On the Berwald-Lagrange scalar curvature in the structuring process of the LB-monolayer

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Abstract. The Berwald – Lagrange curvature with respect to the vector field in the two dimensional geometric dynamics of the Langmuir – Blodgett monolayer is determined. It is shown, that the metric of the monolayer space is of the scalar curvature and the sign of the Berwald – Lagrange curvature governs the growth of geodesic deviations within the monolayer, which in turn gives qualitative information about the sign of compressibility. Computer-drawn graphics and physical illustrate the relation between the geometric structure and the underlying physical background.

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Key words: 2D-monolayer Lagrangian, Barthel connection, Berwald linear connection, Berwald – Lagrange scalar curvature, compressibility, phase transition of first order.

1 Introduction

In the paper we utilize a geometrical approach to describe a structurization into Langmuir – Blodgett (LB) monolayers [1, 6, 9]. We assume that the usual physical time is defined on the interval $[0, \infty)$. Consider an open set $\mathbb{D} \subset \mathbb{R}^2$, endowed with the polar coordinates (r, φ), where $r > 0$ and $\varphi \in [0, 2\pi)$. We further consider the vector bundle

$$\mathbb{R} \times T\mathbb{R}^2 \xrightarrow{Id \times \pi} \mathbb{R} \times \mathbb{R}^2,$$

where $T\mathbb{R}^2$ is the tangent bundle of the plane, which is locally endowed with the bundle coordinates $(t, x^1, x^2, y^1, y^2) := (t, r, \varphi, \dot{r}, \dot{\varphi})$. We remind that on the vector bundle $\mathbb{R} \times T\mathbb{R}^2$ the transformation of coordinates are (the Einstein convention of summation is used throughout this work, and the Latin letters i, j, $k, l, q, s, \ldots$ take values in the set $\{1, 2\}$):

$$\widetilde{t} = t, \quad \widetilde{x}^q = \widetilde{x}^q(x^s), \quad \widetilde{y}^q = \frac{\partial \widetilde{x}^q}{\partial x^s} y^s, \tag{1.1}$$

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where rank $(\partial\widetilde{x}^p/\partial x^q) = 2$. Using the special function

$$f(z) \stackrel{def}{=} -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt,$$

we study several distinguished geometrical properties of the time-dependent Lagrangian (governing the 2D-motion of a particle of the monolayer [4, 5, 10]) $L : \mathbb{R} \times T\mathbb{R}^2 \to \mathbb{R}$, which is defined by

$$L(t, r, \dot{r}, \dot{\varphi}) = \frac{m}{2}\dot{r}^2 + \frac{mr^2}{2}\dot{\varphi}^2 \underbrace{-pr^5|V|e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-1} + U(t, r)}_{U_s(t,r)}, \tag{1.2}$$

where we have the following physical meanings: **(1)** m is the *mass* of the particle; **(2)** V is the *LB-monolayer compressing rate*; **(3)** p is a *constant monolayer parameter* given by the physical formula

$$p = \frac{\pi^2 q^2}{\varepsilon\varepsilon_0} \frac{\rho_0^2}{R_0^2};$$

(4) $U_s(t, r)$ is an *electro-capillarity potential energy*, including the monomolecular layer function

$$\begin{aligned} U(t, r) \quad &= \quad p\left\{\left[-\frac{4}{3}r^5 + \frac{16}{15}(|V|t)r^4 + \frac{1}{30}(|V|t)^2r^3 + \frac{1}{45}(|V|t)^3r^2 + \right.\right. \\ &\left.\left. + \frac{1}{45}(|V|t)^4 r + \frac{2}{45}(|V|t)^5\right] e^{\frac{2|V|t}{r}} - \frac{4}{45}\frac{(|V|t)^6}{r} f\left(\frac{2|V|t}{r}\right)\right\}. \end{aligned}$$

In what follows, we particularize several general geometrical ideas developed by Miron and Anastasiei in classical Lagrangian geometry on tangent bundles (see [7]) to the 2D-monolayer physical Lagrangian (1.2). To this aim, we consider projections of the tangent space regarded a 2D-slices which are spanned by a flag $\{y; X\}$. The considered curvature of each slice is the Finsler space flag curvature [3]. The metric produced by the 2D-monolayer Lagrangian is of Berwald-Lagrange curvature and is similar to the flag curvature.

The goal of the paper is to show that the Berwald-Lagrange metric is of scalar curvature and that the sign of the Berwald-Lagrange curvature governs the growth of geodesic deviations in the monolayer, which in turn gives qualitative information about the behavior of compressibility κ within the phase transition of first order.

2 The canonical nonlinear connection

The *fundamental vertical metrical d-tensor* produced by the 2D-monolayer Lagrangian (1.2) is given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}.$$

By direct computations, one gets the metrical d-tensor g_{ij}, whose associated matrix has the form

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \dfrac{m - 2pr^5|V|e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-3}}{2} & 0 \\ 0 & \dfrac{mr^2}{2} \end{pmatrix}. \tag{2.1}$$

Remark 2.1. In order to have $\det g \neq 0$, we assume that $g_{11} \neq 0$.

The matrix $g = (g_{ij})$ admits the inverse $g^{-1} = (g^{jk})$, whose entries are

$$g^{-1} = \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} \dfrac{2}{m - 2pr^5|V|e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-3}} & 0 \\ 0 & \dfrac{2}{mr^2} \end{pmatrix}. \tag{2.2}$$

Using the general semispray expression (cf., e.g., [7]), we infer the following geometrical result:

Proposition 2.1. *The energy action functional*

$$\begin{aligned} \mathbb{E}(t, r(t), \varphi(t)) &= \int_a^b L dt = \int_a^b \left[\frac{m}{2}\left(\frac{dr}{dt}\right)^2 + \frac{mr^2}{2}\left(\frac{d\varphi}{dt}\right)^2 - \right. \\ & \left. -pr^5|V|e^{\frac{2|V|t}{r}} \cdot \left(\frac{dr}{dt}\right)^{-1} + U(t,r) \right] dt, \end{aligned}$$

associated to the 2D-monolayer Lagrangian (1.2), produces on the vector fibre bundle $\mathbb{R} \times T\mathbb{R}^2$ *the* canonical semispray $\mathcal{G} = \left(G^i\right)_{i=\overline{1,2}}$, *whose components are*

$$\begin{aligned} G^1 = & \frac{pr^3|V|e^{\frac{2|V|t}{r}}\left(5r\dot{r}^{-1} - 2|V|t\dot{r}^{-1} + |V|r\dot{r}^{-2}\right) - \dfrac{1}{2}\dfrac{\partial U}{\partial r} - \dfrac{mr}{2}\dot{\varphi}^2}{m - 2pr^5|V|e^{\frac{2|V|t}{r}} \cdot \dot{r}^{-3}} \overset{approx}{\approx} \\ \overset{approx}{\approx} & -\frac{1}{2}\frac{|V|}{r}\dot{r} + \left(\frac{|V|t}{r^2} - \frac{5}{2r}\right)\dot{r}^2 - \frac{\dot{r}^3}{|V|}\left[\frac{5}{3}r^{-1} - \frac{26(|V|t)}{15}r^{-2} + \right. \\ & + \frac{61(|V|t)^2}{120}r^{-3} + \frac{(|V|t)^3}{180}r^{-4} + \frac{(|V|t)^4}{180}r^{-5} + \frac{(|V|t)^5}{90}r^{-6} - \\ & \left. - \frac{(|V|t)^6}{45}r^{-7}e^{-\frac{2|V|t}{r}} f\left(\frac{2|V|t}{r}\right)\right] + \frac{m}{4p|V|}r^{-4}e^{-\frac{2|V|t}{r}}\dot{r}^3\dot{\varphi}^2, \qquad G^2 = \frac{\dot{r}}{r}\dot{\varphi}. \end{aligned}$$

Proof. The Euler-Lagrange equations of the energy action functional $\mathbb{E}$ can be written in the equivalent form

$$\frac{d^2x^i}{dt^2} + 2G^i\left(t, x^k, y^k\right) = 0, \qquad y^k = \frac{dx^k}{dt},$$

where the local components

$$G^i \overset{def}{=} \frac{g^{is}}{4}\left[\frac{\partial^2 L}{\partial x^q \partial y^s}y^q - \frac{\partial L}{\partial x^s} + \frac{\partial^2 L}{\partial t \partial y^s}\right]$$

represent a semispray on the vector bundle $\mathbb{R} \times T\mathbb{R}^2$. □

The approximate polynomial form of the semispray $\mathcal{G}$ yields the canonical nonlinear connection

$$N = \left(N_j^i = \frac{\partial G^i}{\partial y^j} \right).$$

Consequently, by direct computations, we obtain the following important geometrical result:

Corollary 2.2. *The canonical nonlinear connection produced by the 2D-monolayer Lagrangian (1.2) has the following approximate components:*

$$N_1^1 = -\frac{1}{2}\frac{|V|}{r} + \left(\frac{2|V|t}{r^2} - \frac{5}{r}\right)\dot{r} - \mathcal{U}(t,r)\,\dot{r}^2 + \frac{3me^{-\frac{2|V|t}{r}}}{4p|V|r^4}\dot{r}^2\dot{\varphi}^2,$$
$$N_2^1 = \frac{me^{-\frac{2|V|t}{r}}}{2p|V|r^4}\dot{r}^3\dot{\varphi}, \qquad N_1^2 = \frac{\dot{\varphi}}{r}, \qquad N_2^2 = \frac{\dot{r}}{r},$$

where

$$\begin{aligned}\mathcal{U}(t,r) \;=\; & \frac{1}{|V|}\left[5r^{-1} - \frac{26(|V|t)}{5}r^{-2} + \frac{61(|V|t)^2}{40}r^{-3} + \frac{(|V|t)^3}{60}r^{-4} + \right. \\ & \left. + \frac{(|V|t)^4}{60}r^{-5} + \frac{(|V|t)^5}{30}r^{-6} - \frac{(|V|t)^6}{15}r^{-7}e^{-\frac{2|V|t}{r}} f\left(\frac{2|V|t}{r}\right)\right].\end{aligned}$$

3 The Berwald-Lagrange curvature relative to the vector field

The nonlinear connection from Corollary 2.2 is useful to construct the dual *adapted bases* of distinguished vector fields

$$\left\{\frac{\partial}{\partial t}\;;\;\frac{\delta}{\delta r}\;;\;\frac{\delta}{\delta\varphi}\;;\;\frac{\partial}{\partial\dot{r}}\;;\;\frac{\partial}{\partial\dot{\varphi}}\right\} \subset \mathcal{X}(E) \tag{3.1}$$

and distinguished covector fields

$$\{dt\;;\;dr\;;\;d\varphi\;;\;\delta\dot{r}\;;\;\delta\dot{\varphi}\} \subset \mathcal{X}^*(E), \tag{3.2}$$

where $E = \mathbb{R} \times T\mathbb{R}^2$, and we set

$$\frac{\delta}{\delta r} = \frac{\partial}{\partial r} - N_1^1\frac{\partial}{\partial\dot{r}} - \frac{\dot{\varphi}}{r}\frac{\partial}{\partial\dot{\varphi}}, \qquad \frac{\delta}{\delta\varphi} = \frac{\partial}{\partial\varphi} - \frac{me^{-\frac{2|V|t}{r}}}{2p|V|r^4}\dot{r}^3\dot{\varphi}\frac{\partial}{\partial\dot{r}} - \frac{\dot{r}}{r}\frac{\partial}{\partial\dot{\varphi}},$$
$$\delta\dot{r} = d\dot{r} + N_1^1 dr + \frac{me^{-\frac{2|V|t}{r}}}{2p|V|r^4}\dot{r}^3\dot{\varphi}d\varphi, \qquad \delta\dot{\varphi} = d\dot{\varphi} + \frac{\dot{\varphi}}{r}dr + \frac{\dot{r}}{r}d\varphi.$$

Remark 3.1. Under a change of coordinates (1.1), the elements of the adapted bases (3.1) and (3.2) transform as classical tensors.

According to the geometrical framework developed in [7], the Berwald N-linear connection produced by the 2D-monolayer Lagrangian (1.2) on the vector bundle $\mathbb{R} \times T\mathbb{R}^2$ is defined by the local components (for more details, see [7] and [8])

$$\mathtt{B}\Gamma\left(N\right) = \left(L^i_{jk} := B^i_{jk},\ C^i_{jk} = 0\right),$$

where $(i, j, k = \overline{1, 2})$,

$$B^i_{jk} = \frac{\partial N^i_j}{\partial y^k}. \tag{3.3}$$

Using the formulas (3.3), by direct partial derivation, we get

Proposition 3.1. *The Berwald N-linear connection of the 2D-monolayer Lagrangian (1.2) has the following approximate components:*

$$\begin{gathered} B^1_{11} = \left[\frac{2|V|t}{r^2} - \frac{5}{r} - 2\mathcal{U}(t,r)\dot{r} + \frac{3me^{-\frac{2|V|t}{r}}}{2p|V|r^4}\dot{r}\dot{\varphi}^2\right], \\ B^1_{12} = B^1_{21} = \frac{3me^{-\frac{2|V|t}{r}}}{2p|V|r^4}\dot{r}^2\dot{\varphi}, \qquad B^1_{22} = \frac{me^{-\frac{2|V|t}{r}}}{2p|V|r^4}\dot{r}^3, \\ B^2_{12} = B^2_{21} = \frac{1}{r}, \qquad B^2_{11} = B^2_{22} = 0. \end{gathered} \tag{3.4}$$

Corollary 3.2. *The Berwald N-linear connection $\mathtt{B}\Gamma(N)$ of the 2D-monolayer Lagrangian (1.2) has the following approximate adapted local (hv)-curvature d-tensors:*

$$B^2_{111} = B^1_{222} = B^2_{222} = B^2_{112} = B^2_{121} = B^2_{211} = B^2_{122} = B^2_{212} = B^2_{221} = 0,$$

$$\begin{aligned} B^1_{111} &= \frac{3me^{-\frac{2|V|t}{r}}}{2p\,|V|\,r^4}\dot{\varphi}^2 - 2\mathcal{U}(t,r), \\ B^1_{112} = B^1_{121} = B^1_{211} &= \frac{3me^{-\frac{2|V|t}{r}}}{p\,|V|\,r^4}\dot{r}\dot{\varphi}, \\ B^1_{122} = B^1_{212} = B^1_{221} &= \frac{3me^{-\frac{2|V|t}{r}}}{2p\,|V|\,r^4}\dot{r}^2. \end{aligned} \tag{3.5}$$

Proof. The Berwald N-linear connection $\mathtt{B}\Gamma(N)$ is characterized by the following *sixteen* (hv)-curvature d-tensors:

$$B^i_{jkl} = \frac{\partial B^i_{jk}}{\partial y^l} = \frac{\partial^2 N^i_j}{\partial y^k \partial y^l} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l},$$

where $(y^1, y^2) = (\dot{r}, \dot{\varphi})$. Consequently, using the connection components (3.4), we find the local Berwald (hv)-curvature d-tensors (3.5). □

Now, one can define a natural Lagrangian extension K of the flag curvature from Finsler spaces by means of (see [8, p. 54]):

$$K := K\left(t, r, \varphi, \dot{r}, \dot{\varphi};\ X\right) = K\left(t, x, y\ ;\ X\right) \overset{def}{=} \frac{B_{hijk}y^h X^i y^j X^k}{\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right)y^h X^i y^j X^k}, \tag{3.6}$$

which is called the Berwald–Lagrange curvature with respect to the vector field X:

$$X(t,r,\varphi) = X_r \frac{\partial}{\partial r} + X_\varphi \frac{\partial}{\partial \varphi} \neq 0,$$

where $B_{hijk} = g_{is} B^s_{hjk}$, $X^1 = X_r$ and $X^2 = X_\varphi$.

4 The Berwald–Lagrange curvature in monolayer compressibility

In this section we establish the relation between certain Finsler geometric invariants and matter parameters in the phase transition of first order.

Proposition 4.1. *The Berwald-Lagrange curvature $K(x, y; X)$ with respect to the monolayer field X does not depend on $X(r, \phi)$ and may be approximated by the expressions*
(4.1)

$$K(t,r,\varphi,\dot r,\dot\varphi;\ X) = \begin{cases} \infty, & \text{for } X = \dot r \frac{\partial}{\partial r} + \dot\varphi \frac{\partial}{\partial \varphi};\ \dot r,\ \dot\varphi \neq 0; \\ \left[\frac{18}{pr^6|V|e^{\frac{2|V|t}{r}}} - \frac{4\mathcal{U}(t,r)}{mr^2\dot\varphi^2}\right] \cdot \dot r^2, & \text{for } X = X_r \frac{\partial}{\partial r},\ X_r \neq 0; \\ 0, & \text{for } X = X_\varphi \frac{\partial}{\partial \varphi},\ X_\varphi \neq 0. \end{cases}$$

Proof. Using the formulas $B_{hijk} = g_{is} B^s_{hjk}$, where B^s_{hjk} are given by (3.5), direct computations yield

$$B_{1211} = B_{1212} = B_{1221} = B_{1222} = B_{2211} = B_{2212} = B_{2221} = B_{2122} = B_{2222} = 0,$$

$$B_{1111} = \frac{m - 2pr^5|V|e^{\frac{2|V|t}{r}} \cdot \dot r^{-3}}{2} \cdot \left[\frac{3me^{-\frac{2|V|t}{r}}}{2p\,|V|\,r^4}\dot\varphi^2 - 2\mathcal{U}(t,r)\right],$$

$$B_{1112} = B_{1121} = B_{2111} = \frac{m - 2pr^5|V|e^{\frac{2|V|t}{r}} \cdot \dot r^{-3}}{2} \cdot \frac{3me^{-\frac{2|V|t}{r}}}{p\,|V|\,r^4}\dot r\dot\varphi,$$

$$B_{1122} = B_{2112} = B_{2121} = \frac{m - 2pr^5|V|e^{\frac{2|V|t}{r}} \cdot \dot r^{-3}}{4} \cdot \frac{3me^{-\frac{2|V|t}{r}}}{p\,|V|\,r^4}\dot r^2.$$

Taking into account only the non-zero components of B_{hijk} and the fact that $g_{12} = g_{21} = 0$, we get

$$K = \frac{B_{1111}\dot r^2 {X_r}^2 + 2B_{1112}\dot r\dot\varphi {X_r}^2 + B_{1122}\dot\varphi^2 {X_r}^2 + B_{1112}\dot r^2 X_r X_\varphi + 2B_{1122}\dot\varphi\dot r X_r X_\varphi}{g_{11}g_{22}\left(X_\varphi \dot r - X_r\dot\varphi\right)^2}.$$

The substitution of $B_{1111},\ B_{1112},\ B_{1122},\ g_{11}$ and g_{22} in explicit form into the previous expression leads to

$$K = \frac{\dot r X_r}{(X_\varphi \dot r - X_r \dot\varphi)^2}\left[\left(\frac{18\dot\varphi^2}{pr^6|V|e^{\frac{2|V|t}{r}}} - \frac{4\mathcal{U}(t,r)}{mr^2}\right)\dot r X_r + \frac{12}{pr^6|V|e^{\frac{2|V|t}{r}}}\dot r^2\dot\varphi X_\varphi\right]. \tag{4.2}$$

The numerator in the left hand side of (4.2) is zero at $X = (\frac{\partial}{\partial r},\ \frac{\partial}{\partial \varphi})$, while subjected to the 1-form $\omega = (\dot r,\ \dot\varphi)$. Therefore $K = \infty$ for the field X.

The right hand side of the expression (4.2) equals to zero at $X_r = 0$, and therefore $K = 0$ for the field $X = (0, X_\phi)$.

At $X = (X_r, 0)$, the expression (4.2) reduces to

$$K_{flag_1} = \left(\frac{18}{pr^6|V|e^{\frac{2|V|t}{r}}} - \frac{4\mathcal{U}(t,r)}{mr^2\dot{\varphi}^2} \right) \dot{r}^2. \tag{4.3}$$

Since any tangent 2D-vector $(\dot{r},\ \dot{\varphi})$ can be decomposed into $(X_r, 0)$ and $(0, X_\phi)$:

$$X = (\dot{r},\ \dot{\varphi}) = (X_r, 0) + (0, X_\phi), \tag{4.4}$$

we find all the curvatures of slices over all the flags $\{y_0, X_0\}$. We note that the curvature $K(x, y_0; X_0) \equiv K(x;\ \{y_0,\ X_0\})$ is everywhere infinite, except at the slices of the tangent space given by the flags $flag_i,\ i = 1,\ 2$: $flag_1 = \{\dot{\varphi}, X_r\}$ and $flag_2 = \{\dot{r}, X_\phi\}$. Moreover, since any tangent 2D-vector can be split by $(X_r, 0)$ and $(0, X_\phi)$ via the formula (4.4), the expression (4.1) becomes the curvature for the slices for all the flags:

$$\begin{aligned} K\left(t, r, \varphi, \dot{r}, \dot{\varphi};\ X_{\vec{r}}\right) &= \lim_{\{y_0,\ X_0\}\to\{\vec{r}, X_{\vec{r}}\}} K(x;\ \{y_0,\ X_0\}) \\ &= \lim_{\{y_0,\ X_0\}\to\{\vec{r}, X_{\vec{r}}\}} \left\{ \tfrac{\dot{r}X_r}{(X_\varphi \dot{r} - X_r\dot{\varphi})^2} \Theta \right\} \Big|_{t\to t_0}, \end{aligned} \tag{4.5}$$

where

$$\Theta = \left(\frac{18\dot{\varphi}^2}{pr^6|V|e^{\frac{2|V|t}{r}}} - \frac{4\mathcal{U}(t, r(t))}{mr^2} \right) \dot{r}X_r + \frac{12}{pr^6|V|e^{\frac{2|V|t}{r}}} \dot{r}^2\dot{\varphi}X_\varphi.$$

Hence, the monolayer curvature $K(x, y_0; X_0)$ does not depend on the vector $X_0 \in \widetilde{TM}$ of the slit tangent space $\widetilde{TM} = TM \setminus \{0\}$, and it becomes of scalar type. $\square$

Now, since the Berwald–Lagrange curvature $K(x, y; X)$ with respect to the monolayer field is of scalar type, we may try to unify the expressions of all the curvatures of slices of all flags $\{y,\ X\}$ into one single expression. We note that properties similar to (4.5), are exhibited as well by the generalized Dirac function $\delta_D(\vec{r}(t) - \vec{r}(t_0))$, which also takes infinite values under the increase of $\vec{r}(t)$ at any arbitrary moment of time t, except at t_0. For this reason, the flag limit from (4.5) can be regarded by means of the Dirac δ-function:

$$\begin{aligned} K\left(t, r, \varphi, \dot{r}, \dot{\varphi};\ X\right) &= \lim_{\{y_0,\ X_0\}\to\{\vec{r},\ X_{\vec{r}}\}} K(x;\ \{y_0,\ X_0\}) \\ = \left\{ \dot{r}X_r \frac{1}{(X_\varphi\dot{r} - X_r\dot{\varphi})^2} \left[\left(\frac{18\dot{\varphi}^2}{pr^6|V|e^{\frac{2|V|t}{r}}} - \frac{4\mathcal{U}(t, r(t))}{mr^2} \right) \dot{r}X_r + \frac{12}{pr^6|V|e^{\frac{2|V|t}{r}}} \dot{r}^2\dot{\varphi}X_\varphi \right] \right\} \Big|_{t\to t_0} \\ \times\delta_D(\dot{r}(t) - \dot{r}(t_0)). \end{aligned} \tag{4.6}$$

This approximation holds true, while describing the beginning of the compression process:

$$\dot{r} \to 0,\ t \to t_0 = 0,\ r \gg |V|t. \tag{4.7}$$

We can plot the 4-dimensional hypersurfaces of *constant flag curvature* (which depend on the four variables $(t, r, \dot{r}, \dot{\varphi})$)

$$\Sigma : \left[\frac{18}{pr^6|V|e^{\frac{2|V|t}{r}}} - \frac{4\mathcal{U}(t,r)}{mr^2\dot{\varphi}^2} \right] \dot{r}^2 = K_{flag_1},$$

where $K_{flag_1} \in \{-1, 0, 1\}$, $m = 47 \times 10^{-26}$ kg, $p = 8.93 \times 10^9$, $|V| = 0.3 \times 10^{-3}$ m/s, $r = 0.1$ m, $t = 0.01$ s. In Fig. 1 we plot the hypersurfaces (curves) of constant

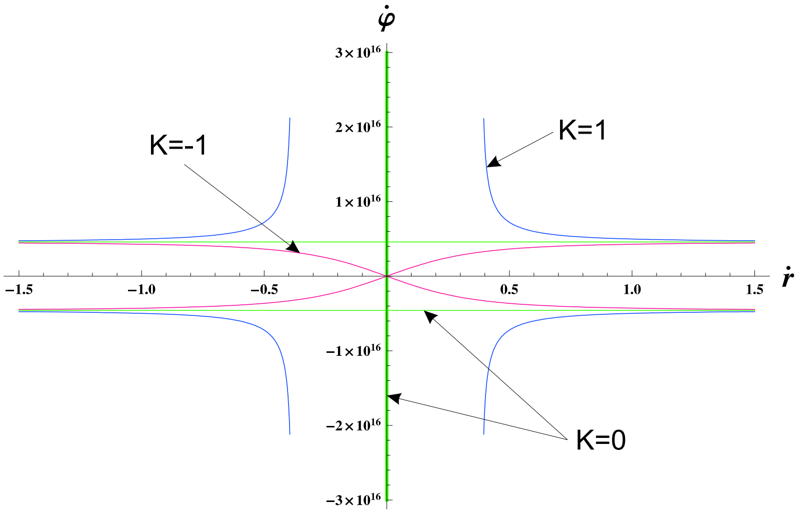

Figure 1: **Curves of constant flag curvature $K_{flag_1} = K \in \{0, \pm 1\}$.**

scalar curvature of Berwald-Lagrange type K_{flag_1} relative to the radial vector field for $K_{flag_1} = -1,\ 0,\ 1$. We see, that the space may be split into regions with positive and negative curvature, while the curvature K_{flag_1} is negative only over a finite interval of values taken by $\dot{\varphi}$, at whose extremities it changes its sign into the opposite one.

As it can be seen in Fig. 2, the magnitude of the curvature K_{flag_1} increases with the increase of the absolute value of the radial component of the velocity $\dot{r}$. Moreover, the expression of K_{flag_1} may diverge for $\dot{\varphi} \to 0$ and $\dot{r} \neq 0$.

We shall find the expression of K_{flag_1} at the examined limit (4.7):

$$\lim_{\dot{r} \to 0} K_{flag_1} = \frac{1}{|V|r^3} \left[\frac{18}{pr^3} - \frac{20}{m\dot{\varphi}^2} \right] \dot{r}^2. \tag{4.8}$$

From the expression (4.8) we see, that in the given physical situation, the curvature K_{flag_1}, remaining small: $K_{flag_1} \to 0$, may change its sign. It follows that we can assume that at the beginning of the structurization process of the monolayer one can observe a change of sign of the scalar Berwald – Lagrange curvature for decreasing

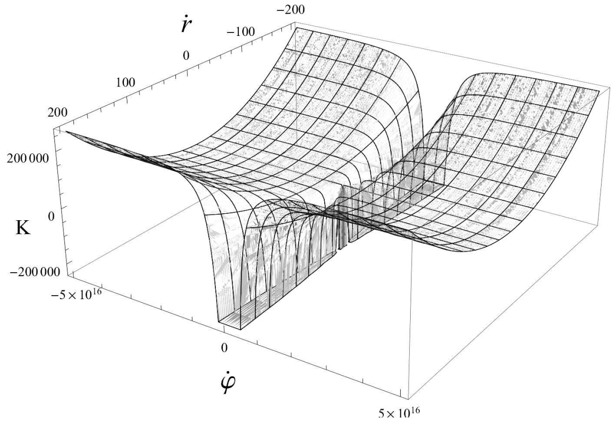

Figure 2: **The dependence of the flag curvature $K_{flag_1} =: K$ on the velocity.**

monolayer area (see Fig. 3) at the point whose coordinate r is:

$$r = \sqrt[3]{\frac{9m\dot{\varphi}^2}{10p}}.$$

The shape of the dependence of the scalar curvature K_{flag_1} on the time is represented

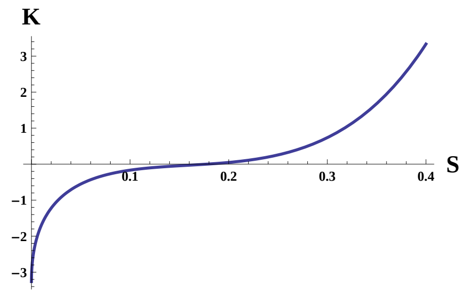

Figure 3: **The dependence of the flag curvature K_{flag_1}, denoted by K on the area of the monolayer S.**

in Fig. 4. From the previous considerations, we can conclude that the motion of particles along closed trajectories ($K_{flag_1} > 0$) changes into motion along diverging ones ($K_{flag_1} < 0$).

We assume, that there exists a relation between the scalar curvature of the monolayer and the thermodynamical parameters – in particular, the matter compressibility κ, which is always positive. The change of sign for κ shows that the system loses its stability and passes to another equilibrium state. Then we may conclude that in the neighborhood of the phase transition, the monolayer may lose its stability and pass to a metastable state.

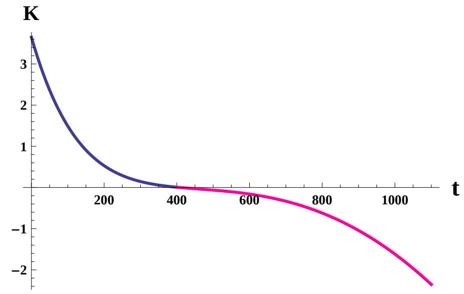

Figure 4: **The dependence of the flag curvature K_{flag_1}, denoted by K, on time.**

In [2] the total trace of Berwald curvature tensor

$$B_C(x,y,V) = g^{ij} B^k_{ikj}$$

has been studied for the monolayer space. It was observed that the integral of B_C at large values of V diverges and differs by sign for stable and metastable states respectively. Hence, it behaves like the compressibility κ, as first order derivative $\frac{\partial s}{\partial \tilde{\pi}}$ of the $s - \tilde{\pi}$–isotherm with respect to $\tilde{\pi}$ in a region of phase transition. This allows us to assume the following relation between B_C and $\tilde{\pi}$ in a neighborhood of phase transition [2]:

$$\kappa = -\frac{1}{s}\frac{\partial s}{\partial \tilde{\pi}} \propto -\frac{1}{s}\int B_C \, d^2 r. \tag{4.9}$$

We note that B_C is everywhere 0, except for the region of phase transition, which occurs at the moment t_0. Therefore the expression (4.9) can be decomposed into series in the neighborhood of phase transition:

$$\kappa \propto -\int \frac{\partial}{\partial \vec{r}} B_C \, d^2 r. \tag{4.10}$$

By using the Fourier transform, we examine the expression (4.10) in $\widetilde{TM}$:

$$\kappa \propto -\int B_C(X_{\vec{r}}) e^{iX_{\vec{r}}\cdot\vec{r}} X_r \, d\vec{r}\, dX_{\vec{r}} = -2\pi \int \dot{r} X_r B_C(X_r) e^{i\vec{X}_{\vec{r}}\cdot\vec{r}} \, dt \, dX_{\vec{r}}. \tag{4.11}$$

According to the definition of K (3.6) we may assume that the divergence of the flag curvature is conditioned by an anomaly in the behavior of the Fourier spectrum of $B_c(X_{\vec{r}})$ in the region of the phase transition. Therefore, the comparison of the expressions (4.6) and (4.11) allows us to propose the following expression for the compressibility κ:

$$\kappa \propto -2\pi \int \dot{r} X_r \frac{1}{(X_\varphi \dot{r} - X_r \dot{\varphi})^2} \left[\left(\frac{18\dot{\varphi}^2}{pr^6 |V| e^{\frac{2|V|t}{r}}} - \frac{4\mathcal{U}(t,r)}{mr^2} \right) \dot{r} X_r + \frac{12}{pr^6|V|e^{\frac{2|V|t}{r}}} \dot{r}^2 \dot{\varphi} X_\varphi \right] \times \delta_D(\dot{r}(t) - \dot{r}(t_0)) \, dt \, dX_{\vec{r}}. \tag{4.12}$$

According to the chosen approximation (4.7), we express the tangent vector $\vec{\dot{r}}$ by means of the acceleration a_u: $\dot{r} = a_u t$. Therefore, by using the properties of the function δ, we can re-express (4.12) as

$$\kappa \propto -\frac{2\pi}{a_u}\int \dot{r}X_r\frac{1}{(X_\varphi\dot{r}-X_r\dot{\varphi})^2}\left[\left(\frac{18\dot{\varphi}^2}{pr^6|V|e^{\frac{2|V|t}{r}}}-\frac{4\mathcal{U}(t,r)}{mr^2}\right)\dot{r}X_r+\frac{12}{pr^6|V|e^{\frac{2|V|t}{r}}}\dot{r}^2\dot{\varphi}X_\varphi\right] \times\delta_D(t-t_0)\,dt\,dX_{\vec{r}}. \tag{4.13}$$

By integration of (4.13) by t, we infer:

$$\kappa \propto -\frac{2\pi}{a_u} \times\int\left\{\dot{r}X_r\frac{1}{(X_\varphi\dot{r}-X_r\dot{\varphi})^2}\left[\left(\frac{18\dot{\varphi}^2}{pr^6|V|e^{\frac{2|V|t}{r}}}-\frac{4\mathcal{U}(t,r)}{mr^2}\right)\dot{r}X_r+\frac{12}{pr^6|V|e^{\frac{2|V|t}{r}}}\dot{r}^2\dot{\varphi}X_\varphi\right]\right\}\Bigg|_{t\to t_0} \times dX_{\vec{r}}. \tag{4.14}$$

According to (4.4), the integration over the tangent space of $d\vec{X}_{\vec{r}}$ may be replaced by the summation over the flags $flag_i$, $i=1,\ 2$:

$$\kappa \propto -\frac{2\pi}{a_u}\sum_{i=1}^{2}K(x;\ flag_i). \tag{4.15}$$

Since the curvature K_{flag_2} vanishes, we finally obtain the compressibility in the neighborhood of the phase transition:

$$\kappa \propto -\frac{2\pi}{a_u}K_{flag_1} = -\frac{2\pi}{a_u}\left[\left(\frac{18}{pr^6|V|e^{\frac{2|V|t}{r}}}-\frac{4\mathcal{U}(t,r)}{mr^2\dot{\varphi}^2}\right)\dot{r}^2\right]\Bigg|_{t\to t_0}. \tag{4.16}$$

In Fig. 5 there are represented the regions of values of $\varphi,\ t$, in which the matter is in stable and metastable states, with positive or negative compressibility κ, accordingly. Fig. 5. allow a qualitative characterization of the behavior of the trajectories during the process of phase transition of first kind. In region I in Fig. 5 the phase variation $\Delta\varphi$ for $t\to-\infty$ migrates towards high values. In this case the phase increase is always positive, since the tangent to the trajectory vector trigonometrically rotates. The positivity of κ infers the stability of the motion along such a trajectory.

In region III in Fig. 5, the phase variation $\Delta\varphi$ for $t\to+\infty$ migrates towards small values. This means that the phase increase is always negative, and the tangent to trajectory vector rotates clockwise. Due to the positivity of κ, the motion along such a trajectory is stable as well.

In region II of Fig. 5 the phase variation $\Delta\varphi$ changes sign due to a jump in its values: from -1×10^{16} for $t\to-\infty$ to $+1\times10^{16}$ for $t\to+\infty$. As it can be seen in Fig. 5, the trajectories within this region have an instanton-like character. The phase increase changes sign and, accordingly, the tangent to the trajectory vector changes its sense by a jump. Since κ is negative, the matter is in metastable state. Hence, in metastable state, the trajectories of the matter particles have an instanton-like character.

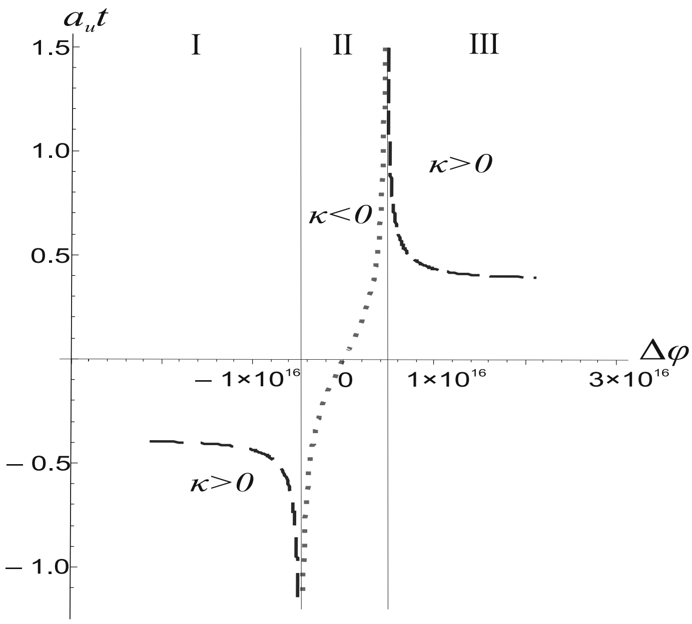

Figure 5: **The contour graph of the compressibility κ in the neighborhood of the phase transition of the first kind. The thin vertical lines separate the regions which have compressibility of opposite signs.**

5 Conclusion

The present work studies the Berwald-Lagrange curvature associated to a monolayer. It is shown that the metric of the monolayer space is of scalar curvature. It is shown that the sign of the Berwald–Lagrange curvature governs the growth of geodesic deviations within the monolayer, which in turn gives qualitative information about the sign of compressibility κ. In the region of phase transition of first kind, the trajectories of the matter particles have an instanton-like character, and the matter is in metastable state.

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References

[1] S. Acharya, J. P. Hill, K. Ariga, *Soft Langmuir–Blodgett technique for hard nanomaterials*, Adv. Mater. 21 (2009), 2959–2981.

[2] V. Balan, H. V. Grushevskaya, N. G. Krylova, A. Oana, *Finsler geometrization of classic theory for fields on the interphase boundary including monomolecular 2D-system*, arXiv:1211.6824v1 [math-ph] 29 Nov 2012.

[3] D. Bao, C. Robles, *Riemann–Finsler Geometry*, MSRI Publications 2004, 197-259.

[4] H. V. Grushevskaya, N. G. Krylova, *Application of Finsler geometry to instanton-like process of LB-nanostructures emerging in monolayer*, In: Proc. of VIII-th International Bolyai – Gauss – Lobachevsky Conference on Non-Euclidean Geometry in Modern Physics and Mathematics, 22 – 25 May 2012, Uzhgorod, Ukraine (Ed. A.M. Zavilopulo), Inst. of Electronic Physics, Uzhgorod 2012, 44.

[5] H. V. Grushevskaya, N. G. Krylova, *Effects of Finsler geometry in physics surface phenomena: case of monolayer systems*, HNGP, 1 (15), 8 (2011), 128–146.

[6] V. V. Hrushevsky, H. V. Krylova, *Thermodynamics of phase states in Langmuir Blodgett monolayers*, In: "*Low-dimensional systems-2*" (Eds. S.A. Maskevich et al.), Grodno State University, Grodno 4 (2005), 30–36.

[7] R. Miron, M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Academic Publishers, Dordrecht 1994.

[8] R. Miron, D. Hrimiuc, H. Shimada, S. V. Sabău, *The Geometry of Hamilton and Lagrange Spaces*, Kluwer Academic Publishers, Dordrecht 2001.

[9] H. Motschmann, H. Möhwald, *Langmuir–Blodgett Films*, Handbook of Applied Surface and Colloid Chemistry (Ed. Krister Holmberg), Ch. 28 (2001), 629–648.

[10] M. Neagu, N. G. Krylova, H. V. Grushevskaya, *Jet theoretical Yang-Mills energy in the geometric dynamics of two-dimensional monolayer*, J. Math. Phys. 54 (2013), 031508.

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