Application of the credibility theory based on important mathematical properties

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Abstract. In this article we introduce the original Bühlmann model, which involves only one isolated contract. We present the best linear credibility estimators for this model and we consider the as application of the optimal credibility estimator of Bühlmann, the recursive credibility model (our motivation for introducing this model was that we want that the new claims to have more weight than the older claims). The optimal estimator of Bühlmann has been criticized because it gives the same weight of the claim amounts for all previous years; intuitively one should believe that the new claims should have more weight than the old claims. However, as the claim amounts of different years were assumed to be exchangeable, it was only reasonable that the claim amounts should have equal weights. The recursive credibility estimation is an attempt to amend this intuitive weakness, and thus an application of the original credibility model of Bühlmann. The present results are obtained by using important mathematical properties of the probability theory, namely properties of conditional expectations and conditional (co-)variances.

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Introduction

In Section 1 we present Bühlmann's original model, which implies only one isolated contract. The original Bühlmann model gives the optimal linear credibility estimate for the risk premium of this case (see Subsection 1.1.). This estimator has been criticized because it gives the same weight of the claim amounts for all previous years; intuitively one should believe that the new claims should have more weight than the old claims. However, as the claim amounts of different years were assumed to be exchangeable, it was only reasonable that the claim amounts should have equal weights. The following model (which is called "Recursive credibility estimation" - see Section 2) is an attempt to amend this intuitive weakness, and thus an application of the original credibility model of Bühlmann.

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1 The original credibility model of Bühlmann

In this model, we consider one contract with unknown and fixed risk parameter θ, during a period of t years. The yearly claim amounts are noted by $X_1, \ldots, X_t$. The risk parameter θ is supposed to be taken from some structure distribution $U(\cdot)$. It is assumed that, for given $\theta = \theta$, the claims are conditionally independent and identically distributed with known common distribution function $F_{X|\theta}(x, \theta)$. For this model we want to estimate the net premium $\mu(\theta) = E[X_r|\theta = \theta]$, $r = \overline{1, t}$ as well as X_{t+1} for a contract with risk parameter θ.

1.1 Bühlmann's optimal credibility estimator

Suppose that $X_1, \ldots, X_t$ are random variables with finite variation, which are, for given $\theta = \theta$, conditionally independent and identically distributed with already known common distribution function $F_{X|\theta}(x, \theta)$. The structure distribution function is $U(\theta) = P[\theta \leq \theta]$. Let D, which represent the set of non-homogeneous linear combinations $g(\cdot)$ of the observable random variables $X_1, X_2, \ldots, X_t$:

$$g(\underline{X}') = c_0 + c_1 X_1 + c_2 X_2 + \ldots + c_t X_t.$$

Then the solution of the problem:

$$\underset{g \in D}{Min}\, E\{[\mu(\theta) - g(X_1, \ldots, X_t)]^2\} \tag{1.1}$$

is

$$g(X_1, \ldots, X_t) = z\overline{X} + (1 - z)\, m, \tag{1.2}$$

where $\underline{X}' = (X_1, \ldots, X_t)$ is the vector of observations, $z = at/(s^2 + at)$, is the resulting credibility factor, $\overline{X} = \frac{1}{t}\sum_{i=1}^{t} X_i$ is the individual estimator, and a, s^2 and m are the structural parameters defined as in (1.4):

$$\begin{aligned} m &= E\,[X_r] = E[\mu(\theta)], && r = \overline{1, t} \\ a &= Var\{E[X_r|\theta]\} = Var[\mu(\theta)], && r = \overline{1, t}, \\ \sigma^2\,(\theta) &= Var[X_r|\theta = \theta], && r = \overline{1, t}, \\ s^2 &= E\{Var[X_r|\theta]\} = E[\sigma^2(\theta)], && r = \overline{1, t}. \end{aligned} \tag{1.3}$$

If $\mu(\theta)$ is replaced by X_{t+1} in (1.1), exactly the same solution (1.2) is obtained, since the co-variances with $\underline{X}'$ are the same.

For proof see [1] of the references, pages 7 to 20).

We end Subsection 1.1 giving two important mathematical properties obtained for the optimal credibility result of this model (see Subsection 1.2) and demonstrated by probability theory and credibility theory.

1.2 Important mathematical properties for optimal credibility result

We consider credibility estimators under more general conditions. We assume that the real random variables $X_1, X_2, \ldots, X_t$ are known, and that we want to estimate the unknown real random variable X_{t+1} with $\widetilde{X}_{t+1}$, the credibility estimator of X_{t+1} based on $\underline{X}'$, that is, the best estimator of the form $a_0 + \sum_{r=1}^{t} a_r X_r$, according to quadratic loss (or the same, the linear estimator of X_{t+1}, based on $\underline{X}'$, with minimal mean squared error). We assume that $Var(X_r) < \infty$, for $r = \overline{1,t}$.

Property 1. A linear estimator $\dot{X}_{t+1} = a_0 + \sum_{r=1}^{t} a_r X_r$ of X_{t+1} is a credibility estimator if and only if it satisfies the normal equations:

$$E\left(\dot{X}_{t+1}\right) = E\left(X_{t+1}\right), \tag{1.4}$$

$$Cov\left(\dot{X}_{t+1}, X_r\right) = Cov(X_{t+1}, X_r),\ \forall r = \overline{1,t}. \tag{1.5}$$

For proof see [1] of the references, pages 47 to 55.

Property 2. The credibility estimator $\widetilde{X}_{t+1}$ of X_{t+1} based on $\underline{X}'$, satisfies:

$$Cov\left(X_{t+1}, \widetilde{X}_{t+1}\right) = Var\left(\widetilde{X}_{t+1}\right) = Var\left(X_{t+1}\right) - Var\left(X_{t+1} - \widetilde{X}_{t+1}\right).$$

For proof see [1] of the references, pages 56 to 58.

2 The recursive credibility estimation, an application of the original credibility model of Bühlmann

We will analyze the credibility estimator (1.2). This estimator has been criticized because it gives the same weight of the claim amounts for all previous years; intuitively one should believe that the new claims should have more weight than the old claims. However, as the claim amounts of different years were assumed to be exchangeable, it was only reasonable that the claim amounts should have equal weights. The following model (which is called "The recursive credibility estimation") is an attempt to amend this intuitive weakness, and thus an application of the original credibility model of Bühlmann.

We assume that $X_1, X_2, \ldots$ are conditionally independent given an unknown random sequence $\theta = \{\theta_i\}_{i=1}^{+\infty}$, and that for all i X_i depends on θ only through θ_i. This means that for each year i there is a separate risk parameter θ_i containing the risk characteristics of the policy in that year. The original credibility model of Bühlmann appears as a special case by assuming that $\theta_i = \theta_1$ for all i. We assume that: $E(X_i|\theta_i) = \mu(\theta_i)$ with the function μ independent of i. As well, we assume that:

$$Cov[\mu(\theta_i), \mu(\theta_j)] = \rho^{|i-j|}\lambda,$$

with $0 < \rho < 1$ and $\lambda > 0$ (λ bigger than zero), means that the correlation between the claim amounts from different years decreases when the time distance between years increases, which is intuitively appealing. Furthermore we suppose that: $\mu = E[\mu(\theta_i)]$, $\phi = E[Var(X_i|\theta_i)] > 0$, $\lambda = Var[\mu(\theta_i)]$ for all i. Our motivation for introducing the present model was that we wanted that the new claims to have more weight than the older claims. The following result (see Result 2) shows that this desire has been satisfied.

Let $\widetilde{X}_{t+1}$ be the credibility estimator of X_{t+1} or $\mu(\theta_{t+1})$ based on $\underline{X}'$. We introduce the estimation error:

$$\psi_{t+1} = E\left\{\left[\mu(\theta_{t+1}) - \widetilde{X}_{t+1}\right]^2\right\}$$

of $\widetilde{X}_{t+1}$ considered as estimator of $\mu(\theta_{t+1})$. Unfortunately, it seems to be impossible to find a "nice" expression for $\widetilde{X}_{t+1}$. However, the following result (see Result 1) gives a simple recursive procedure for the updating of $\widetilde{X}_{t+1}$, and as a by-product, we also get a recursive updating of the estimation error ψ_{t+1}.

Result 1. We have:

$$\widetilde{X}_{t+1} = \rho\left[\frac{\psi_t}{\psi_t + \varphi} X_t + \frac{\varphi}{\psi_t + \varphi}\widetilde{X}_t\right] + (1-\rho)\mu, \tag{2.1}$$

$$\psi_{t+1} = \rho^2 \frac{\psi_t \varphi}{\psi_t + \varphi} + \left(1 - \rho^2\right)\lambda, \tag{2.2}$$

for $t = 1, 2, 3, \ldots$ with $\widetilde{X}_1 = \mu$ and $\psi_1 = \lambda$.

Result 2. Suppose that the coefficients $\alpha_{t0}, \alpha_{t1}, \ldots, \alpha_{tt}$ are defined by

$$\widetilde{X}_{t+1} = \hat{\mu}(\theta_{t+1}) = \alpha_{t0} + \sum_{j=1}^{t} \alpha_{tj} X_j$$

and assume that $\rho < 1$. Then we obtain:

$$0 < \alpha_{t1} < \alpha_{t2} < \ldots < \alpha_{tt} < 1.$$

For the Result *1*, the proof is given below. Since $\widetilde{X}_1$ is *the credibility estimator* of X_1 we deduce that $\widetilde{X}_1$ is the linear estimator of X_1 with minimal mean squared error, that is $\widetilde{X}_1 = g$ (where $g^* \in$ R, scalar constant), which minimizes:

$$E\left[(X_1 - g)^2\right] \overset{not.}{=} Q(g).$$

We must have:

$$Q'(g) = 0. \tag{2.3}$$

But:

$$Q(g) = E\left(X_1^2 + g^2 - 2X_1 g\right) = E\left(X_1^2\right) + g^2 - 2gE(X_1) = E\left(X_1^2\right) + g^2 - 2g\mu, \tag{2.4}$$

because

$$\mu = E[\mu(\theta_i)] = E[E(X_i|\theta_i)] = E(X_i), \quad \text{for } i = 1, 2, 3, \ldots. \tag{2.5}$$

From (2.3) and (2.4), we have:

$$Q'(g) = 0 \overset{(2.7)}{\Leftrightarrow} 2g - 2\mu = 0 \Leftrightarrow g = \mu. \tag{2.6}$$

It is clear that:

$$Q''(g)|_{g=\mu} > 0, \text{ because } Q''(g)|_{g=\mu} = (2g - 2\mu)'_g\Big|_{g=\mu} = 2|_{g=\mu} = 2 > 0.$$

Therefore $\widetilde{X}_1 = \mu$. Furthermore:

$$\psi_1 \overset{def.}{=} E\left\{[\mu(\theta_1) - \widetilde{X}_1]^2\right\} \overset{(2.11)}{=} E\left\{[\mu(\theta_1) - \widetilde{X}_1]^2\right\} - [E(X_1) - E(\widetilde{X}_1)]^2, \tag{2.7}$$

because the credibility estimator $\widetilde{X}_1$ of X_1 satisfies the normal equation (1.4) of Subsection 1.2 and thus we have:

$$E\left(\widetilde{X}_1\right) = E(X_1). \tag{2.8}$$

Relations (2.5), (2.7) and (2.8) imply:

$$\begin{aligned}
\psi_1 &= E\left\{\left[\mu(\theta_1) - \widetilde{X}_1\right]^2\right\} - \left[\mu - E\left(\widetilde{X}_1\right)\right]^2 = \\
&= E\left\{\left[\mu(\theta_1) - \widetilde{X}_1\right]^2\right\} - \left\{E[\mu(\theta_1)] - E\left(\widetilde{X}_1\right)\right\}^2 = \\
&= E\left\{\left[\mu(\theta_1) - \widetilde{X}_1\right]^2\right\} - E^2\left[\mu(\theta_1) - \widetilde{X}_1\right] \overset{def.}{=} Var\left[\mu(\theta_1) - \widetilde{X}_1\right] = \\
&= Var[\mu(\theta_1) - \mu] = Var[\mu(\theta_1)] = Cov[\mu(\theta_1), \mu(\theta_1)] = \rho^{|1-1|}\lambda = \lambda.
\end{aligned} \tag{2.9}$$

Now, let $t \geq 1$, and let:

$$\widetilde{X}_{t+1} = a_0 + a_1 X_t + a_2 \widetilde{X}_t, \tag{2.10}$$

where a_0, a_1, a_2 are constants. The form (2.10) of $\widetilde{X}_{t+1}$ will be established, if we determine a_0, a_1, a_2, such that the normal equations (1.4) and (1.5) to be satisfied. Since the credibility estimator $\widetilde{X}_{t+1}$ of X_{t+1} or $\mu(\theta_{t+1})$ based on $\underline{X}'$, satisfies the normal equations (1.5) of Subsection 1.2, we have:

$$
\begin{aligned}
&Cov\left(\widetilde{X}_{t+1}, X_j\right) = Cov\left(X_{t+1}, X_j\right), \forall j=\overline{1,t} \Leftrightarrow \\
&\Leftrightarrow Cov\left(a_0 + a_1 X_t + a_2\, \widetilde{X}_t, X_j\right) = \\
&= Cov\left(X_{t+1}, X_j\right), \forall j=\overline{1,t} \Leftrightarrow Cov\left(a_0, X_j\right) + a_1 Cov\left(X_t, X_j\right) + \\
&+ a_2 Cov\left(\widetilde{X}_t\ ,\ , X_j\right) = Cov\left(X_{t+1}, X_j\right), \forall j=\overline{1,t} \Leftrightarrow \\
&\Leftrightarrow 0 + a_1 Cov\left(X_t, X_j\right) + a_2 Cov\left(\widetilde{X}_t, X_j\right) = \\
&= Cov\left(X_{t+1}, X_j\right), \forall j=\overline{1,t} \Leftrightarrow a_1 Cov\left(X_t, X_j\right) + a_2 Cov\left(\widetilde{X}_t, X_j\right) = \\
&= Cov\left(X_{t+1},\ , X_j\right), \quad \forall j=\overline{1,t}.
\end{aligned} \tag{2.11}
$$

For $j=\overline{1,t}$ take place the following relations:

$$
\begin{aligned}
&Cov\left(X_{t+1}, X_j\right) = E\left(X_{t+1} X_j\right) - E\left(X_{t+1}\right) E\left(X_j\right) = \\
&= E\left[E\left(X_{t+1} X_j \mid \theta_{t+1}, \theta_j\right)\right] - E\left[E\left(X_{t+1} \mid \theta_{t+1}\right)\right] E\left[E\left(X_j \mid \theta_j\right)\right] = \\
&= E\left[E\left(X_{t+1} \mid \theta_{t+1}\right) E\left(X_j \mid \theta_j\right)\right] - E\left[\mu\left(\theta_{t+1}\right)\right] \cdot E\left[\mu\left(\theta_j\right)\right] = \\
&= E\left[\mu\left(\theta_{t+1}\right) \mu\left(\theta_j\right)\right] - E\left[\mu\left(\theta_{t+1}\right)\right] E\left[\mu\left(\theta_j\right)\right] = \\
&= Cov\left[\mu\left(\theta_{t+1}\right), \mu\left(\theta_j\right)\right] \overset{(2.1)}{=} \rho^{|t+1-j|}\lambda = \rho^{t+1-j}\lambda.
\end{aligned}
$$

From the above we notice that:

$$
Cov\left(X_{t+1}, X_j\right) = \rho^{t+1-j}\lambda, \quad \forall j=\overline{1,t}. \tag{2.12}
$$

Relations (2.11) and (2.12) imply:

$$
a_1 Cov\left(X_t, X_j\right) + a_2 Cov\left(\widetilde{X}_t, X_j\right) = \rho^{t+1-j}\lambda, \quad \forall j=\overline{1,t}. \tag{2.13}
$$

Since the credibility estimator $\widetilde{X}_t$ of X_t based on $X_1, X_2, \ldots, X_{t-1}$, satisfies the normal equations (1.5) of Subsection 1.2, we have:

$$
Cov\left(\widetilde{X}_t, X_j\right) = Cov\left(X_t, X_j\right), \forall j=\overline{1,t-1}, \tag{2.14}
$$

and thus, introducing (2.14) in (2.13), we obtain:

$$
a_1 Cov\left(X_t, X_j\right) + a_2 Cov\left(X_t, X_j\right) = \rho^{t+1-j}\lambda, \forall j=\overline{1,t-1}, \tag{2.15}
$$

that is

$$
\begin{aligned}
&\left(a_1 + a_2\right) Cov\left(X_t, X_j\right) = \rho^{t+1-j}\lambda,\ \forall j=\overline{1,t-1} \Leftrightarrow \left(a_1 + a_2\right)\rho^{t-j}\lambda = \\
&= \rho^{t+1-j}\lambda, \quad \forall j=\overline{1,t-1} \Leftrightarrow a_1 + a_2 = \rho \Leftrightarrow a_1 = \rho - a_2,
\end{aligned} \tag{2.16}
$$

where, for $j = \overline{1, t-1}$, we considered that:

$$Cov(X_t, X_j) = E(X_t X_j) - E(X_t)E(X_j) = E[E(X_t X_j | \theta_t, \theta_j)] -$$
$$-E[E(X_t|\theta_t)] \cdot E[E(X_j|\theta_j)] = E[E(X_t|\theta_t)E(X_j|\theta_j)] - E[\mu(\theta_t)]E[\mu(\theta_j)] =$$
$$= E[\mu(\theta_t)\mu(\theta_j)] - E[\mu(\theta_t)]E[\mu(\theta_j)] = Cov[\mu(\theta_t),\ \mu(\theta_j)] = \rho^{|t-j|}\lambda = \rho^{t-j}\lambda.$$

Inserting (2.16) and $j = t$ in (2.13), we obtain:

$$\begin{aligned}(\rho - a_2)Cov(X_t, X_t) + a_2 Cov\left(\widetilde{X}_t, X_t\right) = \rho^{t+1-t}\lambda \Leftrightarrow \\ \Leftrightarrow (\rho - a_2)Var(X_t) + a_2 \cdot Cov\left(\widetilde{X}_t, X_t\right) = \rho\lambda \Leftrightarrow \\ \Leftrightarrow (\rho - a_2)(\lambda + \varphi) + a_2 Cov\left(\widetilde{X}_t, X_t\right) = \rho\lambda,\end{aligned} \tag{2.17}$$

where we considered that:

$$\begin{aligned}Var(X_t) = Var[E(X_t|\theta_t)] + E[Var(X_t|\theta_t)] = Var[\mu(\theta_t)] + \varphi = \\ = Cov[\mu(\theta_t), \mu(\theta_t)] + \varphi = \rho^{|t-t|}\lambda + \varphi = \lambda + \varphi,\end{aligned} \tag{2.18}$$

$$\begin{aligned}
&\psi_t \overset{def.}{=} E\left\{\left[\mu(\theta_t) - \widetilde{X}_t\right]^2\right\} \overset{(2.23)}{=} E\left\{\left[\mu(\theta_t) - \widetilde{X}_t\right]^2\right\} - \\
&- \left[E(X_t) - E\left(\widetilde{X}_t\right)\right]^2 \overset{(2.8)}{=} E\left\{\left[\mu(\theta_t) - \widetilde{X}_t\right]^2\right\} - \\
&- \left[\mu - E\left(\widetilde{X}_t\right)\right]^2 = E\left\{\left[\mu(\theta_t) - \widetilde{X}_t\right]^2\right\} - \\
&- \left\{E[\mu(\theta_t)] - E\left(\widetilde{X}_t\right)\right\}^2 = E\left\{\left[\mu(\theta_t) - \widetilde{X}_t\right]^2\right\} - E^2\left[\mu(\theta_t) - \widetilde{X}_t\right] = \\
&= E\left[\mu^2(\theta_t) + \widetilde{X}_t^2 - 2\mu(\theta_t)\widetilde{X}_t\right] - \left\{E[\mu(\theta_t)] - E\left(\widetilde{X}_t\right)\right\}^2 = \\
&= E\left[\mu^2(\theta_t)\right] + E\left(\widetilde{X}_t^2\right) - 2E\left[\mu(\theta_t)\widetilde{X}_t\right] - \\
&- \left\{E^2[\mu(\theta_t)] + E^2\left(\widetilde{X}_t\right) - 2E[\mu(\theta_t)] \cdot E\left(\widetilde{X}_t\right)\right\} = \\
&= \left\{E\left[\mu^2(\theta_t)\right] - E^2[\mu(\theta_t)]\right\} + \left[E\left(\widetilde{X}_t^2\right) - E^2\left(\widetilde{X}_t\right)\right] - \\
&-2\left\{E\left[\mu(\theta_t)\cdot \widetilde{X}_t\right] - E[\mu(\theta_t)]E\left(\widetilde{X}_t\right)\right\} = \\
&= Var[\mu(\theta_t)] + Var\left(\widetilde{X}_t\right) - 2Cov\left[\mu(\theta_t), \widetilde{X}_t\right] \overset{(2.24)}{=} \\
&\overset{(2.24)}{=} Cov[\mu(\theta_t), \mu(\theta_t)] + Cov\left(\widetilde{X}_t, X_t\right) - 2Cov\left[\widetilde{X}_t, \mu(\theta_t)\right] = \\
&= \rho^{|t-t|}\lambda + Cov\left(\widetilde{X}_t\ , X_t\right) - 2Cov\left[\widetilde{X}_t, \mu(\theta_t)\right] = \\
&= \lambda + Cov\left(\widetilde{X}_t, X_t\right) - 2Cov\left[\widetilde{X}_t, \mu(\theta_t)\right],
\end{aligned} \tag{2.19}$$

with:

$$E\left(\widetilde{X}_t\right)=E\left(X_t\right), \tag{2.20}$$

because the credibility estimator $\widetilde{X}_t$ of X_t based on $X_1, X_2, \ldots, X_{t-1}$, satisfies the normal equation (1.4) of Subsection 1.2;

$$Var\left(\widetilde{X}_t\right)=Cov\left(X_t,\widetilde{X}_t\right)=Cov\left(\widetilde{X}_t,X_t\right), \tag{2.21}$$

according to the property *2* of Subsection *1.2*. Also we may write:

$$\begin{aligned}
&Cov\left(\widetilde{X}_t,X_t\right)=E\left(\widetilde{X}_t\,X_t\right)-E\left(\widetilde{X}_t\right)E\left(X_t\right)=\\
&=E\left[E\left(\widetilde{X}_t\,X_t\middle|\,\theta_t\right)\right]-E\left(\widetilde{X}_t\right)\cdot E\left[E\left(X_t|\,\theta_t\right)\right]=\\
&=E\left[\widetilde{X}_t\,E\left(X_t|\,\theta_t\right)\right]-E\left(\widetilde{X}_t\right)E\left[\mu\left(\theta_t\right)\right]=\\
&=E\left[\widetilde{X}_t\,\mu\left(\theta_t\right)\right]-E\left(\widetilde{X}_t\right)\cdot E\left[\mu\left(\theta_t\right)\right]=\\
&=Cov\left[\widetilde{X}_t,\mu\left(\theta_t\right)\right]\Leftrightarrow Cov\left[\widetilde{X}_t,\mu\left(\theta_t\right)\right]=Cov\left(\widetilde{X}_t,X_t\right).
\end{aligned} \tag{2.22}$$

Relations (2.19) and (2.22) lead to:

$$\psi_t=\lambda+Cov\left(\widetilde{X}_t,X_t\right)-2Cov\left(\widetilde{X}_t,X_t\right)=\lambda-Cov\left(\widetilde{X}_t,X_t\right), \tag{2.23}$$

and thus we have:

$$Cov\left(\widetilde{X}_t,X_t\right)=\lambda-\psi_t. \tag{2.24}$$

Relation (2.17) becomes:

$$\left(\rho-a_2\right)\left(\lambda+\varphi\right)+a_2\left(\lambda-\psi_t\right)=\rho\lambda, \tag{2.25}$$

taking in consideration (2.24). Making the calculations in (2.25), we obtain:

$$\rho\lambda+\rho\varphi-a_2\lambda-a_2\varphi+a_2\lambda-a_2\psi_t=\rho\lambda\Leftrightarrow a_2\left(\psi_t+\varphi\right)=\rho\varphi\Leftrightarrow a_2=\frac{\rho\varphi}{\psi_t+\varphi} \tag{2.26}$$

and so:

$$a_1\underset{(2.29)}{\overset{(2.19)}{=}}\rho-\frac{\rho\varphi}{\psi_t+\varphi}=\frac{\rho\psi_t}{\psi_t+\varphi}. \tag{2.27}$$

Since the credibility estimator $\widetilde{X}_{t+1}$ of X_{t+1} based on $X_1, X_2, \ldots, X_t$, satisfies the normal equations (1.4) of Subsection 1.2, we have:

$$\begin{aligned}
&E\left(\widetilde{X}_{t+1}\right)=E\left(X_{t+1}\right)\underset{(2.13)}{\overset{(2.8)}{\Leftrightarrow}}E\left(a_0+a_1X_t+a_2\,\widetilde{X}_t\right)=\mu\\
&\Leftrightarrow a_0+a_1E\left(X_t\right)+a_2\cdot E\left(\widetilde{X}_t\right)=\mu\overset{(2.8)}{\Leftrightarrow}a_0+a_1\mu+a_2E\left(\widetilde{X}_t\right)=\\
&=\mu\underset{(2.8)}{\overset{(2.23)}{\Leftrightarrow}}a_0+a_1\mu+a_2\mu=\mu\Leftrightarrow a_0+\left(a_1+a_2\right)\mu=\mu\underset{(2.32)}{\Leftrightarrow}\\
&\underset{(2.32)}{\Leftrightarrow}a_0+\rho\mu=\mu\Leftrightarrow a_0=\left(1-\rho\right)\mu,
\end{aligned} \tag{2.28}$$

where we considered that (2.16) can be written in the following form:

(2.29)
$$a_1 + a_2 = \rho$$

denoted by. So (2.2) is proved (see (2.10), (2.28), (2.27) and (2.26)). It remains to show that ψ_t satisfies the recursion (2.2). We have:

(2.30)
$$\psi_{t+1} \overset{def.}{=} E\left\{\left[\mu\left(\theta_{t+1}\right) - \widetilde{X}_{t+1}\right]^2\right\} \overset{(2.34)}{=}$$
$$= E\left\{\left[\mu\left(\theta_{t+1}\right) - \widetilde{X}_{t+1}\right]^2\right\} - \left[E\left(X_{t+1}\right) - E\left(\widetilde{X}_{t+1}\right)\right]^2 \overset{(2.8)}{=}$$
$$= E\left\{\left[\mu\left(\theta_{t+1}\right) - \widetilde{X}_{t+1}\right]^2\right\} - \left[\mu - E\left(\widetilde{X}_{t+1}\right)\right]^2 =$$
$$= E\left\{\left[\mu\left(\theta_{t+1}\right) - \widetilde{X}_{t+1}\right]^2\right\} - \left\{E\left[\mu\left(\theta_{t+1}\right)\right] - E\left(\widetilde{X}_{t+1}\right)\right\}^2 =$$
$$= E\left\{\left[\mu\left(\theta_{t+1}\right) - \widetilde{X}_{t+1}\right]^2\right\} - E^2\left[\mu\left(\theta_{t+1}\right) - \widetilde{X}_{t+1}\right] \overset{(2.13)}{=}$$
$$\overset{(2.13)}{=} E\left\{\left[\mu\left(\theta_{t+1}\right) - a_0 - a_1 X_t - a_2 \widetilde{X}_t\right]^2\right\} -$$
$$-E^2[\mu\left(\theta_{t+1}\right) - a_0 - a_1 X_t - a_2 \widetilde{X}_t] = E[\mu^2(\theta_{t+1}) + a_0^2 + a_1^2 X_t^2 +$$
$$+a_2^2 \widetilde{X}_t^2 - 2a_0\mu\left(\theta_{t+1}\right) - 2a_1\mu\left(\theta_{t+1}\right) X_t - 2a_2\mu\left(\theta_{t+1}\right) \widetilde{X}_t + 2a_0 a_1 X_t + 2a_0 a_2 \widetilde{X}_t +$$
$$+2a_1 a_2 X_t \widetilde{X}_t] - \{E^2[\mu(\theta_{t+1})] + a_0^2 + a_1^2 E^2(X_t) + a_2^2 E^2(\widetilde{X}_t) - 2a_0\cdot$$
$$\cdot E[\mu(\theta_{t+1})] - 2a_1 E[\mu(\theta_{t+1})]E(X_t) - 2a_2 E[\mu(\theta_{t+1})]E(\widetilde{X}_t) + 2a_0 a_1 E(X_t) +$$
$$+2a_0 a_2 E(\widetilde{X}_t) + 2a_1 a_2 E(X_t)E(\widetilde{X}_t)\} = E[\mu^2(\theta_{t+1})] + a_0^2 + a_1^2 E(X_t^2) +$$
$$+a_2^2 E(\widetilde{X}_t^2) - 2a_0 E[\mu(\theta_{t+1})] - 2a_1 E[\mu(\theta_{t+1}) X_t] - 2a_2 E[\mu(\theta_{t+1}) \widetilde{X}_t] + 2a_0 a_1\cdot$$
$$\cdot E(X_t) + 2a_0 a_2 E(\widetilde{X}_t) + 2a_1 a_2 E(X_t \widetilde{X}_t) - E^2[\mu(\theta_{t+1})] - a_0^2 - a_1^2 E^2(X_t) -$$
$$-a_2^2 \cdot E^2(\widetilde{X}_t) + 2a_0 E[\mu(\theta_{t+1})] + 2a_1 E[\mu(\theta_{t+1})]E(X_t) + 2a_2 E[\mu(\theta_{t+1})]E(\widetilde{X}_t) -$$
$$-2a_0 a_1 E(X_t) - 2a_0 a_2 E(\widetilde{X}_t) - 2a_1 a_2 E(X_t)E(\widetilde{X}_t) \underset{(2.23)}{\overset{(2.8)}{=}} \left\{E\left[\mu^2\left(\theta_{t+1}\right)\right] -\right.$$
$$\left.-E^2\left[\mu\left(\theta_{t+1}\right)\right]\right\} + (a_0^2 - a_0^2) + a_1^2[E(X_t{}^2) - E^2(X_t)] + a_2^2[E(\widetilde{X}_t^2) - E^2(\widetilde{X}_t)] -$$
$$-2a_0\mu + 2a_0\mu - 2a_1\{E[\mu(\theta_{t+1})X_t] - E[\mu(\theta_{t+1})]E(X_t)\} - 2a_2\{E[\mu(\theta_{t+1})\cdot \widetilde{X}_t]\} -$$
$$-E[\mu(\theta_{t+1})]E(\widetilde{X}_t) + 2a_0 a_1\mu - 2a_0 a_1\mu + 2a_0 a_2\mu - 2a_0 a_2\mu +$$
$$+2a_1 a_2[E(X_t \widetilde{X}_t) - E(X_t)E(\widetilde{X}_t)] = Var[\mu(\theta_{t+1})] + a_1^2 Var(X_t) + a_2^2\cdot$$
$$\cdot Var(\widetilde{X}_t) - 2a_1 Cov[\mu(\theta_{t+1}), X_t] - 2a_2 Cov[\mu(\theta_{t+1}), \widetilde{X}_t] + 2a_1 a_2 Cov(X_t, \widetilde{X}_t),$$

with:

$$E(\widetilde{X}_{t+1}) = E(X_{t+1}), \tag{2.31}$$

because the credibility estimator $\widetilde{X}_{t+1}$ of X_{t+1} based on $X_1, X_2, \ldots, X_t$, satisfies the normal equation (1.4) of Subsection 1.2. But

$$Var\left[\mu\left(\theta_{t+1}\right)\right] = Cov\left[\mu\left(\theta_{t+1}\right), \mu\left(\theta_{t+1}\right)\right] \overset{(2.1)}{=} \rho^{|t+1-t-1|}\lambda = \lambda, \tag{2.32}$$

$$Var(\widetilde{X}_t) \overset{(2.24)}{=} Cov(\widetilde{X}_t, X_t) \overset{(2.27)}{=} \lambda - \psi_t, \tag{2.33}$$

$$\begin{aligned} &Cov\left[\mu\left(\theta_{t+1}\right), X_t\right] = E\left[\mu\left(\theta_{t+1}\right) X_t\right] - E\left[\mu\left(\theta_{t+1}\right)\right] E\left(X_t\right) = \\ &= E\left\{E\left[\mu\left(\theta_{t+1}\right) X_t\right] \middle| \theta_t\right\} - E\left[\mu\left(\theta_{t+1}\right)\right] E\left[E\left(X_t \middle| \theta_t\right)\right] = \\ &= E\left[\mu\left(\theta_{t+1}\right) E\left(X_t \middle| \theta_t\right)\right] - E\left[\mu\left(\theta_{t+1}\right)\right] E\left[\mu\left(\theta_t\right)\right] = \\ &= E\left[\mu\left(\theta_{t+1}\right) \mu\left(\theta_t\right)\right] - E\left[\mu\left(\theta_{t+1}\right)\right] E\left[\mu\left(\theta_t\right)\right] = \\ &= Cov\left[\mu\left(\theta_{t+1}\right), \mu\left(\theta_t\right)\right] \overset{(2.1)}{=} \rho^{|t+1-t|}\lambda = \rho\lambda, \end{aligned} \tag{2.34}$$

$$\begin{aligned} &Cov[\mu\left(\theta_{t+1}\right), \widetilde{X}_t] \overset{(2.39)}{=} Cov\left[\mu\left(\theta_{t+1}\right), a_0 + \sum_{j=1}^{t-1} a_j X_j\right] = \\ &= Cov\left[\mu\left(\theta_{t+1}\right), a_0\right] + \sum_{j=1}^{t-1} a_j \cdot Cov\left[\mu\left(\theta_{t+1}\right), X_j\right] \overset{(2.40)}{=} \\ &\overset{(2.40)}{=} 0 + \sum_{j=1}^{t-1} a_j \rho Cov\left[\mu\left(\theta_t\right), X_j\right] = \rho \sum_{j=1}^{t-1} a_j Cov\left[\mu\left(\theta_t\right), X_j\right] = \\ &= \rho\left\{Cov\left[\mu\left(\theta_t\right), a_0\right] + Cov\left[\mu\left(\theta_t\right), \sum_{j=1}^{t-1} a_j X_j\right]\right\} = \\ &= \rho Cov\left[\mu\left(\theta_t\right), a_0 + \sum_{j=1}^{t-1} a_j X_j\right] \overset{(2.39)}{=} \rho Cov\left[\mu\left(\theta_t\right), \widetilde{X}_t\right] = \\ &= \rho Cov[\widetilde{X}_t, \mu\left(\theta_t\right)] \underset{(2.27)}{\overset{(2.25)}{=}} \rho\left(\lambda - \psi_t\right), \end{aligned} \tag{2.35}$$

where

$$\widetilde{X}_t = a_0 + \sum_{j=1}^{t-1} a_j X_j, \tag{2.36}$$

because the credibility estimator $\widetilde{X}_t$ of X_t based on $X_1, X_2, \ldots, X_{t-1}$, is a non-homogeneous linear combination of the observable random variables $X_1, X_2, \ldots, X_{t-1}$;

for $j = \overline{1, t-1}$ we have

$$\begin{aligned}
&Cov\left[\mu\left(\theta_{t+1}\right), X_j\right] = E\left[\mu\left(\theta_{t+1}\right) X_j\right] - E\left[\mu\left(\theta_{t+1}\right)\right] E\left(X_j\right) = \\
&= E\left\{E\left[\mu\left(\theta_{t+1}\right) X_j \middle| \theta_j\right]\right\} - E\left[\mu\left(\theta_{t+1}\right)\right] E\left[E\left(X_j \middle| \theta_j\right)\right] = \\
&= E\left[\mu\left(\theta_{t+1}\right) E\left(X_j \middle| \theta_j\right)\right] - E\left[\mu\left(\theta_{t+1}\right)\right] E\left[\mu\left(\theta_j\right)\right] = \\
&= E\left[\mu\left(\theta_{t+1}\right) \mu\left(\theta_j\right)\right] - E\left[\mu\left(\theta_{t+1}\right)\right] E\left[\mu\left(\theta_j\right)\right] = \\
&= Cov\left[\mu\left(\theta_{t+1}\right), \mu\left(\theta_j\right)\right] \overset{(2.1)}{=} \rho^{|t+1-j|}\lambda = \rho^{t+1-j}\lambda = \rho\left[\rho^{t-j}\lambda\right] \overset{(2.1)}{=} \\
&\overset{(2.1)}{=} \rho Cov\left[\mu\left(\theta_t\right), \mu\left(\theta_j\right)\right] \overset{(2.41)}{=} \rho Cov\left[\mu\left(\theta_t\right), X_j\right],
\end{aligned} \tag{2.37}$$

where we considered that

$$\begin{aligned}
&Cov\left[\mu\left(\theta_t\right), \mu\left(\theta_j\right)\right] = E\left[\mu\left(\theta_t\right) \mu\left(\theta_j\right)\right] - E\left[\mu\left(\theta_t\right)\right] M\left[\mu\left(\theta_j\right)\right] = \\
&= E\left\{E\left[\mu\left(\theta_t\right) X_j \middle| \theta_j\right]\right\} - E\left[\mu\left(\theta_t\right)\right] E\left[E\left(X_j \middle| \theta_j\right)\right] = \\
&= E\left\{E\left[\mu\left(\theta_t\right) X_j \middle| \theta_j\right]\right\} - E\left[\mu\left(\theta_t\right)\right] E\left(X_j\right) = E\left[\mu\left(\theta_t\right) X_j\right] - \\
&- E\left[\mu\left(\theta_t\right)\right] E\left(X_j\right) = Cov\left[\mu\left(\theta_t\right), X_j\right], \quad j = \overline{1, t-1}.
\end{aligned} \tag{2.38}$$

From (2.30), (2.32), (2.18), (2.33), (2.34), (2.35) and (2.24) we obtain:

$$\begin{aligned}
&\psi_{t+1} = \lambda + a_1^2\left(\lambda + \varphi\right) + a_2^2\left(\lambda - \psi_t\right) - 2a_1\rho\lambda - 2a_2\rho\left(\lambda - \psi_t\right) + 2a_1a_2(\lambda - \psi_t) \overset{(2.19)}{=} \\
&\overset{(2.19)}{=} \lambda + a_1^2\left(\lambda + \varphi\right) + a_2^2\left(\lambda - \psi_t\right) - 2a_1\rho\lambda - 2a_2\rho\left(\lambda - \psi_t\right) + 2\left(\rho - a_2\right) a_2 \cdot \left(\lambda - \psi_t\right) = \\
&= \lambda + a_1^2\lambda + a_1^2\varphi + a_2^2\left(\lambda - \psi_t\right) - 2a_1\rho\lambda - 2a_2\rho\left(\lambda - \psi_t\right) + 2a_2\rho \cdot \left(\lambda - \psi_t\right) - \\
&- 2a_2^2\left(\lambda - \psi_t\right) = \lambda + a_1^2\lambda + a_1^2\varphi - a_2^2\left(\lambda - \psi_t\right) - 2a_1\rho\lambda = \lambda + a_1^2\lambda + \\
&+ a_1^2\varphi - a_2^2\lambda + a_2^2\psi_t - 2a_1\rho\lambda = \lambda\left(1 + a_1^2 - a_2^2 - 2a_1\rho\right) + a_1^2\varphi + a_2^2\psi_t \overset{(2.32)}{=} \\
&\overset{(2.32)}{=} \lambda[1 + a_1^2 - a_2^2 - 2a_1\left(a_1 + a_2\right)] + a_1^2\varphi + a_2^2\psi_t = \lambda(1 + a_1^2 - a_2^2 - 2a_1^2 - 2a_1 \cdot a_2) + \\
&+ a_1^2\varphi + a_2^2\psi_t = \lambda(1 - a_1^2 - a_2^2 - 2a_1a_2) + a_1^2\varphi + a_2^2\psi_t = \lambda[1 - (a_1^2 + a_2^2 + 2a_1a_2)] + \\
&+ a_1^2\varphi + a_2^2\psi_t = \lambda[1 - (a_1 + a_2)^2] + a_1^2\varphi + a_2^2\psi_t \overset{(2.32)}{=} \lambda(1 - \rho^2) + a_1^2 \cdot \varphi + a_2^2\psi_t \underset{(2.30)}{\overset{(2.29)}{=}} \\
&\underset{(2.30)}{\overset{(2.29)}{=}} (1 - \rho^2)\lambda + \frac{\rho^2\psi_t^2}{(\psi_t + \varphi)^2}\varphi + \frac{\rho^2\varphi^2}{(\psi_t + \varphi)^2}\psi_t = \\
&= (1 - \rho^2)\lambda + \frac{\rho^2\psi_t^2\varphi + \rho^2\varphi^2\psi_t}{(\psi_t + \varphi)^2} = (1 - \rho^2)\lambda + \rho^2\varphi\psi_t\frac{\psi_t + \varphi}{(\psi_t + \varphi)^2} = \\
&= (1 - \rho^2)\lambda + \rho^2\frac{\varphi\psi_t}{\psi_t + \varphi} = (1 - \rho^2)\lambda + \rho^2\frac{\varphi\psi_t}{\psi_t + \varphi} = \rho^2\frac{\psi_t\varphi}{\psi_t + \varphi} + (1 - \rho^2)\lambda,
\end{aligned}$$

and thus (2.2) is proved. This completes the proof of Result 1. We end this paper, giving a conclusion on the expression of $\widetilde{X}_{t+1}$ deduced from the recursion (2.1). For Result 2, the proof is given below. The existence of the coefficients α_{tj}, $j = \overline{0, t}$ with the property of Result 2 is obtained by applying of "t times" of the recurrence relation (2.1) as follows:

$$
\begin{aligned}
(2.39)\quad & \widetilde{X}_{t+1} = \frac{\rho\psi_t}{\psi_t+\varphi}X_t + \frac{\rho\varphi}{\psi_t+\varphi}\widetilde{X}_t + (1-\rho)\mu = \frac{\rho\psi_t}{\psi_t+\varphi}X_t + \\
& + \frac{\rho\varphi}{\psi_t+\varphi}\left[\frac{\rho\psi_{t-1}}{\psi_{t-1}+\varphi}X_{t-1} + \frac{\rho\varphi}{\psi_{t-1}+\varphi}\widetilde{X}_{t-1} + (1-\rho)\mu\right] + (1-\rho)\mu = \\
& = \frac{\rho\psi_t}{\psi_t+\varphi}X_t + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\psi_{t-1}}{\psi_{t-1}+\varphi}X_{t-1} + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\varphi}{\psi_{t-1}+\varphi}\widetilde{X}_{t-1} + \\
& + \frac{\rho\varphi}{\psi_t+\varphi}(1-\rho)\mu + (1-\rho)\mu = \frac{\rho\psi_t}{\psi_t+\varphi}X_t + \frac{\rho\varphi}{\psi_t+\varphi}\cdot\frac{\rho\psi_{t-1}}{\psi_{t-1}+\varphi}X_{t-1} + \\
& + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\varphi}{\psi_{t-1}+\varphi}\left[\frac{\rho\psi_{t-2}}{\psi_{t-2}+\varphi}X_{t-2} + \frac{\rho\varphi}{\psi_{t-2}+\varphi}\widetilde{X}_{t-2} + (1-\rho)\mu\right] + \\
& + \frac{\rho\varphi}{\psi_t+\varphi}(1-\rho)\mu + (1-\rho)\mu = \frac{\rho\psi_t}{\psi_t+\varphi}X_t + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\psi_{t-1}}{\psi_{t-1}+\varphi}X_{t-1} + \\
& + \frac{\rho\varphi}{\psi_t+\varphi}\cdot\frac{\rho\varphi}{\psi_{t-1}+\varphi}\frac{\rho\psi_{t-2}}{\psi_{t-2}+\varphi}X_{t-2} + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\varphi}{\psi_{t-1}+\varphi}\frac{\rho\varphi}{\psi_{t-2}+\varphi}\widetilde{X}_{t-2} + \\
& + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\varphi}{\psi_{t-1}+\varphi}\cdot(1-\rho)\mu + \frac{\rho\varphi}{\psi_t+\varphi}(1-\rho)\mu + (1-\rho)\mu = \\
& = \frac{\rho\psi_t}{\psi_t+\varphi}X_t + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\psi_{t-1}}{\psi_{t-1}+\varphi}X_{t-1} + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\varphi}{\psi_{t-1}+\varphi}\frac{\rho\psi_{t-2}}{\psi_{t-2}+\varphi}X_{t-2} + \\
& + \ldots + \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\varphi}{\psi_{t-1}+\varphi}\cdot\ldots\cdot\frac{\rho\varphi}{\psi_2+\varphi}\frac{\rho\psi_1}{\psi_1+\varphi}X_1 + (1-\rho)\mu + \\
& + (1-\rho)\mu\frac{\rho\varphi}{\psi_t+\varphi} + (1-\rho)\mu\frac{\rho^2\varphi^2}{(\psi_t+\varphi)(\psi_{t-1}+\varphi)} + \\
& + \ldots + (1-\rho)\mu\cdot\frac{\rho^{t-1}\varphi^{t-1}}{(\psi_t+\varphi)(\psi_{t-1}+\varphi)\cdot\ldots\cdot(\psi_2+\varphi)}.
\end{aligned}
$$

From (2.39) it is clear that:

$$
\alpha_{t0} = (1-\rho)\mu + (1-\rho)\mu\frac{\rho\varphi}{\psi_t+\varphi} + (1-\rho)\mu\frac{\rho^2\varphi^2}{(\psi_t+\varphi)(\psi_{t-1}+\varphi)} + \ldots + (1-\rho)\mu\cdot
$$

$$
\cdot\frac{\rho^{t-1}\varphi^{t-1}}{(\psi_t+\varphi)(\psi_{t-1}+\varphi)\ldots(\psi_2+\varphi)},
$$

$$
\alpha_{t1} = \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\varphi}{\psi_{t-1}+\varphi}\cdot\ldots\cdot\frac{\rho\varphi}{\psi_2+\varphi}\frac{\rho\psi_1}{\psi_1+\varphi},
$$

$$
\vdots
$$

$$
\alpha_{t,t-2} = \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\varphi}{\psi_{t-1}+\varphi}\frac{\rho\psi_{t-2}}{\psi_{t-2}+\varphi},
$$

$$
\alpha_{t,t-1} = \frac{\rho\varphi}{\psi_t+\varphi}\frac{\rho\psi_{t-1}}{\psi_{t-1}+\varphi},
$$

$$\alpha_{t,t} = \frac{\rho\psi_t}{\psi_t + \varphi},$$

that is:

(2.40) $$\alpha_{tj} = \delta_j \alpha_{t,j+1}, \quad j = \overline{1, t-1}.$$

with

$$\begin{aligned}
\delta_j &= \frac{\rho \dfrac{\psi_j}{\psi_j + \varphi} \rho \dfrac{\varphi}{\psi_{j+1} + \varphi}}{\rho \dfrac{\psi_{j+1}}{\psi_{j+1} + \varphi}} = \frac{\rho \psi_j \varphi}{\psi_{j+1} (\psi_j + \varphi)} \overset{(2.3)}{=} \\
&= \frac{\rho \psi_j \varphi}{\left[\rho^2 \dfrac{\psi_j \varphi}{\psi_j + \varphi} + \left(1 - \rho^2\right) \lambda\right]} \cdot \frac{1}{(\psi_j + \varphi)} = \\
&= \frac{\rho \psi_j \varphi}{\rho^2 \psi_j \varphi + (1 - \rho^2) \lambda (\psi_j + \varphi)}, \quad j = \overline{1, t-1}.
\end{aligned} \tag{2.41}$$

From (2.2) we see that:

(2.42) $$\psi_j > 0, \quad \forall j = \overline{1, t-1},$$

because $\psi_j \overset{(2.3)}{=} \rho^2 \dfrac{\psi_{j-1}\varphi}{\psi_{j-1} + \varphi} + \left(1 - \rho^2\right) \lambda \geq \left(1 - \rho^2\right) \lambda > 0$, $j = \overline{1, t-1}$ under the assumptions of the beginning of this section: $0 < \rho < 1$, $\lambda > 0$,

$$\psi_{j-1} \overset{def.}{=} E\left\{\left[\mu(\theta_{j-1}) - \widetilde{X}_{j-1}\right]^2\right\} \geq 0$$

with $j = \overline{1, t-1}$, $\varphi > 0$. Considering (2.42), we may divide the numerator and the denominator in (2.41) by ψ_j, and thus we obtain:

(2.43) $$\delta_j = \frac{\rho\varphi}{\rho^2 \varphi + (1 - \rho^2) \lambda \left(1 + \dfrac{\varphi}{\psi_j}\right)}; \quad j = \overline{1, t-1}.$$

The credibility estimator $\widetilde{X}_j$ of $\mu(\theta_j)$ based on $X_1, X_2, \ldots, X_{j-1}$ is a non-homogeneous linear combination of the observable random variables $X_1, X_2, \ldots, X_{j-1}$ with minimal mean squared error, which means that the solution of the following minimization problem: $\underset{\dot{X}_j = a_0 + \sum_{r=1}^{j-1} a_r X_r}{Min} E\left\{\left[\mu(\theta_j) - \dot{X}_j\right]^2\right\}$ is $\widetilde{X}_j$, that is:

$$E\left\{\left[\mu(\theta_j) - \widetilde{X}_j\right]^2\right\} = \underset{\dot{X}_j = a_0 + \sum_{r=1}^{j-1} a_r X_r}{Min} E\left\{\left[\mu(\theta_j) - \dot{X}_j\right]^2\right\} \leq$$

$$\leq E\{[\mu(\theta_j) - \dot{X}_j]^2\}, \quad \forall \dot{X}_j = a_0 + \sum_{r=1}^{j-1} a_r X_r,$$

and from here in particular for $\dot{X}_j = \mu =$ (constant), we obtain:

$$E\left\{\left[\mu\left(\theta_j\right)-\widetilde{X}_j\right]^2\right\} \leq E\left\{\left[\mu\left(\theta_j\right)-\mu\right]^2\right\}, \quad \text{where } j=\overline{1,t-1}. \tag{2.44}$$

We have:

$$\begin{aligned}
&E\left\{[\mu(\theta_j)-\mu]^2\right\} = E\left\{[\mu(\theta_j)-\mu]^2\right\} - \{E[\mu(\theta_j)]-\mu\}^2 = \\
&= E\left\{[\mu(\theta_j)-\mu]^2\right\} - \{E\left[\mu(\theta_j)\right] - E(\mu)\}^2 = \\
&= E\left\{[\mu(\theta_j)-\mu]^2\right\} - E^2\left[\mu(\theta_j)-\mu\right] = E\left[\mu^2\left(\theta_j\right)+\mu^2-2\mu\mu\left(\theta_j\right)\right] - \\
&- \left\{E^2\left[\mu\left(\theta_j\right)\right] + E^2\left(\mu\right) - 2E\left[\mu\left(\theta_j\right)\right]E\left[\mu\right]\right\} = \\
&= E\left[\mu^2\left(\theta_j\right)\right] + \mu^2 - 2\mu E\left[\mu\left(\theta_j\right)\right] - E^2\left[\mu\left(\theta_j\right)\right] - \mu^2 + 2\mu E\left[\mu\left(\theta_j\right)\right] = \\
&= E\left[\mu^2\left(\theta_j\right)\right] - E^2\left[\mu\left(\theta_j\right)\right] = Var\left[\mu\left(\theta_j\right)\right] = \\
&= Cov\left[\mu\left(\theta_j\right),\mu\left(\theta_j\right)\right] \overset{(2.1)}{=} \rho^{|j-j|}\lambda = \lambda, \quad \text{where } j=\overline{1,t-1}.
\end{aligned} \tag{2.45}$$

Relations (2.44) and (2.45) lead to:

$$\psi_j \overset{def.}{=} E\left\{\left[\mu\left(\theta_j\right)-\widetilde{X}_j\right]^2\right\} \leq \lambda, \tag{2.46}$$

where $j=\overline{1,t-1}$. So, from (2.46) we have:

$$\begin{aligned}
&\frac{1}{\psi_j} \geq \frac{1}{\lambda} \Leftrightarrow \frac{\varphi}{\psi_j} \geq \frac{\varphi}{\lambda} \Leftrightarrow 1+\frac{\varphi}{\psi_j} \geq 1+\frac{\varphi}{\lambda} \Leftrightarrow \\
&\Leftrightarrow \rho^2\varphi + (1-\rho^2)\lambda\left(1+\frac{\varphi}{\psi_j}\right) \geq \rho^2\varphi + \left(1-\rho^2\right)\lambda\left(1+\frac{\varphi}{\lambda}\right) \Leftrightarrow \\
&\Leftrightarrow \delta_j \overset{(2.46)}{=} \frac{\rho\varphi}{\rho^2\varphi + (1-\rho^2)\lambda\left(1+\dfrac{\varphi}{\psi_j}\right)} \leq \\
&\leq \frac{\rho\varphi}{\rho^2\varphi + (1-\rho^2)\lambda\left(1+\dfrac{\varphi}{\lambda}\right)} = \frac{\rho\varphi}{\rho^2\varphi + (1-\rho^2)(\lambda+\varphi)} = \\
&= \frac{\rho\varphi}{\rho^2\varphi + \lambda + \varphi - \rho^2\lambda - \rho^2\varphi} = \frac{\rho\varphi}{\varphi + (1-\rho^2)\lambda} < \frac{\varphi}{\varphi + (1-\rho^2)\lambda} < 1,
\end{aligned} \tag{2.47}$$

with $j=\overline{1,t-1}$ and where we considered the assumptions of the beginning of this section: $0<\rho<1$, $\lambda>0$, $\varphi>0$. On the other hand, it is clear that:

$$\delta_j > 0; \quad j=\overline{1,t-1}, \tag{2.48}$$

because take place the relations (2.41), (2.42) and the assumptions of the beginning of this section: $0<\rho<1$, $\lambda>0$, $\varphi>0$. Inequalities (2.47) and (2.48) imply:

$$0<\delta_j<1; \quad j=\overline{1,t-1}. \tag{2.49}$$

Also we see that:

$$0 \overset{(2.55)}{<} \alpha_{tt} = \frac{\rho\psi_t}{\psi_t + \varphi} = \rho\frac{\psi_t}{\psi_t + \varphi} \underset{(2.54)}{<} 1 \cdot 1 = 1, \tag{2.50}$$

where we considered the assumptions of the beginning of this section: $0 < \rho < 1$, $\varphi > 0$ and the below relations:

$$\frac{\psi_t}{\psi_t + \varphi} < 1, \tag{2.51}$$

$$\psi_t \overset{(2.3)}{=} \rho^2 \frac{\psi_{t-1}\varphi}{\psi_{t-1} + \varphi} + \left(1 - \rho^2\right)\lambda \geq \left(1 - \rho^2\right)\lambda > 0, \tag{2.52}$$

under the assumptions of the beginning of this section: $0 < \rho < 1$, $\lambda > 0$,

$$\psi_{t-1} \overset{def.}{=} E\left\{\left[\mu\left(\theta_{t-1}\right) - \widetilde{X}_{t-1}\right]^2\right\} \geq 0,\ \varphi > 0.$$

From (2.40) we obtain:

$$\delta_j = \frac{\alpha_{tj}}{\alpha_{t,j+1}}; \quad j = \overline{1, t-1}, \tag{2.53}$$

and since (2.49) take place. Thus:

$$0 < \frac{\alpha_{tj}}{\alpha_{t,j+1}} < 1; \quad j = \overline{1, t-1}. \tag{2.54}$$

But:

$$\alpha_{t1} = \delta_1\alpha_{t2} = \delta_1\delta_2\alpha_{t3} = \delta_1\delta_2\delta_3\alpha_{t4} = \ldots = \delta_1\delta_2\delta_3 \cdot \ldots \cdot \delta_{t-1}\alpha_{tt} > 0, \tag{2.55}$$

$$\alpha_{t2} = \delta_2\alpha_{t3} = \delta_2\delta_3\alpha_{t4} = \ldots = \delta_2\delta_3 \cdot \ldots \cdot \delta_{t-1}\alpha_{tt} > 0, \tag{2.56}$$

$$\vdots$$

$$\alpha_{t,t-1} = \delta_{t-1}\alpha_{tt} > 0. \tag{2.57}$$

Relations (2.55), (2.56), …, (2.57) together with the relations (2.55), lead to:

$$0 < \alpha_{tj} < \alpha_{t,j+1}; \quad j = \overline{1, t-1},$$

that is:

$$0 < \alpha_{t1} < \alpha_{t2} < \alpha_{t3} < \ldots < \alpha_{tt} < 1.$$

This completes the proof of Result 2.

3 Conclusions

The results show that the credibility theory is really a useful tool – perhaps the only existing tool – for such insurance applications.

The fact that it is based on complicated mathematics, involving conditional expectations and conditional (co-)variances, needs not bother the user more than it does when he applies statistical tools like SAS, GLIM, discriminant analysis, and scoring models. These techniques can be applied by anybody on his own field of endeavor, be it economics, medicine or insurance.

The main results of the paper are: the two results obtained for the non-homogeneous linear credibility model and demonstrated by probability theory and credibility theory (see Section 2). The first result gives a simple recursive procedure for the updating of $\widetilde{X}_{t+1}$ (the credibility estimator of X_{t+1} or $\mu(\theta_{t+1})$ based on $\underline{X'}$) and as a by-product, we also get a recursive updating of ψ_{t+1} (the estimation error of $\widetilde{X}_{t+1}$). We finished this paper, giving the recursive credibility model (see Result 2 of Section 2), which is an application of the original credibility model of Bühlmann. Our motivation for introducing the recursive credibility model was that we wanted that the new claims to have more weight than the older claims. The second result (see Result 2 of Section 2) shows that this desire has been satisfied.

So in this paper we gave some results of the credibility theory, obtained by using important mathematical properties of the probability theory, more precisely properties of conditional expectations and conditional (co-)variances.

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