On some quadratic Hamilton-Poisson systems

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Abstract. We identify a class of quadratic Hamilton-Poisson systems on the three-dimensional Euclidean Lie-Poisson space. Specifically, we consider systems that are both homogeneous and for which the underlying quadratic form is positive semidefinite. Any such system is shown to be equivalent to one of four normal forms (of which two are parametrized families of systems). For the cases with non-trivial dynamics, the stability nature of the equilibrium states is fully investigated. Furthermore, we find explicit expressions for the integral curves (in terms of Jacobi elliptic functions).

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1 Introduction

The dual space $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ has a natural Poisson structure, the (minus) Lie-Poisson structure of $\mathfrak{g}^*$; this Poisson space is denoted by $\mathfrak{g}^*_-$. A quadratic Hamilton-Poisson system on $\mathfrak{g}^*_-$ is given by $H_{A,\mathcal{Q}} = p(A) + \mathcal{Q}(p)$. Here $A \in \mathfrak{g}$ and $\mathcal{Q}$ is a quadratic form on $\mathfrak{g}^*$. Such Hamilton-Poisson systems have been studied in the last few decades (see, e.g., [5, 6, 10, 7, 25, 26, 9, 11]).

In this paper we consider only quadratic Hamilton-Poisson systems on the Euclidean Lie-Poisson space $\mathfrak{se}\,(2)^*_-$. Moreover, we shall restrict to systems that are both *homogeneous* (i.e., $A = 0$) and for which the quadratic form $\mathcal{Q}$ is *positive semidefinite*. We show that any such system is equivalent to one of the systems

$$H_0(p) = 0 \qquad\qquad H_1(p) = \tfrac{1}{2}\, p_1^2$$
$$H_{2,\alpha}(p) = \tfrac{1}{2}\left(p_2^2 + \alpha\, p_3^2\right) \qquad\qquad H_{3,\alpha}(p) = \tfrac{1}{2}\,\alpha\, p_3^2$$

for some $\alpha > 0$. H_0 and H_1 have almost trivial dynamics and so we shall omit any discussion of these systems. The stability nature of all equilibrium states of (the Hamiltonian vector fields) $\vec{H}_{2,\alpha}$ and $\vec{H}_{3,\alpha}$ is investigated by the energy-Casimir method. Furthermore, explicit expressions for the integral curves of these vector fields are found in terms of Jacobi elliptic functions.

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A general left-invariant control affine system on the Euclidean group $\mathsf{SE}(2)$ has the form $\dot{g} = g\,(A + u_1 B_1 + \cdots + u_\ell B_\ell)$, where $A, B_1, \ldots, B_\ell \in \mathfrak{se}(2)$, $1 \leq \ell \leq 3$. (The elements $B_1, \ldots, B_\ell$ are assumed to be linearly independent.) Specific left-invariant optimal control problems on $\mathsf{SE}(2)$, associated with the above mentioned control systems, have been studied by several authors (see, e.g., [14, 13, 24, 22, 21, 23]). One can associate to drift-free systems the following class of optimal control problems (with quadratic cost)

$$\left.\begin{array}{c} \dot{g} = g\,(u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g(0) = g_0, \quad g(T) = g_1 \\ g \in \mathsf{SE}(2), \quad u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell, \quad 1 \leq \ell \leq 3 \\ \mathcal{J} = \frac{1}{2} \displaystyle\int_0^T u(t)^\top Q\, u(t)\, dt \to \min . \end{array}\right\} \tag{1.1}$$

Here Q is an $\ell \times \ell$ *positive definite* matrix. Each of these problems is lifted, via the Pontryagin Maximum Principle (cf. [3, 14]), to a homogeneous (positive semidefinite) quadratic Hamilton-Poisson system on $\mathfrak{se}(2)^*_-$. The extremal controls can then be found in terms of integral curves of the associated Hamiltonian vector field. Hamilton-Poisson systems corresponding to lifted optimal control problems on $\mathsf{SE}(2)$ have been investigated in [1, 2].

2 Preliminaries

2.1 The Lie-Poisson structure

Let $\mathfrak{g}$ be a (real) Lie algebra. The dual space $\mathfrak{g}^*$ has a natural Poisson structure, the *(minus) Lie-Poisson structure* (see, e.g., [16, 17]). This structure is given by

$$\{F, G\}\,(p) = -p\,([dF(p), dG(p)])$$

for $p \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. Here $dF(p)$ is a linear function on $\mathfrak{g}^*$ and so is (identified with) an element of $\mathfrak{g}$. A function $C \in C^\infty(\mathfrak{g}^*)$ is a *Casimir function* if $\{C, F\} = 0$ for all $F \in C^\infty(\mathfrak{g}^*)$.

To each function $H \in C^\infty(\mathfrak{g}^*)$ we may associate a unique vector field $\vec{H}$ on $\mathfrak{g}^*$ such that $\vec{H}[F] = \{F, H\}$. This vector field is called the *Hamiltonian vector field* associated with H. Two vector fields $\vec{F}$ and $\vec{G}$ are *compatible* with a diffeomorphism $\phi : \mathfrak{g}^* \to \mathfrak{g}^*$ if the push-forward $\phi_* \vec{F}$ equals $\vec{G}$ (i.e., $T_p\phi \cdot \vec{F}(p) = \vec{G}(\phi(p))$ for $p \in \mathfrak{g}^*$). The diffeomorphism ϕ establishes a one-to-one correspondence between the integral curves of $\vec{F}$ and $\vec{G}$.

A linear automorphism $\psi : \mathfrak{g}^* \to \mathfrak{g}^*$ is a *linear Poisson automorphism* if $\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$ for all $F, G \in C^\infty(\mathfrak{g}^*)$. Linear Poisson automorphisms are exactly the dual maps of Lie algebra automorphisms. We say that two Hamilton-Poisson systems are *equivalent* if the vector fields associated with them are compatible with a linear Poisson automorphism.

Remark 2.1. Let $H \in C^\infty(\mathfrak{g}^*)$ and let ψ be a linear Poisson automorphism. The systems $H \circ \psi$ and H are equivalent as the associated vector fields are compatible with ψ. For any Casimir function C and any function $\chi \in C^\infty(\mathbb{R})$, the vector fields associated with H and $H + \chi(C)$, respectively, are identical. Hence the systems $H \circ \psi$ and $H + \chi(C)$ are equivalent.

2.2 The energy-Casimir method

The *energy-Casimir method* [12] gives sufficient conditions for Lyapunov stability of equilibrium states of certain types of Hamilton-Poisson dynamical systems (cf. [17, 20]). The method is restricted to certain types of systems, since its implementation relies on an abundant supply of Casimir functions.

The standard energy-Casimir method states that if z_e is an equilibrium point of a Hamiltonian vector field $\vec{H}$ (associated with an energy function H) and if there exists a Casimir function C such that z_e is a critical point of $H + C$ and $d^2(H + C)(z_e)$ is (positive or negative) definite, then z_e is Lyapunov stable.

Ortega and Ratiu have obtained a generalization of the standard energy-Casimir method (cf. [18, 19]). This extended version states that if $C = \lambda_1 C_1 + \cdots + \lambda_k C_k$, where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $C_1, \ldots, C_k$ are conserved quantities (i.e., they Poisson commute with the energy function H), then definiteness of $d^2(\lambda_0 H + C)(z_e)$, $\lambda_0 \in \mathbb{R}$ is only required on the intersection (subspace) $W = \ker dH(z_e) \cap \ker dC_1(z_e) \cap \cdots \cap \ker dC_k(z_e)$.

2.3 Jacobi elliptic functions

Given the modulus $k \in [0, 1]$, the basic *Jacobi elliptic functions* $\mathrm{sn}(\cdot, k)$, $\mathrm{cn}(\cdot, k)$, and $\mathrm{dn}\,(\cdot, k)$ can be defined as

$$\begin{aligned}
\mathrm{sn}(x, k) &= \sin\,\mathrm{am}(x, k) \\
\mathrm{cn}(x, k) &= \cos\,\mathrm{am}(x, k) \\
\mathrm{dn}(x, k) &= \sqrt{1 - k^2 \sin^2 \mathrm{am}(x, k)}
\end{aligned}$$

where $\mathrm{am}(\cdot, k) = F(\cdot, k)^{-1}$ is the amplitude and $F(\varphi, k) = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2 \sin^2 t}}\cdot$ (For the degenerate cases $k = 0$ and $k = 1$ we recover the circular functions and the hyperbolic functions, respectively.) Simple elliptic integrals can be expressed in terms of appropriate inverse (elliptic) functions. The following formulas hold true (see [4, 15]):

$$\int_x^b \frac{dt}{\sqrt{(a^2 + t^2)(b^2 - t^2)}} = \tfrac{1}{\sqrt{a^2+b^2}}\,\mathrm{cn}^{-1}\left(\tfrac{1}{b}\,x,\ \tfrac{b}{\sqrt{a^2+b^2}}\right), \quad 0 \le x \le b \tag{2.1}$$

$$\int_x^a \frac{dt}{\sqrt{(a^2 - t^2)(t^2 - b^2)}} = \tfrac{1}{a}\,\mathrm{dn}^{-1}\left(\tfrac{1}{a}\,x,\ \tfrac{\sqrt{a^2-b^2}}{a}\right), \quad b \le x \le a. \tag{2.2}$$

3 Hamilton-Poisson systems on $\mathfrak{se}\,(2)^*_-$

3.1 The Lie-Poisson space $\mathfrak{se}\,(2)^*_-$

The Euclidean Lie algebra is given by

$$\mathfrak{se}\,(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} \,:\, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

be the standard basis of $\mathfrak{se}(2)$. Then the bracket operation is given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = 0$. Let (E_1^*, E_2^*, E_3^*) denote the dual of the standard basis. An element $p = p_1 E_1^* + p_2 E_2^* + p_3 E_3^*$ will be written as $p = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}$. The group of linear Poisson automorphisms of $\mathfrak{se}(2)^*_-$ is given by

$$\left\{ p \mapsto p \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \; : \; x, y, v, w \in \mathbb{R},\, x^2 + y^2 \neq 0,\, \varsigma = \pm 1 \right\}.$$

Now consider a Hamiltonian H on $\mathfrak{se}(2)^*_-$. The equations of motion take the following form

$$\dot{p}_i = -p([E_i, dH(p)]), \quad i = 1, 2, 3$$

or, explicitly,

$$\begin{cases} \dot{p}_1 = \dfrac{\partial H}{\partial p_3}\, p_2 \\ \dot{p}_2 = -\dfrac{\partial H}{\partial p_3}\, p_1 \\ \dot{p}_3 = \dfrac{\partial H}{\partial p_2}\, p_1 - \dfrac{\partial H}{\partial p_1}\, p_2. \end{cases}$$

We note that $C : \mathfrak{se}(2)^* \to \mathbb{R}$, $C(p) = p_1^2 + p_2^2$ is a Casimir function.

3.2 Equivalence of systems

Consider a (homogeneous) quadratic Hamilton-Poisson system $H_Q(p) = p\,Q\,p^\top$, where Q is a positive semidefinite 3×3 matrix.

Theorem 3.1. *Let $H_0(p) = 0$, $H_1(p) = \frac{1}{2} p_1^2$, $H_{2,\alpha}(p) = \frac{1}{2}\left(p_2^2 + \alpha\, p_3^2\right)$, and $H_{3,\alpha}(p) = \frac{1}{2}\alpha\, p_3^2$. There exist a linear Poisson automorphism ψ and constants $\alpha, \gamma > 0$ such that*

$$(H_Q + C) \circ \psi = H + \gamma\, C$$

for some $H \in \{H_0, H_1, H_{2,\alpha}, H_{3,\alpha}\}$.

Proof. Let

$$Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}.$$

First, let us assume $a_3 = 0$. The 2×2 principle minors of Q are then $a_1 a_2 - b_1^2$, $-b_2^2$, and $-b_3^2$. As Q is positive semidefinite (shortly PSD), all principle minors are nonnegative. Thus $b_2 = b_3 = 0$. Now,

$$\psi_1 : p \mapsto p\,\Psi_1, \qquad \Psi_1 = \begin{bmatrix} 1 & y & 0 \\ -y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism for all $y \in \mathbb{R}$. Also,

$$((H_Q + C) \circ \psi_1)(p) = p\,\Psi_1\,(Q + C)\,\Psi_1^\top p^\top$$

$$\Psi_1\,(Q + C)\,\Psi_1^\top = \begin{bmatrix} a'_{1,y} & b_1 + y\,(a_2 - a_1) - y^2 b_1 & 0 \\ b_1 + y\,(a_2 - a_1) - y^2 b_1 & a'_{2,y} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

for some $a'_{1,y}, a'_{2,y} \in \mathbb{R}$. Furthermore, $b_1 + y\,(a_2 - a_1) - y^2 b_1 = 0$ has a real solution for y. Thus there exists $y \in \mathbb{R}$ such that

$$((H_Q + C) \circ \psi_1)(p) = a'_{1,y} p_1^2 + a'_{2,y} p_2^2.$$

Note that $\begin{bmatrix} a_1 & b_1 \\ b_1 & a_2 \end{bmatrix}$ is PSD and so $\begin{bmatrix} 1 + a_1 & b_1 \\ b_1 & 1 + a_2 \end{bmatrix}$ is positive definite (shortly PD). We have

$$\begin{bmatrix} 1 & y \\ -y & 1 \end{bmatrix} \begin{bmatrix} 1 + a_1 & b_1 \\ b_1 & 1 + a_2 \end{bmatrix} \begin{bmatrix} 1 & y \\ -y & 1 \end{bmatrix}^\top = \begin{bmatrix} a'_{1,y} & 0 \\ 0 & a'_{2,y} \end{bmatrix}.$$

Thus $a'_{1,y}$, $a'_{2,y} > 0$.

Suppose $a'_{1,y} - a'_{2,y} = 0$. Then $(H_Q + C) \circ \psi_1 = H_0 + \gamma\,C$, where $\gamma = a'_{1,y} > 0$. Now suppose $a'_{1,y} - a'_{2,y} \neq 0$. Then

$$\psi_2 : p \mapsto p\,\Psi_2, \qquad \Psi_2 = \begin{bmatrix} \frac{1}{\sqrt{2|a'_{1,y} - a'_{2,y}|}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2|a'_{1,y} - a'_{2,y}|}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\begin{aligned} \big((H_Q + C) \circ (\psi_1 \circ \psi_2)\big)(p) &= \tfrac{a'_{1,y}}{2|a'_{1,y} - a'_{2,y}|} p_1^2 + \tfrac{a'_{2,y}}{2|a'_{1,y} - a'_{2,y}|} p_2^2 \\ &= \tfrac{a'_{1,y} - a'_{2,y}}{2|a'_{1,y} - a'_{2,y}|} p_1^2 + \tfrac{a'_{2,y}}{2|a'_{1,y} - a'_{2,y}|} C(p). \end{aligned}$$

Suppose $a'_{1,y} - a'_{2,y} > 0$. Then $(H_Q + C) \circ \psi_1 = H_1 + \gamma\,C$, where $\gamma = \frac{a'_{2,y}}{2|a'_{1,y} - a'_{2,y}|} > 0$. Suppose $a'_{1,y} - a'_{2,y} < 0$. Then

$$\psi_3 : p \mapsto p\,\Psi_3, \qquad \Psi_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\begin{aligned} \big((H_Q + C) \circ (\psi_1 \circ \psi_2 \circ \psi_3)\big)(p) &= \tfrac{a'_{2,y}}{2|a'_{1,y} - a'_{2,y}|} p_1^2 + \tfrac{a'_{1,y}}{2|a'_{1,y} - a'_{2,y}|} p_2^2 \\ &= \tfrac{a'_{2,y} - a'_{1,y}}{2|a'_{1,y} - a'_{2,y}|} p_1^2 + \tfrac{a'_{1,y}}{2|a'_{1,y} - a'_{2,y}|} C(p). \end{aligned}$$

Therefore $(H_Q + C) \circ (\psi_1 \circ \psi_2 \circ \psi_3) = H_1 + \gamma\,C$, where $\gamma = \frac{a'_{1,y}}{2|a'_{1,y} - a'_{2,y}|} > 0$.

On the other hand suppose $a_3 \neq 0$. Then

$$\psi_4 : p \mapsto p\Psi_4, \qquad \Psi_4 = \begin{bmatrix} 1 & 0 & -\frac{b_2}{a_3} \\ 0 & 1 & -\frac{b_3}{a_3} \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\big((H_Q + C) \circ \psi_4\big)(p) = p\,\Psi_4\,(Q + C)\,\Psi_4^\top\,p^\top$$
$$\Psi_4\,(Q + C)\,\Psi_4^\top = \begin{bmatrix} 1 + a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & 1 + a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

Now $\Psi_4\,(Q + C)\,\Psi_4^\top = \Psi_4\,Q\,\Psi_4^\top + C$. Hence $\Psi_4\,(Q + C)\,\Psi_4^\top$ is PD.

Let

$$Q' = \Psi_4\,(Q + C)\,\Psi_4^\top = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

for some $a_1', a_2', b_1' \in \mathbb{R}$. As before, there exists a linear Poisson automorphism ψ_1 such that

$$\begin{aligned} \big((H_Q + C) \circ (\psi_4 \circ \psi_1)\big)(p) &= a_1'' p_1^2 + a_2'' p_2^2 + a_3 p_3^2 \\ &= (a_2'' - a_1'')p_2^2 + a_3 p_3^2 + a_1'' C(p) \end{aligned}$$

for some a_1'', $a_2'' > 0$.

Suppose $a_2'' - a_1'' = 0$. Then $(H_Q + C) \circ (\psi_4 \circ \psi_1) = H_{3,\alpha} + \gamma\,C$, where $\alpha = 2a_3 > 0$ and $\gamma = a_1'' > 0$. Now suppose $a_2'' - a_1'' \neq 0$. Then

$$\psi_5 : p \mapsto p\,\Psi_5, \qquad \Psi_5 = \begin{bmatrix} \frac{1}{\sqrt{2|a_2''-a_1''|}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2|a_2''-a_1''|}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a linear Poisson automorphism such that

$$\big((H_Q + C) \circ (\psi_4 \circ \psi_1 \circ \psi_5)\big)(p) = \tfrac{a_2''-a_1''}{2|a_2''-a_1''|} p_2^2 + a_3 p_3^2 + \tfrac{a_1''}{2|a_2''-a_1''|} C(p).$$

Suppose $a_2'' - a_1'' > 0$. Then $(H_Q + C) \circ (\psi_4 \circ \psi_1 \circ \psi_5) = H_{2,\alpha} + \gamma\,C$, where $\alpha = 2a_3 > 0$ and $\gamma = \frac{a_1''}{2|a_2''-a_1''|} > 0$. Suppose $a_2'' - a_1'' < 0$. Then

$$\big((H_Q + C) \circ (\psi_4 \circ \psi_1 \circ \psi_5 \circ \psi_3)\big)(p) = \tfrac{a_1''-a_2''}{2|a_2''-a_1''|} p_2^2 + a_3 p_3^2 + \tfrac{a_2''}{2|a_2''-a_1''|} C(p).$$

Therefore $(H_Q + C) \circ (\psi_4 \circ \psi_1 \circ \psi_5 \circ \psi_3) = H_{2,\alpha} + \gamma\,C$, where $\alpha = 2a_3 > 0$ and $\gamma = \frac{a_2''}{2|a_2''-a_1''|} > 0$. $\square$

Corollary 3.2. *H_Q is equivalent to H_0, H_1, $H_{2,\alpha}$, or $H_{3,\alpha}$.*

Accordingly, the integral curves of $\vec{H}_Q$ are simply the images, under some linear Poisson automorphism, of the integral curves of $\vec{H}_0$, $\vec{H}_1$, $\vec{H}_{2,\alpha}$, or $\vec{H}_{3,\alpha}$. The system H_0 is trivial, i.e., $\vec{H}_0(p) = 0$. The equations of motion for H_1 are $\dot{p}_1 = \dot{p}_2 = 0$, $\dot{p}_3 = -p_1 p_2$. On the other hand, the equations of motion for $H_{2,\alpha}$ and $H_{3,\alpha}$ are, respectively,

$$\begin{cases} \dot{p}_1 = \alpha\, p_2 p_3 \\ \dot{p}_2 = -\alpha\, p_1 p_3 \\ \dot{p}_3 = p_1 p_2 \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{p}_1 = \alpha\, p_2 p_3 \\ \dot{p}_2 = -\alpha\, p_1 p_3 \\ \dot{p}_3 = 0. \end{cases} \tag{3.2}$$

4 Stability

We investigate the stability nature of the dynamical system (3.1), i.e., $\vec{H}_{2,\alpha}$. The equilibrium states are

$$e_1^\nu = (\nu, 0, 0), \quad e_2^\nu = (0, \nu, 0) \quad \text{and} \quad e_3^\mu = (0, 0, \mu)$$

where $\mu, \nu \in \mathbb{R}$, $\nu \neq 0$.

Theorem 4.1. *The equilibrium states have the following behaviour.*

(i) Each equilibrium state e_1^ν is stable.

(ii) Each equilibrium state e_2^ν is unstable.

(iii) Each equilibrium state e_3^μ is stable.

Proof. The linearization of the system is given by

$$\begin{bmatrix} 0 & \alpha\, p_3 & \alpha\, p_2 \\ -\alpha\, p_3 & 0 & -\alpha\, p_1 \\ p_2 & p_1 & 0 \end{bmatrix}.$$

Let $H_\chi = H_{2,\alpha} + \chi(C)$ be an energy-Casimir function, i.e.,

$$H_\chi(p_1, p_2, p_3) = \tfrac{1}{2}\, p_2^2 + \tfrac{1}{2}\, \alpha\, p_3^2 + \chi(p_1^2 + p_2^2)$$

where $\chi \in C^\infty(\mathbb{R})$. The derivative is given by

$$dH_\chi = \begin{bmatrix} 2p_1 \dot{\chi}(p_1^2 + p_2^2) & p_2 + 2p_2 \dot{\chi}(p_1^2 + p_2^2) & \alpha\, p_3 \end{bmatrix}.$$

The derivative dH_χ vanishes at e_1^ν if and only if $\dot{\chi}(\nu^2) = 0$. Then the Hessian (at e_1^ν)

$$d^2 H_\chi(\nu, 0, 0) = \operatorname{diag}\left(4\nu^2 \ddot{\chi}(\nu^2),\ 1,\ \alpha\right)$$

is positive definite if and only if $\ddot{\chi}(\nu^2) > 0$. The function $\chi(x) = \frac{1}{2}x^2 - x\nu^2$ satisfies these requirements. Hence, by the standard energy-Casimir method, e_1^ν is stable.

The linearization of the system at e_2^ν has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{\alpha}$. Thus e_2^ν is unstable.

Let $H_\lambda = \lambda_0 H_{2,\alpha} + \lambda_1 C$, where $\lambda_0 = 0, \lambda_1 = 1$. Suppose $\mu \neq 0$. Then $dH_\lambda(0,0,\mu) = \begin{bmatrix} 2p_1 & 2p_2 & 0 \end{bmatrix}_{(0,0,\mu)} = 0$ and $d^2 H_\lambda(0,0,\mu) = \operatorname{diag}(2,\, 2,\, 0)$. Also,

$$W = \ker dH_{2,\alpha}(e_3^\mu) \cap \ker dC(e_3^\mu) = \operatorname{span}\,\{(1,0,0),(0,1,0)\}$$

and so $d^2 H_\lambda(0,0,\mu)\big|_{W\times W} = \operatorname{diag}(2,\, 2)$ is positive definite. Hence, by the extended energy-Casimir method, e_3^μ, $\mu \neq 0$ is stable. On the other hand, suppose $\mu = 0$. We have $d\,(H + C)(\mathsf{e}_3^0) = 0$ and $d^2\,(H + C)(\mathsf{e}_3^0) = \operatorname{diag}(2,3,\alpha)$. Thus the state e_3^0 is stable. □

The equilibrium states for the dynamical system (3.2), i.e., $\vec{H}_{3,\alpha}$, are

$$e_1^{\mu,\nu} = (\mu,\nu,0) \neq 0 \quad \text{and} \quad e_3^\mu = (0,0,\mu)$$

where $\mu,\nu \in \mathbb{R}$. We have the following result.

Theorem 4.2. *The equilibrium states have the following behaviour.*

(i) Each equilibrium state $\mathsf{e}_1^{\mu,\nu}$ *is unstable.*

(ii) Each equilibrium state e_3^μ *is stable.*

5 Explicit integration

The equations of motion (3.1) and (3.2) can be integrated by Jacobi elliptic functions. We obtain explicit expressions for the integral curves of $\vec{H}_{2,\alpha}$ and $\vec{H}_{3,\alpha}$ (but shall omit any discussion of constant solutions).

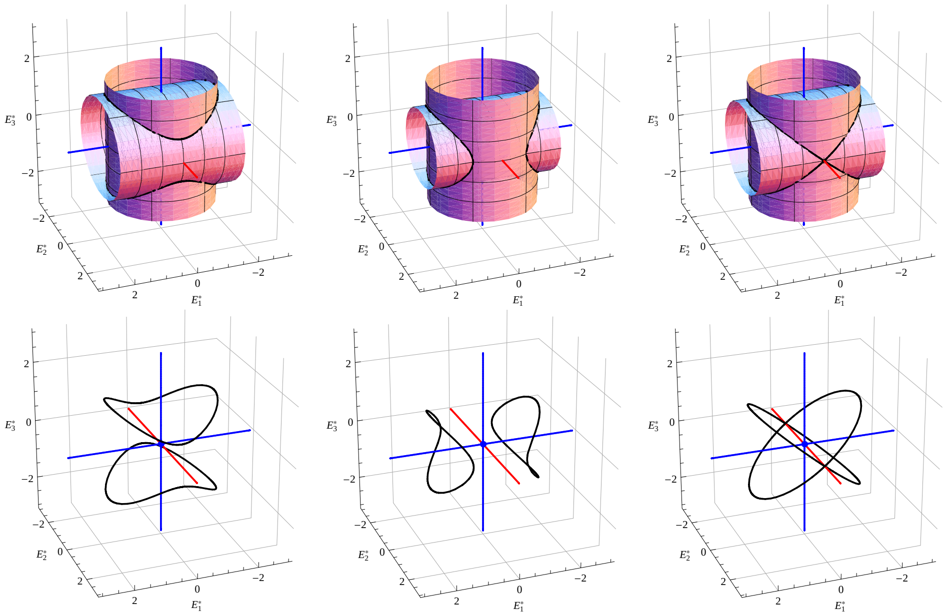
Consider an integral curve $p(\cdot) : (-\varepsilon,\varepsilon) \to \mathfrak{se}\,(2)^*$ of $\vec{H}_{2,\alpha}$. Let $c_0 = C(p(0)) > 0$ and $h_0 = H_{2,\alpha}(p(0)) > 0$. (If $c_0 = 0$ or $h_0 = 0$, then $p(\cdot)$ is a constant solution.) There are three typical cases for the integral curves of $\vec{H}_{2,\alpha}$ corresponding to (a) $c_0 < 2h_0$, (b) $c_0 = 2h_0$, and (c) $c_0 > 2h_0$. In Figure 1, we graph the level sets of $H_{2,\alpha}$ and C and their intersection for some suitable values of h_0, c_0, and α. The stable equilibrium points (illustrated in blue) and unstable equilibrium points (illustrated in red) are also plotted in each case.

We begin with case (a).

Theorem 5.1. *Suppose* $p(\cdot) : (-\varepsilon,\varepsilon) \to \mathfrak{se}(2)^*$ *is an integral curve of* $\vec{H}_{2,\alpha}$ *such that* $H_{2,\alpha}(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$, *and* $c_0 < 2h_0$. *Then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1,1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for* $t \in (-\varepsilon,\varepsilon)$, *where*

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\,\operatorname{cn}(\Omega\, t,\, k) \\ \bar{p}_2(t) = \sigma\sqrt{c_0}\,\operatorname{sn}(\Omega\, t,\, k) \\ \bar{p}_3(t) = -\sigma\,\sqrt{\frac{2h_0}{\alpha}}\,\operatorname{dn}(\Omega\, t,\, k)\,. \end{cases}$$

Here $k = \sqrt{\frac{c_0}{2h_0}}$ *and* $\Omega = \sqrt{2\alpha h_0}$.



(a) $c_0 < 2h_0$ (b) $c_0 = 2h_0$ (c) $c_0 > 2h_0$

Figure 1: Typical cases for $\vec{H}_{2,\alpha}$

Proof. Suppose $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_{2,\alpha}$ satisfying the conditions of the theorem. Then, as C and $H_{2,\alpha}$ are constants of motion, we get

$$\bar{p}_2(t)^2 = c_0 - \bar{p}_1(t)^2 \quad \text{and} \quad \bar{p}_3^2(t) = \tfrac{1}{\alpha}(\bar{p}_1(t)^2 - 2h_0 - c_0).$$

Substituting these expressions into the equation $\left(\frac{d}{dt}\bar{p}_1(t)\right)^2 = \alpha^2\bar{p}_2(t)^2\bar{p}_3(t)^2$ yields a separable differential equation. Hence we have

$$\sqrt{\alpha}\, t = \int \frac{d\bar{p}_1}{\sqrt{(\bar{p}_1^2 + 2h_0 - c_0)(c_0 - \bar{p}_1^2)}}\,\cdot$$

Applying the integral formula (2.1) we get $\bar{p}_1(t) = \sqrt{c_0}\,\mathrm{cn}\,(\Omega\, t,\, k)$, where $k = \sqrt{\frac{c_0}{2h_0}}$ and $\Omega = \sqrt{2\alpha h_0}$. Then $c_0\,\mathrm{cn}\,(\Omega\, t,\, k)^2 + \bar{p}_2(t)^2 = c_0$ and so $\bar{p}_2(t) = \sigma\sqrt{c_0}\,\mathrm{sn}\,(\Omega\, t,\, k)$ for some $\sigma \in \{-1, 1\}$. Thus $\frac{d}{dt}\bar{p}_3(t) = \sigma\, c_0\,\mathrm{cn}\,(\Omega\, t,\, k)\,\mathrm{sn}\,(\Omega\, t,\, k)$ and so $\bar{p}_3(t) = -\sigma\sqrt{\frac{2h_0}{\alpha}}\,\mathrm{dn}\,(\Omega\, t,\, k)$. An easy calculation shows that $\frac{d}{dt}\,\bar{p}(t) = \vec{H}_{2,\alpha}(\bar{p}(t))$ for $\sigma \in \{-1, 1\}$. Thus $\bar{p}(\cdot)$ is an integral curve of $\vec{H}_{2,\alpha}$ for any $0 < c_0 < 2h_0$ and $\alpha > 0$.

We claim that any integral curve $p(\cdot)$ (as described in the statement of the theorem) must be of the form $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ (see Figure 1a). Let $\sigma = \mathrm{sgn}(p_3(0))$. We may assume $\sigma \neq 0$. Note that $(\bar{p}_1(t), \bar{p}_2(t))$ parametrizes the circle $S = \{(x, y) \,:\, x^2 + y^2 = c_0\}$. We have $p_1(0)^2 + p_2(0)^2 = c_0$, i.e., $(p_1(0), p_2(0)) \in S$. Therefore, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_1(t_0) = p_1(0)$ and $\bar{p}_2(t_0) = p_2(0)$. Accordingly, we get

$$p_3(0)^2 = \tfrac{2}{\alpha}(h_0 - \tfrac{1}{2}p_2(0)^2) = \tfrac{2}{\alpha}(h_0 - \tfrac{1}{2}\bar{p}_2(t_0)^2) = \bar{p}_3(t_0)^2.$$

Hence, as $\operatorname{sgn}(p_3(t_0)) = \sigma = \operatorname{sgn}(p_3(0))$, we have $p_3(0) = \bar{p}_3(t_0)$. Thus the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. $\square$

By utilizing formula (2.2), a similar argument can be made for case (c).

Theorem 5.2. *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}_{2,\alpha}$ such that $H_{2,\alpha}(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$, and $c_0 > 2h_0$. Then there exist $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{c_0}\, \operatorname{dn}(\Omega\, t,\, k) \\ \bar{p}_2(t) = \sqrt{2h_0}\, \operatorname{sn}(\Omega\, t,\, k) \\ \bar{p}_3(t) = -\sigma\sqrt{\frac{2h_0}{\alpha}}\, \operatorname{cn}(\Omega\, t,\, k)\, . \end{cases}$$

Here $k = \sqrt{\frac{2h_0}{c_0}}$ and $\Omega = \sqrt{\alpha c_0}$.

Next we consider case (b).

Proposition 5.3. *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}_{2,\alpha}$ such that $H_{2,\alpha}(p(0)) = h_0 > 0$, $C(p(0)) = c_0 > 0$, and $c_0 = 2h_0$. Then there exist $t_0 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \sigma_1\sqrt{c_0}\, \operatorname{sech}(\sqrt{\alpha c_0}\, t) \\ \bar{p}_2(t) = \sigma_1\sigma_2\sqrt{c_0}\, \tanh(\sqrt{\alpha c_0}\, t) \\ \bar{p}_3(t) = -\sigma_2\sqrt{\frac{c_0}{\alpha}}\, \operatorname{sech}(\sqrt{\alpha c_0}\, t)\, . \end{cases}$$

Proof. By limiting $h_0 \to \frac{1}{2}c_0$ in Theorem 5.1 (or Theorem 5.2), and allowing for possible changes in sign, we obtain the following prospective integral curve for $\vec{H}_{2,\alpha}$

$$\begin{aligned} \bar{p}_1(t) &= \varsigma_1\, \sqrt{c_0}\, \operatorname{sech}(\Omega\, t) \\ \bar{p}_2(t) &= \varsigma_2\, \sqrt{c_0}\, \tanh(\Omega\, t) \\ \bar{p}_3(t) &= \varsigma_3\, \sqrt{\frac{c_0}{\alpha}}\, \operatorname{sech}(\Omega\, t) \end{aligned}$$

where $\Omega = \sqrt{\alpha c_0}$ and $\varsigma_1, \varsigma_2, \varsigma_3 \in \{-1, 1\}$. We investigate under which conditions $\bar{p}(\cdot)$ is an integral curve. Now

$$\frac{d}{dt}\bar{p}_1(t) - \alpha\, \bar{p}_2(t)\bar{p}_3(t) = -\left(\varsigma_1 + \varsigma_2\varsigma_3\right)\sqrt{c_0}\, \Omega \operatorname{sech}(\Omega\, t) \tanh(\Omega\, t).$$

Therefore, if $\varsigma_1 = \sigma_1$, $\varsigma_2 = \sigma_1\sigma_2$, $\varsigma_3 = -\sigma_2$, and $\sigma_1, \sigma_2 \in \{-1, 1\}$, then $\frac{d}{dt}\bar{p}_1(t) = \alpha\, \bar{p}_2(t)\bar{p}_3(t)$. It is easy to verify that in this case $\bar{p}(\cdot)$ is indeed an integral curve. Any integral curve $p(\cdot)$ (as described in the statement of the theorem) is then of the form $p(t) = \bar{p}(t + t_0)$ for some $\sigma_1, \sigma_2 \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ (see Figure 1b). $\square$

Finally, let us consider the integral curves of $\vec{H}_{3,\alpha}$. There is only one typical case. As before, we graph the level sets of $H_{3,\alpha}$ and C and their intersection in Figure 2. A straightforward computation gives the solutions.

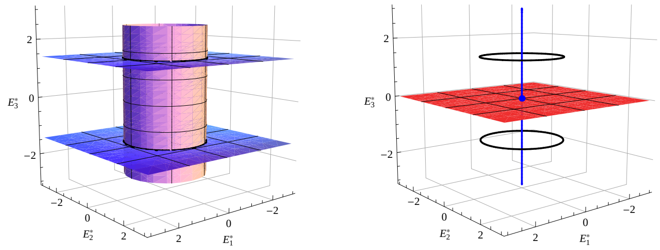


Figure 2: Typical case for $\vec{H}_{3,\alpha}$

Proposition 5.4. *Suppose* $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ *is an integral curve of* $\vec{H}_{3,\alpha}$ *such that* $H_{3,\alpha}(p(0)) = h_0 > 0$ *and* $C(p(0)) = c_0 > 0$*. Then there exist* $t_0 \in \mathbb{R}$ *and* $\sigma \in \{-1, 1\}$ *such that* $p(t) = \bar{p}(t + t_0)$ *for* $t \in (-\varepsilon, \varepsilon)$*, where*

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\,\sin(\sqrt{2\alpha h_0}\,t) \\ \bar{p}_2(t) = \sigma\sqrt{c_0}\,\cos(\sqrt{2\alpha h_0}\,t) \\ \bar{p}_3(t) = \sigma\sqrt{\frac{2h_0}{\alpha}}\,. \end{cases}$$

6 Concluding remark

Quadratic Hamilton-Poisson systems on Lie-Poisson spaces can be linked to invariant optimal control problems on Lie groups (with affine dynamics and quadratic cost). The Pontryagin Maximum Principle provides necessary conditions for optimality of trajectory-control pairs $(g(\cdot), u(\cdot))$. For an optimal control problem (1.1), it turns out that every (normal) extremal control is given by $u(t) = Q^{-1}\,\mathbf{B}^\top\,p(t)^\top$ ([8]). Here $\mathbf{B} = \begin{bmatrix} B_1 & \cdots & B_\ell \end{bmatrix}$ is a $3 \times \ell$ matrix and $p(\cdot) : [0, T] \to \mathfrak{se}\,(2)^*$ is an integral curve of the system $H(p) = \frac{1}{2}\,p\,\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top\,p^\top$. Notice that $\frac{1}{2}\,\mathbf{B}\,Q^{-1}\,\mathbf{B}^\top$ is a 3×3 positive semidefinite matrix of rank ℓ. Thus each extremal control is the image (under a linear map) of an integral curve of one of the systems considered in this paper.

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