On the local spectral properties of operator systems in Banach spaces

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Abstract. This paper contains some results of the theory of spectral and S-spectral systems of commuting operators. The restrictions and the quotients of spectral systems with respect to invariant subspaces are studied here and it proves that they are S-spectral systems. We also prove that if $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ is spectral system, then every operator a_i $(1 \leq i \leq n)$ is spectral, respectively if a is S-spectral system, then every operator a_i $(1 \leq i \leq n)$ is S_i-spectral, where $S_i = \pi_i S$ and π_i is the projection corresponding to the index. We remark that if a system $a = (a_1, a_2, ..., a_n)$ is S-spectral, then a is also S-decomposable.

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1 Introduction

In this paper we recall several notations and definitions from the specialized literature, which will be further needed.

Let X be a Banach space, let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X and let $\mathcal{P}_X$ be the set of the projectors on X.

Let $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ be a system of commuting operators (i.e. $a_i a_j = a_j a_i$, $1 \leq i, j \leq n$), let Y be a subspace of X invariant to a (i.e. $a_j Y \subset Y$, $1 \leq j \leq n$), let $b = a|Y = (a_1|Y, a_2|Y, ..., a_n|Y)$ be the restriction system of a to Y and let $\dot{a} = (\dot{a}_1, \dot{a}_2, ..., \dot{a}_n)$ be the quotient system induced by a on the quotient space $\dot{X} = X|Y$.

The system $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ is said to be *nonsingular* on X if the Koszul complex $E(a, X)$ is exact, where

$E(X, a) : 0 \to X = \Lambda^n[\sigma, X] \xrightarrow{\delta_n} \Lambda^{n-1}[\sigma, X] \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_3} \Lambda^2[\sigma, X] \xrightarrow{\delta_2} \Lambda^1[\sigma, X] \xrightarrow{\delta_1} \Lambda^0[\sigma, X] = X \to 0$

or, equivalently, the complex $F(a, X)$ is exact, where

$F(X, a) : 0 \to X = \Lambda^0[\sigma, X] \xrightarrow{\delta^0} \Lambda^1[\sigma, X] \xrightarrow{\delta^1} \Lambda^2[\sigma, X] \xrightarrow{\delta^2} ... \xrightarrow{\delta^{n-2}} \Lambda^{n-1}[\sigma, X] \xrightarrow{\delta^{n-1}} \Lambda^n[\sigma, X] = X \to 0$ (see [16], [20]).

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The complement in $\mathbb{C}^n$ of the set of those elements $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ for which the system $z - a = (z_1 - a_1, z_2 - a_2, \ldots, z_n - a_n)$ is nonsingular on X is said to be the *spectrum of* a *on* X and is denoted by $\sigma(a, X)$ ([17]). The complement of the reunion of all open sets V in $\mathbb{C}^n$ having the property that there is a form $\varphi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, C^\infty(V, X)]$ satisfying the equality $sx = (\alpha \oplus \overline{\partial})\varphi$ is called the *spectrum of* $x \in X$ *with respect to* a and is denoted by $sp(a, x)$ ([17]). The complement in $\mathbb{C}^n$ of the set of all $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ such that there are an open neighborhood V of z and n X-valued analytic functions $f_1, f_2, \ldots, f_n$ on V, satisfying the identity $x \equiv (\zeta_1 - a_1) f_1(\zeta) + \cdots + (\zeta_n - a_n) f_n(\zeta)$, $\zeta \in V$ is said to be *the local analytic spectrum of* x *with respect to* a and is denoted by $\sigma(a, x)$ ([17]). In [15], J. Eschmeier proved that the local spectra of x with respect to a are equal, $sp(a, x) = \sigma(a, x)$.

We shall say that the system $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ verifies the cohomology property (L) if $H^{n-1}(C^\infty(G, X), \alpha \oplus \overline{\partial}) = 0$, for any open set $G \subset \mathbb{C}^n$ ([16], [20]). We take: $X_{[a]}(F) = \{x \in X; sp(a, x) \subset F\}$ and $X_a(F) = \{x \in X; \sigma(a, x) \subset F\}$, where $F \subset \mathbb{C}^n$.

We denote by S_a the complement in $\mathbb{C}^n$ of the set of those points $\omega \in \mathbb{C}^n$ for which there is an open polydisc $D_\omega \ni \omega$ with the property that $H^p(A(D_\omega, X), \alpha_a) = 0$, for $0 \le p \le n-1$ (where $\alpha_a(z) = z - a$, $z \in \mathbb{C}^n$, $A(\Omega, X)$ is the space of all X-valued analytic functions on Ω, $\Omega \subset \mathbb{C}$ open). The set S_a will be called the *analytic spectral residuum* of the system a. If $S_a = \varnothing$, then we say that the system a has the *single-valued extension property* (or a verifies the cohomology property (L)) ([17], [23]).

2 Restrictions and quotients of spectral systems

Lemma 2.1. *If* $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ *is a commuting operator system,* $Y \subset X$ *a closed linear subspace invariant to* a *and* $\dot{a} = (\dot{a}_1, \dot{a}_2, \ldots, \dot{a}_n) \subset \mathbf{B}(\dot{X})$ *is the system induced by* a *on quotient space* $\dot{X} = X/Y$, *then:*

(1) $\sigma(a, X) \subset \sigma(a, Y) \cup \sigma(\dot{a}, \dot{X})$

(2) $\sigma(a, Y) \subset \sigma(a, X) \cup \sigma(\dot{a}, \dot{X})$

(3) $\sigma(\dot{a}, \dot{X}) \subset \sigma(a, X) \cup \sigma(a, Y)$.

Also we have

(4) $\sigma(a, X) \setminus \sigma(a, Y) = \sigma(\dot{a}, \dot{X}) \setminus \sigma(a, Y)$

(5) $\sigma(a, X) \setminus \sigma(\dot{a}, \dot{X}) = \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})$

(6) $\sigma(\dot{a}, \dot{X}) \setminus \sigma(a, X) = \sigma(a, Y) \setminus \sigma(a, X)$

(7) $\sigma(a, X) \cup \sigma(a, Y) = \sigma(a, X) \cup \sigma(\dot{a}, \dot{X}) = \sigma(a, Y) \cup \sigma(\dot{a}, \dot{X}) =$
$= \sigma(a, X) \cup \sigma(a, Y) \cup \sigma(\dot{a}, \dot{X})$.

Proof. The inclusions (1), (2) and (3) follow from Lemma 1.2. ([21]). The assertions and the primary verifications for the case of a single operator have been proved in [6] for the first time, independent of [21]. The equalities (4), (5), (6) and (7) are directly consequences of the inclusions (1), (2), (3). □

Remark 2.1. From the inclusions (1), (2), (3) of the previous lemma, one may easily notice that if a point $z \in \mathbb{C}^n$ belongs to one of the spectrum, then it also belongs to at least one more or to all the three ones, thus it can not belong to only one spectrum.

The previous inclusions have been extended for closed operators in [15].

For the unidimensional case, $n = 1$, $a = T \in \mathbf{B}(X)$, the result is significant for both operators T and $\dot{T}$ (if it is properly formulated, we believe it is also true for systems $n > 1$; see Proposition 2.11).

Definition 2.2. Let $\mathfrak{B}^n_S$ be the family of all Borelian sets B of $\mathbb{C}^n$ that have the property $B \cap S = \varnothing$ or $S \subset B$, where $S \subset \mathbb{C}^n$ is a compact fixed set.

An application $E_S : \mathfrak{B}^n_S \to \mathcal{P}_X$ is called a $(\mathbb{C}^n, X)$ *type S-spectral measure* if the following conditions are verified:

(1) $E_S(\varnothing) = 0$, $E_S(\mathbb{C}^n) = I$

(2) $E_S(B_1 \cap B_2) = E_S(B_1)E_S(B_2)$, $B_1, B_2 \in \mathfrak{B}^n_S$

(3) $E_S\left(\bigcup_{m=1}^{\infty} B_m\right)x = \sum_{m=1}^{\infty} E_S(B_m)x$, $B_m \in \mathfrak{B}^n_S$, $B_p \cap B_m = \varnothing$, if $p \neq m$, $x \in X$.

A commuting system $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ is called S-*spectral* if there is a $(\mathbb{C}^n, X)$ type S-spectral measure E_S such that:

(4) $a_j E_S(B) = E_S(B) a_j$, $B \in \mathfrak{B}^n_S$, $1 \leq j \leq n$

(5) $\sigma(a, E_S(B)X) \subset \overline{B}$, $B \in \mathfrak{B}^n_S$.

In case that $S = \varnothing$, we have $\mathfrak{B}^n_\varnothing = \mathfrak{B}(\mathbb{C}^n)$ (the family of all Borelian sets of $\mathbb{C}^n$), $\varnothing$-spectral measure is spectral measure and $\varnothing$-spectral system is spectral system (see [17]).

Remark 2.3. A commuting system $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ is S-spectral if and only if it is written as a direct sum $a = b \oplus c$, where $b \subset \mathbf{B}(X)$ is a spectral system and $\sigma(c, X) \subset S$.

Proof. Indeed, if a is S-spectral and E_S is the corresponding S-spectral measure, then one easily verifies that the map $E : \mathfrak{B}(\mathbb{C}^n) \to \mathcal{P}_X$ defined by $E(B) = E_S(B \cap \complement S)$, $B \in \mathfrak{B}(\mathbb{C}^n)$, is a spectral measure of system $b = a | E_S(\complement S)X$, while $c = a | E_S(S)X$, $\sigma(c, X) = \sigma(a, E_S(S)X) \subset S$. Conversely, if $b = (b_1, b_2, \dots, b_n) \subset \mathbf{B}(X_1)$ is spectral and $c = (c_1, c_2, \dots, c_n) \subset \mathbf{B}(X_2)$ is non spectral, with $\sigma(c, X_2) \not\supset \sigma(b, X_1)$, by putting $S = \sigma(c, X_2)$, $X = X_1 \oplus X_2$, $a = b \oplus c$, it results that the map $E_S : \mathfrak{B}^n_S \to \mathcal{P}_X$ defined by the equalities $E_S(B) = E(B) \oplus 0$, if $B \cap S = \varnothing$ and $E_S(B) = E(B) \oplus I_2$, if $B \supset S$, $B \in \mathfrak{B}^n_S$, is an S-spectral measure of a (where E is the spectral measure of b and I_2 is the identity operator in X_2). $\square$

Proposition 2.2. *Let $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ be a spectral system and let E be its spectral measure. Then each operator a_i $(1 \leq i \leq n)$ is spectral and its spectral measure E_i is given by the equality $E_i(B) = E(\pi_i^{-1}(B))$, where $B \in \mathfrak{B}(\mathbb{C})$ (the family of all Borelian sets of $\mathbb{C}$) and π_i is the corresponding projection.*

Proof. Let us notice that $\pi_i^{-1}(B) \in \mathfrak{B}(\mathbb{C}^n)$ if $B \in \mathfrak{B}(\mathbb{C})$. Obviously, we have

$$E_i(\varnothing) = E(\pi_i^{-1}(\varnothing)) = E(\varnothing) = 0, E_i(\mathbb{C}) = E(\pi_i^{-1}(\mathbb{C})) = E(\mathbb{C}^n) = I,$$

$$E_i(B_1 \cap B_2) = E(\pi_i^{-1}(B_1 \cap B_2)) = E(\pi_i^{-1}(B_1) \cap \pi_i^{-1}(B_2)) =$$
$$= E(\pi_i^{-1}(B_1))E(\pi_i^{-1}(B_2)) = E_i(B_1)E_i(B_2),$$

for $B_1, B_2 \in \mathfrak{B}(\mathbb{C})$.

If $(B_k)_{k\in\mathbb{N}} \subset \mathfrak{B}(\mathbb{C})$ is a sequence of disjunct Borelian sets, then $(\pi_i^{-1}(B_k))_{k\in\mathbb{N}} \subset \mathfrak{B}(\mathbb{C}^n)$ is also a sequence of disjunct sets, hence for any $x \in X$ we have

$$E_i\left(\bigcup_{k=1}^{\infty} B_k\right)x = E\left(\pi_i^{-1}\left(\bigcup_{k=1}^{\infty} B_k\right)\right)x = E\left(\bigcup_{k=1}^{\infty}(\pi_i^{-1}(B_k))\right)x =$$
$$= \sum_{k=1}^{\infty} E(\pi_i^{-1}(B_k))x = \sum_{k=1}^{\infty} E_i(B_k)x.$$

It further follows that

$$a_iE_i(B) = a_iE(\pi_i^{-1}(B)) = E(\pi_i^{-1}(B))a_i = E_i(B)a_i$$

and

$$\sigma(a_i|E_i(B)X) = \pi_i\sigma(a, E_i(B)X) = \pi_i\sigma(a, E(\pi_i^{-1}(B))X) \subset$$
$$\subset \pi_i(\overline{\pi_i^{-1}(B)}) \subset \pi_i(\pi_i^{-1}(\overline{B})) = \overline{B},$$

for any $B \in \mathfrak{B}(\mathbb{C})$ (for the inclusion $\overline{\pi_i^{-1}(B)} \subset \pi_i^{-1}(\overline{B})$, see [18]), hence a_i is a spectral operator with spectral measure E_i $(1 \le i \le n)$. □

Definition 2.4. For the Banach space X, let $\mathcal{S}(X)$ be the family of all closed linear subspaces of X, let $S \subset \mathbb{C}^n$ be a compact set and let $\mathfrak{F}_S(\mathbb{C}^n)$ be the family of all closed sets $F \subset \mathbb{C}^n$ which have the property: either $F \cap S = \varnothing$ or $F \supset S$.

We shall call S-*spectral capacity* an application $\mathcal{E}_S : \mathfrak{F}_S(\mathbb{C}^n) \to \mathcal{S}(X)$ that satisfies the following properties:

1. $\mathcal{E}_S(\varnothing) = \{0\}$, $\mathcal{E}_S(\mathbb{C}^n) = X$
2. $\mathcal{E}_S\left(\bigcap_{i=1}^{\infty} F_i\right) = \bigcap_{i=1}^{\infty}\mathcal{E}_S(F_i)$, for any sequence $\{F_i\}_{i\in\mathbb{N}} \subset \mathfrak{F}_S(\mathbb{C}^n)$
3. for any finite open S-covering $G_S \bigcup \{G_j\}_{j=1}^m$ of $\mathbb{C}^n$ we have

$$X = \mathcal{E}_S(\overline{G}_S) + \sum_{j=1}^{m} \mathcal{E}_S(\overline{G}_j).$$

A system of commuting operators $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ is said to be S-*decomposable* if there is an S-spectral capacity $\mathcal{E}_S$ such that:

4. $a_j\mathcal{E}_S(F) \subset \mathcal{E}_S(F)$, for any $F \in \mathfrak{F}_S(\mathbb{C}^n)$, $1 \le j \le n$
5. $\sigma(a, \mathcal{E}_S(F)) \subset F$, for any $F \in \mathfrak{F}_S(\mathbb{C}^n)$.

For $S = \varnothing$, $\mathfrak{F}_\varnothing(\mathbb{C}^n) = \mathfrak{F}(\mathbb{C}^n)$ is the family of all closed sets $F \subset \mathbb{C}^n$, the $\varnothing$-spectral capacity is said to be spectral capacity and the $\varnothing$-decomposable system is decomposable (see [17]).

Proposition 2.3. *Let $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ be an S-spectral system and let E_S be its S-spectral measure. Then a is an S-decomposable system, where*

$$\mathcal{E}_S(F) = E_S(F)X, \;\; F \in \mathfrak{F}_S(\mathbb{C}^n)$$

is the S-spectral capacity of a.

Proof. From Remark 2.3, it follows that $a = b \oplus c$, where $b = a|E_S(\mathbb{C}^n \setminus S)X$ is a spectral system, hence decomposable (Proposition 3.1.3, [17]), $c = a|E_S(S)X$, $\sigma(a, \mathcal{E}_S(S)) = \sigma(a, E_S(S)X) = \sigma(c, X) \subset S$.

It can be directly shown that the map $\mathcal{E}_S : \mathfrak{F}_S(\mathbb{C}^n) \to \mathcal{S}(X)$ defined by

$$\mathcal{E}_S(F) = E_S(F)X$$

is an S-spectral capacity for a, since for every $B \in \mathfrak{B}^n_S$, we have $E_S(B) = E_S(B \cap B) = E^2_S(B)$, thus $E_S(B)$ are (linear bounded) projectors on X and $\mathcal{E}_S(F) = E_S(F)X$ are closed subspaces of X, for $F \in \mathfrak{F}_S(\mathbb{C}^n)$. □

Proposition 2.4. *Let $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ be an S-spectral system and let E_S be its S-spectral measure. Then each operator a_i $(1 \le i \le n)$ is S_i-spectral, where $S_i = \pi_i S$ and π_i is the corresponding projection.*

Proof. Since a is S-spectral system, then we have $a = b \oplus c$ (see the proof of Proposition 2.3), where $b = a|E_S(\complement S)X$ is a spectral system, $c = a|E_S(S)X$ and $\sigma(c, E_S(S)X) \subset S$. Then $a_i = b_i \oplus c_i$, where $b_i = a_i|E_S(\complement S)X$ is a spectral operator (Proposition 2.2) and $\sigma(c_i|E_S(S)X) \subset S_i$ $(1 \le i \le n)$. According to Remark 3.1, [8], each operator a_i is S_i-spectral, for $1 \le i \le n$. □

Lemma 2.5. *Let X be a Banach space and let X_1, X_2 be two closed linear subspaces of X such that $X_1 \cap X_2 = \{0\}$ and $X_1 + X_2$ closed. If $Y_i \subset X_i$ $(i = 1, 2)$ are two closed linear subspaces, then $Y_1 + Y_2$ is closed.*
Moreover, if $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ is a decomposable (spectral) system and Z_1, Z_2 are two closed invariant subspaces to a such that $\sigma(a, Z_1) \cap \sigma(a, Z_2) = \varnothing$, then $Z_1 + Z_2$ is closed.

Proof. Indeed, if $(y_n)_n \subset Y_1 + Y_2$, $y_n \to y \in X$, then $y_n = y^1_n + y^2_n$, $y^i_n \in Y_i$ $(i = 1, 2)$. Since $X_1 + X_2$ is closed, by the closed graph theorem, it follows that $y^i_n \to y_i \in Y_i$ $(i = 1, 2)$, hence $y = y_1 + y_2 \in Y_1 + Y_2$, i.e. $Y_1 + Y_2$ is closed.

For the second part of the proof, we have $Z_1 \subset X_a(\sigma(a, Z_1))$, $Z_2 \subset X_a(\sigma(a, Z_2))$, $X_a(\sigma(a, Z_1)) \cap X_a(\sigma(a, Z_2)) \subseteq X_a(\sigma(a, Z_1) \cap \sigma(a, Z_2)) = X_a(\varnothing) = \{0\}$, while

$$X_a(\sigma(a, Z_1)) \oplus X_a(\sigma(a, Z_2)) = X_a(\sigma(a, Z_1) \cup \sigma(a, Z_2))$$

is closed subspace (Proposition 2.2.8, [17]), hence, from the first part of the proof, it results that $Z_1 + Z_2$ is closed. □

Remark 2.5. Let $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ be a decomposable (spectral) system and Z_1, $Z_2 \subset X$ two closed invariant subspaces to a, with $\sigma(a, Z_1) \cap \sigma(a, Z_2) = \varnothing$. Let $\dot{a}$ be the system induced by a on the quotient space $\dot{X} = X/Z_1$ and let $\varphi : X \to \dot{X}$ be the canonical application. Then $Z_1 + Z_2$ is closed, Z_2 can be identified with $\dot{Z}_2 = \varphi(Z_2)$ (since Z_2 and $\dot{Z}_2$ are topologically isomorphic), $a|Z_2$ and $\dot{a}|\dot{Z}_2$ are similar and $\sigma(a, Z_2) = \sigma(\dot{a}, \dot{Z}_2)$.

Lemma 2.6. *Let $T \in \mathbf{B}(X)$, let Y be an invariant subspace to T and let $\dot{T}$ be the operator induced by T on the quotient space $\dot{X} = X/Y$. If T and $\dot{T}$ have the single-valued extension property, then $X_T(\sigma(T|Y) \setminus \sigma(\dot{T})) \subset Y$.*
Similar, if $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$, Y is an invariant subspace to a, $\dot{a} = (\dot{a}_1, \dot{a}_2, \dots, \dot{a}_n) \subset \mathbf{B}(\dot{X})$ is the system induced by a on $\dot{X} = X/Y$ with $S_a = S_{\dot{a}} = \varnothing$, we have $X_{[a]}(\sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})) \subset Y$ (where S_a, $S_{\dot{a}}$ are the analytic spectral residuum of a, respectively $\dot{a}$; see [23]).

Proof. If $x \in X_T(\sigma(T|Y) \setminus \sigma(\dot{T}))$, we have $\sigma_T(x) \subset \sigma(T|Y) \setminus \sigma(\dot{T})$ and

$$\sigma_{\dot{T}}(\dot{x}) \subset \sigma_T(x) \cap \sigma(\dot{T}) \subset (\sigma(T|Y) \setminus \sigma(\dot{T})) \cap \sigma(\dot{T}) = \varnothing ,$$

hence $\dot{x} = \dot{0}$ and consequently $x \in Y$ (because $S_T = \varnothing$ and $S_{\dot{T}} = \varnothing$ imply that $\gamma_T(x) = \sigma_T(x)$, $\gamma_{\dot{T}}(\dot{x}) = \sigma_{\dot{T}}(\dot{x})$ and $\sigma_{\dot{T}}(\dot{x}) \subset \sigma_T(x)$; see Proposition 2.1, [6]).

J. Eschmeier proved in [15] that the local spectra of x with respect to a are equal, i.e. $\sigma(a, x) = sp(a, x)$, for any $x \in X$.

Let now suppose that $x \in X_{[a]}(\sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X}))$, hence $sp(a, x) = \sigma(a, x) \subset \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})$. We make the notation $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{C}^n$ and from

$$(\zeta_1 - a_1)f_1(\zeta) + (\zeta_2 - a_2)f_2(\zeta) + ... + (\zeta_n - a_n)f_n(\zeta) \equiv x$$

with f_j analytic functions ($j = 1, 2, \dots, n$), it follows that

$$(\zeta_1 - \dot{a}_1)\dot{f}_1(\zeta) + (\zeta_2 - \dot{a}_2)\dot{f}_2(\zeta) + ... + (\zeta_n - \dot{a}_n)\dot{f}_n(\zeta) \equiv \dot{x}$$

hence $\sigma(\dot{a}, \dot{x}) \subset \sigma(a, x) \subset \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})$. Then

$$\sigma(\dot{a}, \dot{x}) \subset (\sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})) \cap \sigma(\dot{a}, \dot{X}) = \varnothing$$

hence $\dot{x} = \dot{0}$ and thus $X_{[a]}(\sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})) \subset Y$. □

Lemma 2.7. *Let $a = (a_1, a_2, \dots, a_n) \subset \mathbf{B}(X)$ be a spectral system with its spectral measure E and let $A \subset \mathbb{C}^n$ be Borelian. Then the restriction $b = a|E(A)X$ is a spectral system with the spectral measure E_A given by $E_A(B) = E(A \cap B)$, for any $B \subset \mathbb{C}^n$ Borelian.*

Proof. One easily verifies that E_A is a spectral measure of b; the fact that E_A is a spectral measure for b follows by the equalities

$$b_j E_A(B) = b_j E(A \cap B) = E(A \cap B) b_j = E_A(B) b_j, \ \ 1 \le j \le n$$

where $b_j = a_j|Y$, $Y = E(A)X$ and from the relations

$$\sigma(b, E_A(B)Y) = \sigma(b, E(A \cap B)X) = \sigma(a, E(A \cap B)X) \subset \overline{B}$$

for $B \subset \mathbb{C}^n$ Borelian. □

Proposition 2.8. *Let $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ be a spectral system with the spectral measure E, let Y be a linear closed subspace invariant to a, let $\dot{a} = (\dot{a}_1, \dot{a}_2, \ldots, \dot{a}_n)$ be the system induced by a on $\dot{X} = X/Y$ and let $\varphi : X \to \dot{X}$ be the canonical application. Then $\dot{a} = \dot{b} \oplus \dot{c}$, where $\dot{b} = \dot{a}|\varphi(E(\sigma')X)$ is spectral, $\dot{c} = \dot{a}|\varphi(E(\sigma)X)$, $\sigma = \sigma(a, Y)$, $\sigma' = \sigma(\dot{a}, \dot{X}) \setminus \sigma(a, Y)$ and $\sigma(\dot{c}, \varphi(E(\sigma)X)) \subset S = \sigma(\dot{a}, \dot{X}) \cap \sigma(a, Y)$.*

Proof. The system $a|E(\sigma')X$ is spectral (Lemma 2.7) and since $Y \subset X_{[a]}(\sigma) = E(\sigma)X$ (Proposition 3.1.3 and Theorem 2.2.1, [17]), we have $Y \cap E(\sigma')X = \{0\}$. Because $E(\sigma')X + Y$ is closed (Lemma 2.5), then $E(\sigma')X + Y = E(\sigma')X \oplus Y$ and according to Remark 2.5, $\varphi(E(\sigma')X)$ can be identified with $E(\sigma')X$, respectively $\dot{b} = \dot{a}|\varphi(E(\sigma')X)$ with $a|E(\sigma')X$, meaning $\dot{b}$ is spectral.

It is easily to verify that $\varphi(X_{[a]}(\sigma)) = \dot{X}_{[\dot{a}]}(\sigma) = \dot{X}_{[\dot{a}]}(S)$ is spectral maximal space of $\dot{a}$, consequently

$$\sigma(\dot{c}, \varphi(E(\sigma)X)) = \sigma(\dot{a}, \dot{X}_{[\dot{a}]}(S)) \subset S.$$

□

Proposition 2.9. *Let $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ be a spectral system having the spectral measure E, let Y be a closed invariant subspace to a with $X_a(\sigma) \subset Y$, where $\sigma = \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})$. Let also $S = \sigma(a, Y) \cap \sigma(\dot{a}, \dot{X})$ and $b = a|Y$. Then $b|E(\sigma)Y$ and $b|\overline{X_a(\sigma)}$ are spectral systems and $b = (b|E(\sigma)Y) \oplus (b|E(S)Y)$, with $\sigma(b, E(S)Y) \subset S \cap \sigma(b, Y)$.*

Proof. σ being open in $\sigma(b, Y)$ and also in $\sigma(a, X)$ (because $\sigma = \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X}) = \sigma(a, X) \setminus \sigma(\dot{a}, \dot{X})$; see Lemma 2.1), there is a growing sequence of closed sets $(\sigma_n)_{n \in \mathbb{N}}$ with $\sigma = \bigcup_{n \in \mathbb{N}} \sigma_n$. From the continuity of the measure $E(\cdot)x$, it results that $E(\sigma) = \lim_{n \to \infty} E(\sigma_n)$, therefore $E(\sigma_n)X = X_a(\sigma_n) \subset X_a(\sigma) \subset Y$ (Proposition 3.1.3, [17]) implies that $E(\sigma)X \subset \overline{X_a(\sigma)} \subset Y$. The closed subspaces $E(\sigma)X$ and $\overline{X_a(\sigma)}$ are invariant to a and to spectral measure E, thus $a|E(\sigma)Y$ and $a|\overline{X_a(\sigma)}$ are spectral systems. $E(\sigma)|Y$ and $E(S)|Y$ are projectors on Y, $E(\sigma)Y$ and $E(S)Y$ are closed subspaces and $Y = E(\sigma(a, Y))Y = E(\sigma)Y \oplus E(S)Y$. We also obtain the relations

$$b = (b|E(\sigma)Y) \oplus (b|E(S)Y)$$

$$\sigma(b, E(S)Y) \subset \sigma(b, E(S)X) \cap \sigma(a, Y) \subset \widetilde{S} \cap \sigma(a, Y)$$

(if $A \subset \mathbb{C}$ is bounded, we denote $\widetilde{A} = \mathbb{C} \setminus D^\infty$, where D^∞ is the unbounded component of $\mathbb{C} \setminus A$). □

Theorem 2.10. *Let $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ be a spectral system, with its spectral measure E, let Y be a closed subspace invariant to a such that $X_a(\sigma) \subset Y$, where $\sigma = \sigma(a, Y) \setminus \sigma(\dot{a}, \dot{X})$, $S = \sigma(a, Y) \cap \sigma(\dot{a}, \dot{X})$. Then the systems $a|Y$ and $\dot{a}$ are S-spectral.*

Proof. The assertions follow easily by Propositions 2.8, 2.9 and Remark 2.3. □

Proposition 2.11. *Let $T \in \mathbf{B}(X)$, let Y be a linear closed subspace invariant to T and let $\dot{T}$ be the quotient operator induced by T on the quotient space $\dot{X} = X/Y$. If D^{∞} is the unbounded component and $(D_n)_{n\in\mathbb{N}}$ are the bounded components of $\rho(T)$, where $\rho(T)$ is the resolvent set of T, then $D^{\infty} \cap \sigma(\dot{T}) = \varnothing$ and $D_n \subset \sigma(\dot{T})$ if and only if $D_n \subset \sigma(T|Y)$ (i.e. if and only if there is $\lambda_0 \in D_n$ such that $R(\lambda_0, T)Y \nsubseteq Y$, where $R(\lambda, T) = (\lambda I - T)^{-1}$ is the resolvent of T).*

Proof. I.E. Seroggs (Duke Math. I. 21, 1, 95-111, 1959) proves that $D^{\infty} \cap \sigma(T|Y) = \varnothing$ and $D_n \subset \sigma(T|Y)$ if and only if there is $\lambda_0 \in D_n$ such that $R(\lambda_0, T)Y \nsubseteq Y$. According to Lemma 2.1 and Remark 2.1, $D_n \subset \sigma(\dot{T})$ if only if $D_n \subset \sigma(T|Y)$ and $\lambda \in D^{\infty}$ implies $\lambda \notin \sigma(\dot{T})$ (if $\lambda \in \sigma(\dot{T})$, $\lambda \notin \sigma(T)$ then $\lambda \in \sigma(T|Y)$, contradictions!). □

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