaki [8].

A non-flat Riemannian manifold (M^n, g) is said to be pseudo Ricci symmetric [8] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(1.3) \qquad (\nabla_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X)$$

for any vector field X, Y, Z, where A is a nowhere vanishing 1-form and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional manifold is denoted by $(PRS)_n$. The pseudo Ricci symmetric manifolds have been also studied by Arslan et. al [3], Chaki and Saha [11], De and Mazumder [16], De, Murathan and Özgür [17], Özen [35] and many others.
The relation (1.3) can be written as

$$(1.4) \qquad (\nabla_X Q)(Y) = 2A(X)Q(Y) + A(Y)Q(X) + S(Y, X)\rho,$$

where ρ is the vector field associated to the 1-form A such that $A(X) = g(X, \rho)$ and Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y.

As a weaker version of local symmetry, the notion of locally ϕ-symmetric Sasakian manifolds was introduced by Takahashi [49]. Generalizing the notion of locally ϕ-symmetric Sasakian manifolds, De, Shaikh and Biswas [18] introduced the notion of ϕ-recurrent Sasakian manifolds. In this connection De [13] introduced and studied ϕ-symmetric Kenmotsu manifolds and in [19] De, Yildiz and Yaliniz introduced and studied ϕ-recurrent Kenmotsu manifolds. In this connection it may be mentioned that Shaikh and Hui studied locally ϕ-symmetric β-kenmotsu manifolds [43] and extended generalized ϕ-recurrent β-Kenmotsu Manifolds [44], respectively. Also in [39] Prakash studied concircularly ϕ-recurrent Kenmotsu Manifolds. In [46] Shukla and Shukla studied ϕ-Ricci symmetric Kenmotsu manifolds. Recently the present author [26] studied ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric Kenmotsu manifolds.

Definition 1.1. A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ $(n > 3)$ is said to be *ϕ-pseudo symmetric* [26] if the curvature tensor R satisfies

$$
\begin{aligned}
(1.5) \qquad \phi^2((\nabla_W R)(X,Y)Z) &= 2A(W)R(X,Y)Z + A(X)R(W,Y)Z \\
&+ A(Y)R(X,W)Z + A(Z)R(X,Y)W \\
&+ g(R(X,Y)Z,W)\rho
\end{aligned}
$$

for any vector field X, Y, Z and W, where A is a non-zero 1-form.
In particular, if $A = 0$ then the manifold is said to be ϕ-symmetric [13].

Definition 1.2. A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ $(n > 3)$ is said to be *ϕ-pseudo Ricci symmetric* [26] if the Ricci operator Q satisfies

$$
(1.6) \qquad \phi^2((\nabla_X Q)(Y)) = 2A(X)QY + A(Y)QX + S(Y,X)\rho
$$

for any vector field X, Y, where A is a non-zero 1-form.

In particular, if $A = 0$, then (1.6) turns into the notion of ϕ-Ricci symmetric Kenmotsu manifold introduced by Shukla and Shukla [46].

In [22] Friedmann and Schouten introduced the notion of semisymmetric linear connection on a differentiable manifold. Then in 1932 Hayden [24] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semisymmetric metric connection on a Riemannian manifold has been given by Yano in 1970 [56]. In 1975, Golab introduced the idea of a quarter symmetric linear connection in differentiable manifolds.

A linear connection $\overline{\nabla}$ in an n-dimensional differentiable manifold M is said to be a quarter symmetric connection [23] if its torsion tensor τ of the connection $\overline{\nabla}$ is of the form

$$
\begin{aligned}
(1.7) \qquad \tau(X,Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \\
&= \eta(Y)\phi X - \eta(X)\phi Y,
\end{aligned}
$$

where η is a 1-form and ϕ is a tensor of type (1,1). In particular, if $\phi X = X$ then the quarter symmetric connection reduces to the semisymmetric connection. Thus the notion of quarter symmetric connection generalizes the notion of the semisymmetric connection. Again if the quarter symmetric connection $\overline{\nabla}$ satisfies the condition

$$(\overline{\nabla}_X g)(Y, Z) = 0 \tag{1.8}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, then $\overline{\nabla}$ is said to be a quarter symmetric metric connection. Quarter symmetric metric connection have been studied by many authors in several ways to a different extent such as [1], [2], [4], [25], [27], [28], [30], [31], [32], [33], [41], [42], [45], [50], [54]. Recently Prakasha [40] studied ϕ-symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection.

Motivated by the above studieds the present paper deals with the study of ϕ-pseudo symmetric and ϕ-pseudo Ricci symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of ϕ-pseudo symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection and obtain a necessary and sufficient condition of a ϕ-pseudo symmetric Kenmotsu manifold with respect to quarter symmetric metric connection to be ϕ-pseudo symmetric Kenmotsu manifold with respect to Levi-Civita connection. In section 4, we have studied ϕ-pseudo Ricci symmetric symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection.

2 Preliminaries

A smooth manifold (M^n, g) $(n = 2m + 1 > 3)$ is said to be an almost contact metric manifold [5] if it admits a (1,1) tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \qquad \eta(X) = g(X, \xi), \qquad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following condition holds [29]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

where ∇ denotes the Riemannian connection of g.
In a Kenmotsu manifold, the following relations hold [29]:

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.6}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{2.9}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.10}$$

$$S(\xi, \xi) = -(n-1), \quad \text{i.e.,} \quad Q\xi = -(n-1)\xi, \tag{2.11}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.12}$$

$$(\nabla_W R)(X, Y)\xi = g(X, W)Y - g(Y, W)X - R(X, Y)W \tag{2.13}$$

for any vector field X, Y, Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0,2) such that $g(QX, Y) = S(X, Y)$.
Let M be an n-dimensional Kenmotsu manifold and ∇ be the Levi-Civita connection on M. A quarter symmetric metric connection $\overline{\nabla}$ in a Kenmotsu manifold is defined by ([23], [40])

$$\overline{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{2.14}$$

where H is a tensor of type (1,1) such that

$$H(X, Y) = \frac{1}{2}\big[\tau(X, Y) + \tau'(X, Y) + \tau'(Y, X)\big] \tag{2.15}$$

and

$$g(\tau'(X, Y), Z) = g(\tau(Z, X), Y). \tag{2.16}$$

From (1.7) and (2.16), we get

$$\tau'(X, Y) = g(\phi Y, X)\xi - \eta(X)\phi Y. \tag{2.17}$$

Using (1.7) and (2.17) in (2.15), we obtain

$$H(X, Y) = -\eta(X)\phi Y. \tag{2.18}$$

Hence a quarter symmetric metric connection $\overline{\nabla}$ in a Kenmotsu manifold is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{2.19}$$

If R and $\overline{R}$ are respectively the curvature tensor of Levi-Civita connection ∇ and the quarter symmetric metric connection $\overline{\nabla}$ in a Kenmotsu manifold then we have [40]

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z - 2d\eta(X, Y)\phi Z \\ &+ \big[\eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z)\big]\xi \\ &+ \big[\eta(Y)\phi X - \eta(X)\phi Y\big]\eta(Z). \end{aligned} \tag{2.20}$$

From (2.20) we have

$$\overline{S}(Y,Z) = S(Y,Z) - 2d\eta(\phi Z,Y) + g(\phi Y,Z) + \psi\eta(Y)\eta(Z), \tag{2.21}$$

where $\overline{S}$ and S are respectively the Ricci tensor of a Kenmotsu manifold with respect to the quarter symmetric metric connection and Levi-Civita connection and $\psi = tr.\omega$, where $\omega(X,Y) = g(\phi X,Y)$. From (2.21) it follows that the Ricci tensor with respect to quarter symmetric metric connection is not symmetric.
Also from (2.21), we have

$$\overline{r} = r + 2(n-1), \tag{2.22}$$

where $\overline{r}$ and r are the scalar curvatures with respect to quarter symmetric metric connection and Levi-Civita connection respectively.
From (2.1), (2.2), (2.5), (2.13), (2.19) and (2.20), we get

$$\begin{aligned}(\overline{\nabla}_W\overline{R}(X,Y)\xi &= g(X,W)Y - g(Y,W)X - R(X,Y)W \\ &+ \big[\eta(Y)g(\phi W,X) - \eta(X)g(\phi W,Y)\big]\xi \\ &- \eta(W)\big[\eta(X)Y - \eta(Y)X \\ &+ \eta(X)\phi Y - \eta(Y)\phi X\big].\end{aligned} \tag{2.23}$$

Again from (2.19) and (2.20), we have

$$g((\overline{\nabla}_W\overline{R})(X,Y)Z,U) = -g((\overline{\nabla}_W\overline{R})(X,Y)U,Z). \tag{2.24}$$

Definition 2.1. A Kenmotsu manifold M is said to be η-Einstein if its Ricci tensor S of type (0,2) is of the form

$$S = ag + b\eta\otimes\eta, \tag{2.25}$$

where a, b are smooth functions on M.

3 ϕ-pseudo symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection

Definition 3.1. A Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$ $(n = 2m+1 > 3)$ is said to be *ϕ-pseudo symmetric* with respect to quarter symmetric metric connection if the curvature tensor $\overline{R}$ with respect to quarter symmetric metric connection satisfies

$$\begin{aligned}\phi^2((\overline{\nabla}_W\overline{R})(X,Y)Z) &= 2A(W)\overline{R}(X,Y)Z + A(X)\overline{R}(W,Y)Z \\ &+ A(Y)\overline{R}(X,W)Z + A(Z)\overline{R}(X,Y)W \\ &+ g(\overline{R}(X,Y)Z,W)\rho,\end{aligned} \tag{3.1}$$

for any vector field X, Y, Z and W, where A is a non-zero 1-form.
In particular, if $A = 0$ then the manifold is said to be ϕ-symmetric Kenmotsu manifold with respect to quarter symmetric metric connection [40].

We now consider a Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$ $(n = 2m+1 > 3)$, which is ϕ-pseudo symmetric with respect to quarter symmetric metric connection. Then by virtue of (2.1), it follows from (3.1) that

$$\begin{aligned} &-(\overline{\nabla}_W\overline{R})(X,Y)Z + \eta((\overline{\nabla}_W\overline{R})(X,Y)Z)\xi \\ = \;& 2A(W)\overline{R}(X,Y)Z + A(X)\overline{R}(W,Y)Z + A(Y)\overline{R}(X,W)Z \\ + \;& A(Z)\overline{R}(X,Y)W + g(\overline{R}(X,Y)Z,W)\rho \end{aligned} \tag{3.2}$$

from which it follows that

$$\begin{aligned} &-g((\overline{\nabla}_W\overline{R})(X,Y)Z,U) + \eta((\overline{\nabla}_W\overline{R})(X,Y)Z)\eta(U) \\ = \;& 2A(W)g(\overline{R}(X,Y)Z,U) + A(X)g(\overline{R}(W,Y)Z,U) + A(Y)g(\overline{R}(X,W)Z,U) \\ + \;& A(Z)g(\overline{R}(X,Y)W,U) + g(\overline{R}(X,Y)Z,W)A(U). \end{aligned} \tag{3.3}$$

Taking an orthonormal frame field and then contracting (3.3) over X and U and then using (2.1) and (2.2), we get

$$\begin{aligned} &-(\overline{\nabla}_W\overline{S})(Y,Z) + g((\overline{\nabla}_W\overline{R})(\xi,Y)Z,\xi) \\ = \;& 2A(W)\overline{S}(Y,Z) + A(Y)\overline{S}(W,Z) + A(Z)\overline{S}(Y,W) \\ + \;& A(\overline{R}(W,Y)Z) + A(\overline{R}(W,Z)Y). \end{aligned} \tag{3.4}$$

Using (2.8), (2.23) and (2.24), we have

$$\begin{aligned} g((\overline{\nabla}_W\overline{R})(\xi,Y)Z,\xi) &= -g((\overline{\nabla}_W\overline{R})(\xi,Y)\xi,Z) \\ &= g(\phi W,Y)\eta(Z) + g(\phi Y,Z)\eta(W) \\ &+ \big[g(Y,Z) - \eta(Y)\eta(Z)\big]\eta(W). \end{aligned} \tag{3.5}$$

By virtue of (3.5) it follows from (3.4) that

$$\begin{aligned} (\overline{\nabla}_W\overline{S})(Y,Z) &= -2A(W)\overline{S}(Y,Z) - A(Y)\overline{S}(W,Z) - A(Z)\overline{S}(Y,W) \\ &- A(\overline{R}(W,Y)Z) - A(\overline{R}(W,Z)Y) - g(\phi W,Y)\eta(Z) \\ &- g(\phi Y,Z)\eta(W) - \big[g(Y,Z) - \eta(Y)\eta(Z)\big]\eta(W). \end{aligned} \tag{3.6}$$

This leads to the following:

Theorem 3.1. *A ϕ-pseudo symmetric Kenmotsu manifold with respect to quarter symmetric metric connection is pseudo Ricci symmetric with respect to quarter symmetric metric connection if and only if*

$$\begin{aligned} &A(\overline{R}(W,Y)Z) + A(\overline{R}(W,Z)Y) + g(\phi W,Y)\eta(Z) \\ &+g(\phi Y,Z)\eta(W) + \big[g(Y,Z) + \eta(Y)\eta(Z)\big]\eta(W) = 0. \end{aligned}$$

Setting $Z = \xi$ in (3.4) and using (3.5), we get

$$\begin{aligned} &-(\overline{\nabla}_W\overline{S})(Y,\xi) + g(\phi W,Y) \\ = \;& 2A(W)\overline{S}(Y,\xi) + A(Y)\overline{S}(W,\xi) + A(\xi)\overline{S}(Y,W) \\ + \;& A(\overline{R}(W,Y)\xi) + A(\overline{R}(W,\xi)Y). \end{aligned} \tag{3.7}$$

By virtue of (2.7), (2.8), we have from (2.20) that

$$\overline{R}(X,Y)\xi = \eta(X)[Y - \phi Y] - \eta(Y)[X - \phi X], \tag{3.8}$$

$$\overline{R}(X,\xi)Y = [g(X,Y) - g(\phi X,Y)]\xi - \eta(Y)[X - \phi X]. \tag{3.9}$$

Also in view of (2.10) we get from (2.21) that

$$\overline{S}(Y,\xi) = [\psi - (n-1)]\eta(Y). \tag{3.10}$$

We know that

$$(\overline{\nabla}_W \overline{S})(Y,\xi) = \overline{\nabla}_W \overline{S}(Y,\xi) - \overline{S}(\overline{\nabla}_W Y,\xi) - \overline{S}(Y,\overline{\nabla}_W \xi). \tag{3.11}$$

Using (2.4), (2.10), (2.19) and (2.21) in (3.11) we get

$$\begin{aligned}(\overline{\nabla}_W \overline{S})(Y,\xi) &= -S(Y,W) + 2d\eta(\phi Y,W) - g(\phi Y,W) \\ &+ [\psi - (n-1)]g(Y,W) - \psi\eta(Y)\eta(W).\end{aligned} \tag{3.12}$$

In view of (2.3), (2.21) and (3.8)-(3.10), we have from (3.12) that

$$\begin{aligned}[1 - A(\xi)]S(Y,W) &= [\psi - (n-1) - A(\xi)]g(Y,W) \\ &+ 2A(\xi)g(\phi Y,W) + [\psi - (n-1)][2A(W)\eta(Y) \\ &+ A(Y)\eta(W)] - [1 - A(\xi)]\psi\eta(Y)\eta(W) \\ &+ [A(Y) - A(\phi Y)]\eta(W) \\ &- 2[A(W) - A(\phi W)]\eta(Y).\end{aligned} \tag{3.13}$$

Contracting (3.13) over Y and W, we obtain

$$[1 - A(\xi)]r = (n-1)(\psi - n) + 2(3\psi - 2n + 1)A(\xi). \tag{3.14}$$

This leads to the following:

Theorem 3.2. *In a ϕ-pseudo symmetric Kenmotsu manifold with respect to quarter symmetric metric connection the Ricci tensor and the scalar curvature are respectively given by* (3.13) *and* (3.14).

In particular, if $A = 0$ then (3.13) reduces to

$$S(Y,W) = [\psi - (n-1)]g(Y,W) - \psi\eta(Y)\eta(W),$$

which implies that the manifold under consideration is η-Einstein.
This leads to the following:

Corollary 3.3. *A ϕ-symmetric Kenmotsu manifold with respect to quarter symmetric metric connection is an η-Einstein manifold.*

Using (2.24) in (3.2), we get

$$\begin{aligned}(\overline{\nabla}_W \overline{R})(X,Y)Z &= -g((\overline{\nabla}_W \overline{R})(X,Y)\xi,Z)\xi - 2A(W)\overline{R}(X,Y)Z \\ &- A(X)\overline{R}(W,Y)Z - A(Y)\overline{R}(X,W)Z \\ &- A(Z)\overline{R}(X,Y)W - g(\overline{R}(X,Y)Z,W)\rho.\end{aligned} \tag{3.15}$$

In view of (2.20) and (2.23) it follows from (3.15) that

$$
\begin{aligned}
(\overline{\nabla}_W\overline{R})(X,Y)Z &= \big[R(X,Y,W,Z) \\
&+ g(X,Z)g(Y,W) - g(X,W)g(Y,Z) \\
&+ \eta(X)g(\phi W,Y) - \eta(Y)g(\phi W,X) \\
&+ \{g(Y,Z)+g(\phi Y,Z)\}\eta(W)\eta(X) \\
&- \{g(X,Z)+g(\phi X,Z)\}\eta(W)\eta(Y)\big]\xi \\
&- 2A(W)\big[R(X,Y)Z - 2d\eta(X,Y)\phi Z \\
&+ \{\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)\}\xi \\
&+ \{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z)\big] \\
&- A(X)\big[R(W,Y)Z - 2d\eta(W,Y)\phi Z \\
&+ \{\eta(W)g(\phi Y,Z) - \eta(Y)g(\phi W,Z)\}\xi \\
&+ \{\eta(Y)\phi W - \eta(W)\phi Y\}\eta(Z)\big] \\
&- A(Y)\big[R(X,W)Z - 2d\eta(X,W)\phi Z \\
&+ \{\eta(X)g(\phi W,Z) - \eta(W)g(\phi X,Z)\}\xi \\
&+ \{\eta(W)\phi X - \eta(X)\phi W\}\eta(Z)\big] \\
&- A(Z)\big[R(X,Y)W - 2d\eta(X,Y)\phi W \\
&+ \{\eta(X)g(\phi Y,W) - \eta(Y)g(\phi X,W)\}\xi \\
&+ \{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(W)\big] \\
&- \big[R(X,Y,Z,W) - 2d\eta(X,Y)g(\phi Z,W) \\
&+ \{\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)\}\eta(W) \\
&+ \{\eta(Y)g(\phi X,W) - \eta(X)g(\phi Y,W)\}\eta(Z)\big]\rho
\end{aligned}
\tag{3.16}
$$

for arbitrary vector fields X, Y, Z and W.
This leads to the following:

Theorem 3.4. *A Kenmotsu manifold is ϕ-pseudo symmetric with respect to quarter symmetric metric connection if and only if the relation* (3.16) *holds.*

Let us take a ϕ-pseudo symmetric Kenmotsu manifold with respect to Levi-Civita connection. Then the relation (1.5) holds. By virtue of (2.1), (2.13) and the relation $g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z)$ it follows from (1.5) that

$$
\begin{aligned}
(\nabla_W R)(X,Y)Z &= \big[R(X,Y,W,Z) + g(X,Z)g(Y,W) \\
&- g(X,W)g(Y,Z)\big]\xi - 2A(W)R(X,Y)Z \\
&- A(X)R(W,Y)Z - A(Y)R(X,W)Z \\
&- A(Z)R(X,Y)W - g(R(X,Y)Z,W)\rho.
\end{aligned}
\tag{3.17}
$$

From (3.16) and (3.17), we can state the following:

Theorem 3.5. *A ϕ-pseudo symmetric Kenmotsu manifold is invariant under quarter* *symmetric metric connection if and only if the relation*

$$\begin{aligned}
& \big[\eta(X)g(\phi W,Y)-\eta(Y)g(\phi W,X)+\{g(Y,Z)+g(\phi Y,Z)\}\eta(W)\eta(X)\\
-\ & \{g(X,Z)+g(\phi X,Z)\}\eta(W)\eta(Y)\big]\xi+2A(W)\big[2d\eta(X,Y)\phi Z\\
-\ & \{\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z)\}\xi-\{\eta(Y)\phi X-\eta(X)\phi Y\}\eta(Z)\big]\\
+\ & A(X)\big[2d\eta(W,Y)\phi Z-\{\eta(W)g(\phi Y,Z)-\eta(Y)g(\phi W,Z)\}\xi\\
-\ & \{\eta(Y)\phi W-\eta(W)\phi Y\}\eta(Z)\big]+A(Y)\big[2d\eta(X,W)\phi Z\\
-\ & \{\eta(X)g(\phi W,Z)-\eta(W)g(\phi X,Z)\}\xi-\{\eta(W)\phi X-\eta(X)\phi W\}\eta(Z)\big]\\
+\ & A(Z)\big[2d\eta(X,Y)\phi W-\{\eta(X)g(\phi Y,W)-\eta(Y)g(\phi X,W)\}\xi\\
-\ & \{\eta(Y)\phi X-\eta(X)\phi Y\}\eta(W)\big]+\big[2d\eta(X,Y)g(\phi Z,W)\\
-\ & \{\eta(X)g(\phi Y,Z)-\eta(Y)g(\phi X,Z)\}\eta(W)\\
-\ & \{\eta(Y)g(\phi X,W)-\eta(X)g(\phi Y,W)\}\eta(Z)\big]\rho=0
\end{aligned}$$

holds for arbitrary vector fields X, Y, Z and W.

4 ϕ-pseudo Ricci symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection

Definition 4.1. A Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$ $(n=2m+1>3)$ is said to be *ϕ-pseudo Ricci symmetric* with respect to quarter symmetric metric connection if the Ricci operator Q satisfies

$$\phi^2((\overline{\nabla}_X\overline{Q})(Y))=2A(X)\overline{Q}Y+A(Y)\overline{Q}X+\overline{S}(Y,X)\rho. \tag{4.1}$$

for any vector field X, Y, where A is a non-zero 1-form.

In particular, if $A=0$, then (4.1) turns into the notion of ϕ-Ricci symmetric Kenmotsu manifold with respect to quarter symmetric metric connection.

Let us take a Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$ $(n=2m+1>3)$, which is ϕ-pseudo Ricci symmetric with respect to quarter symmetric metric connection. Then by virtue of (2.1) it follows from (4.1) that

$$-(\overline{\nabla}_X\overline{Q})(Y)+\eta((\overline{\nabla}_X\overline{Q})(Y))\xi=2A(X)\overline{Q}Y+A(Y)\overline{Q}X+\overline{S}(Y,X)\rho$$

from which it follows that

$$\begin{aligned}
& -g(\overline{\nabla}_X\overline{Q}(Y),Z)+\overline{S}(\overline{\nabla}_XY,Z)+\eta((\overline{\nabla}_X\overline{Q})(Y))\eta(Z)\\
=\ & 2A(X)\overline{S}(Y,Z)+A(Y)\overline{S}(X,Z)+\overline{S}(Y,X)A(Z).
\end{aligned} \tag{4.2}$$

Putting $Y=\xi$ in (4.2) and using (2.4), (2.10), (2.19), (2.21) and (3.10), we get

$$\begin{aligned}
[1-A(\xi)]S(X,Z) =\ & \big[\psi-(n-1)-2A(\xi)\big]g(X,Z)\\
+\ & A(\xi)g(\phi X,Z)+[(\psi+2)A(\xi)-\psi]\eta(X)\eta(Z)\\
+\ & \big[\psi-(n-1)\big]\big[2A(X)\eta(Z)+A(Z)\eta(X)\big].
\end{aligned} \tag{4.3}$$

This leads to the following:

Theorem 4.1. *In a ϕ-pseudo Ricci symmetric Kenmotsu manifold with quarter symmetric metric connection the Ricci tensor is of the form* (4.3).

In particular, if $A = 0$ then from (4.3), we get

$$S(X,Z) = \{\psi - (n-1)\}g(X,Z) - \psi\eta(X)\eta(Z), \tag{4.4}$$

which implies that the manifold under consideration is η-Einstein. This leads the following:

Corollary 4.2. *A ϕ-Ricci symmetric Kenmotsu manifold with quarter symmetric metric connection is an η-Einstein manifold.*

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