Existence of fractional parametric Cauchy problem

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Abstract. By employing the concept of holomorphic flow, we establish the existence solution of fractional parametric Cauchy problem in complex Banach space of the form $D_z^\alpha u_t(z) = f_\alpha(t, z, u_t(z))$, $t \in [0,\infty), 0 \le \alpha < 1$, subject to $u_t(0) = 0$ in sense of Srivastava-Owa fractional operators. Moreover, by using the concept of admissible functions in complex Banach spaces, we show that the solution remains in the unit disk.

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1 Introduction

Fractional calculus and its applications (that is the theory of derivatives and integrals of any arbitrary real or complex order) has importance in several widely diverse areas of mathematical physical, control theory, mechanics and engineering sciences. It generalized the ideas of integer order differentiation and n-fold integration. Fractional derivatives introduce an excellent instrument for the description of general properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description of properties of gases, liquids and rocks, and in many other fields (see [2,14,15,16,18,22]).

The class of fractional differential equations of various types plays important roles and tools not only in mathematics but also in physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. Naturally, such equations required to be solved. Many studies on fractional calculus and fractional differential equations, involving different operators such as Riemann-Liouville operators [13], Erdlyi-Kober operators [5], Weyl-Riesz operators [17], Caputo operators [12] and Grnwald-Letnikov operators [19], have appeared during the past three decades. The existence of holomorphic solutions for different kind of fractional differential equations in complex domain are imposed in [6-9].

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By employing the concept of holomorphic flow, we establish the existence solution of fractional parametric Cauchy problem in sense of Srivastava-Owa fractional operators in complex domain. Moreover, by using the concept of admissible functions in complex Banach spaces, we show that the solution remain in the unit disk.

2 Preliminaries

In [21], Srivastava and Owa, gave definitions for fractional operators (derivative and integral) in the complex z-plane $\mathbb{C}$ as follows:

Definition 2.1. The fractional derivative of order α is defined, for a function $f(z)$, by

$$D_z^\alpha f(z) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\alpha} d\zeta; \quad 0 \le \alpha < 1,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin and the multiplicity of $(z-\zeta)^{-\alpha}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Definition 2.2. The fractional integral of order α is defined, for a function $f(z)$, by

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha > 0,$$

where the function $f(z)$ is analytic in simply-connected region of the complex z-plane ($\mathbb{C}$) containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.

Remark 2.1.

$$D_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} z^{\mu-\alpha}, \;\; \mu > -1; \; 0 \le \alpha < 1$$

and

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \;\; \mu > -1; \; \alpha > 0.$$

Definition 2.3. Let X be a topological space. A family of mappings $\varphi \in C([0,\infty) \times X, X)$ is called a *flow* on X if
(i) $\varphi(0,x) = x, \; \forall x \in X,$
(ii) $\varphi(t+s,x) = \varphi(s,\varphi(t,x)), \; \forall x \in X, \; t,s \in [0,\infty).$

We also write $\varphi(t,x) = \varphi_t(x)$. When φ_t is a flow of holomorphic mappings from a domain $G \subseteq \mathbb{C}$ into itself we say that φ_t is a *holomorphic flow* on G. It was shown in [10], every holomorphic flow on G is univalent. Furthermore, we have the following result which can be found in [10,11]:

Lemma 2.1. *If φ_t is a nontrivial holomorphic flow on $\mathbb{C}$, then every φ_t can have at most one fixed point in $\mathbb{C}$. Moreover, if φ_t does not have fixed points in $\mathbb{C}$, then $\varphi_t(z) = z + Kt$ for some $K \in \mathbb{C}, K \neq 0$, i.e. φ_t is a translation flow.*

Definition 2.4. Let $G(t) \subseteq \mathbb{C}$ be a family of simply connected domains such that

(i) $0 \in G(s) \subseteq G(t)$ if $0 \leq s < t < \infty$,
(ii) $G(t_n) \to G(t)$ if $t_n \to t$, and $G(t_n) \to \mathbb{C}$ if $t_n \to \infty$.
Let $U = \{z : |z| < 1\}$ be the open unit disk, and $f(z,t) : U \times [0,\infty) \to G(t)$ be the univalent function such that $f(0,t) = 0$ and $f'(0,t) > 0$. Then

$$f(z,s) \preceq f(z,t) \quad 0 \leq s < t < \infty.$$

i.e. there exists a univalent map $\varphi(z,s,t)$ from the disk into the disk, fixing 0, so that $f(z,s) = f(\varphi(z,s,t),t)$ with the flow property

$$\varphi(z,s,\tau) = \varphi(\varphi(z,s,t),t,\tau) := \varphi_{s,\tau}(z), \quad \varphi_{t,t}(z) = id_U,$$

where $0 \leq s \leq t \leq \tau < \infty$. The family $f(z,t)$ is called a *Loewner chain.*

3 The fractional Cauchy problems

In this section, we establish the solution for the fractional parametric Cauchy problem of the form

$$D_z^\alpha u_t(z) = f_\alpha(t,z,u_t(z)), \quad z \in U,\ 0 \leq \alpha < 1,\ t \in [0,\infty), \tag{3.1}$$

where $u_t(0) = 0$. Here, we concern about functions of the form

$$f_\alpha(t,z,u_t(z)) = \frac{1}{\Gamma(2-\alpha)}\Big(e^t z^{1-\alpha} + b_2(t) z^{2-\alpha} + ...\Big), \tag{3.2}$$

such that $b_j(0) = 0$, $\forall j = 2,3,...$. These functions (for every fixed t and for some α) are holomorphic univalent map of U onto any subset of $\mathbb{C}$ containing the origin.

Theorem 3.1. *Let the function $f_\alpha : [0,\infty) \times U \times \mathbb{C} \to \mathbb{C}$ be a Loewner chain for some α. Then the problem (3.1) has a solution in $\mathbb{C}$.*

Proof. Assume that U is a topological space containing absolutely converge analytic functions. Let $\mathcal{B}$ be a complex Banach space of all holomorphic bounded functions on the unit disk endow with the sup norm. Define an operator $\Phi_t : \mathcal{B} \longrightarrow \mathcal{B}$ by

$$(\Phi u)_t(z) = I_z^\alpha f_\alpha(t,z,u_t(z)), \quad z \in U,\ t \in [0,\infty). \tag{3.3}$$

This operator is bounded

$$\begin{aligned} |(\Phi u)_t(z)| &= |\frac{1}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} f_\alpha(t,\zeta,u_t(\zeta)) d\zeta| \\ &\leq |f_\alpha|| \frac{1}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} d\zeta| \\ &\leq \frac{|f_\alpha|}{\Gamma(\alpha+1)} := r. \end{aligned}$$

Thus Φu is bounded in the set $S_r := \{w \in \mathcal{B} : \|w\| \leq r, r > 0\}$. By continuity of f, we have Φ_t is continuous on U. Our aim is to apply Lemma 2.1; we prove that Φ_t is holomorphic flow. In view of Remark 2.1, a computation implies

$$(\Phi u)_0(z) = I_z^\alpha f_\alpha(0,z,u_0(z)) = z.$$

Since f is a a Loewner chain, then from Definition 4.1, yields that Φ_t is a holomorphic flow; thus in virtue of Lemma 2.1, Φ_t has at most one fixed point on U. By formula (3.2), we pose that

$$(\Phi u)_t(z) = e^t z + \frac{\Gamma(3-\alpha)}{2\Gamma(2-\alpha)} b_2(t) z^2 + \ldots;$$

hence Φ_t has a fixed point corresponding to the solution of the problem (3.1). This completes the proof. □

Remark 3.1. Problem (3.1) has a locally univalent solution on U. This solution satisfies the following properties:
(i) It takes the form

$$u(t,z) = e^t z + a_2(t) z^2 + \ldots.;$$

(ii) It is absolutely continuous in $t \geq 1$ for all $z \in U$;

Definition 3.1. Let $u_t(z)$ be a solution for the problem (3.1). Then it is called a stable on U, if there exists a positive function $\Psi : [0,\infty) \times U \times U \longrightarrow [0,\infty)$ such that

$$|u_t(z) - u_t(w)| \leq \Psi(t,z,w), \quad z,w \in U.$$

Theorem 3.2. *Let u_t be a solution of problem (3.1). Then u_t is stable on U.*

Proof. By Remark 3.1, we have

$$\begin{aligned} |u_t(z) - u_t(w)| &= |e^t(z-w) + a_2(t)(z^2 - w^2) + \ldots.| \\ &\leq e^t|z-w| + |a_2(t)||z^2 - w^2| + \ldots \\ &\leq e^t|z-w| + |a_2(t)||z-w|^2 + \ldots \\ &:= \Psi(t,z,w), \quad z,w \in U. \end{aligned}$$

Hence u_t is stable. □

4 Extension to Banach spaces

In this section, we shall provide an extension of the solution to holomorphic vector-valued function. Let X, Y represent complex Banach spaces. The class of admissible functions $\mathcal{G}(X,Y)$, consists of those functions $g : [0,\infty) \times U \times X \to Y$ that satisfy the admissibility conditions:

$$\|g_t(z,x)\| \geq 1, \text{ when } \|x\| = 1, \quad z \in U,\ x \in X. \tag{4.1}$$

We need the following results:

Lemma 4.1. [4] *If $f : D \to X$ is holomorphic, then $\|f\|$ is a subharmonic of $z \in D \subset \mathbb{C}$. It follows that $\|f\|$ can have no maximum in D unless $\|f\|$ is of constant value throughout D.*

Theorem 4.1. *Let $f \in \mathcal{G}(X,Y)$. If $u_t : U \to X$ is a holomorphic vector-valued solution for (3.1) defined in the unit disk U, with $u(0) = \Theta$ (the zero vector in X), then*

$$\|f(z,x)\| < 1 \Longrightarrow \|u(z)\| < 1. \tag{4.2}$$

Proof. Assume that $\|u(z)\| \not< 1$ for $z \in U$. Thus, there exists a point $z_0 \in U$ for which $\|u_t(z_0)\| = 1$. According to Lemma 4.1, we have

$$\|u_t(z)\| < 1, \quad z \in U_{r_0} = \{z : |z| < |z_0| = r_0\},$$

and

$$\max_{|z| \le |z_0|} \|u_t(z)\| = \|u_t(z_0)\| = 1.$$

Hence from equation (4.1), we deduce $\|f(z_0, x_\Theta)\| \ge 1$, which contradicts the hypothesis in (4.2), we must have $\|u_t(z)\| < 1$. □

Theorem 4.2. *Let f_α be a contraction holomorphic mapping for the hyperbolic metric space. Then there exists a constant $R < 1$ (depends on α, t and r) such that*

$$|u_t(z)| < R, \quad |z| < r < 1,\ 0 \le t < \infty. \tag{4.3}$$

Proof. It is enough to show that f_α is bounded in U. Assume this is not true. Then there exist two sequences $\{z_n\}, \{t_n\}$ such that $|z_n| < r$, $z_n \to z_0$, $t_n \in [0, \infty)$, $t_n \to t_0$, and $|f_\alpha(t_n, z_n, u_{t_n}(z_n))| \to 1$. Since the map f_α is a contraction for the hyperbolic metric, we have that

$$\rho_U\Big(f_\alpha(t_n, z_n, u_{t_n}(z_n)), f_\alpha(t_n, z_0, u_{t_n}(z_0))\Big) \le \rho_U\Big(z_n, z_0\Big),$$

this implies that

$$|f_\alpha(t_n, z_0, u_{t_n}(z_0))| \to 1.$$

In addition, we have

$$\rho_U\Big(f_\alpha(t, z_0, u_0(z_0)), f_\alpha(t_n, z_0, u_{t_n}(z_0))\Big) \le \rho_U\Big(f_\alpha(t, z_0, u_0(z_0)), z_0\Big) < \infty.$$

This yields that

$$|f_\alpha(t, z_0, u_{t_n}(z_0))| \longrightarrow 1;$$

but

$$f_\alpha(t_n, z_0, u_{t_n}(z_0)) \longrightarrow f_\alpha(t, z_0, u_t(z_0)) \in U,$$

which contradicts the assumption of the theorem; hence there is a constant $R < 1$ satisfying (4.3). □

Theorem 4.3. *Let $f_\alpha(t, z, u_t(z))$ be measurable in $t \in [0, \infty)$ for all $z \in U$, holomorphic in $z \in U$ for all $t \in [0, \infty)$ and for any compact set $U_1 \in U$ and for all $T > 0$ there exists a non-negative function $k \in L^d([0, T], \mathbb{R})$, $d \in [1, \infty]$ (depends on T, U_1) such that*

$$|f_\alpha(t, z, u_t(z))| \le k(t), \quad t \in [0, T].$$

Then there exists a non-negative function $\widetilde{k} \in L^d([0, T], \mathbb{R})$, $d \in [1, \infty]$ such that

$$|f_\alpha(t, z, u_t(z)) - f_\alpha(t, w, u_t(w))| \le \widetilde{k}(t)|u_t(z) - u_t(w)| \tag{4.4}$$

for almost every $t \in [0, T]$.

Proof. Fix compact sets $U_1 := \{z : |z| \le r < 1\}$ and $U_2 := \{u : |u| \le R_1 < R < 1\}$. Assume that f_α is holomorphic in U_1 and U_2 and continuous in ∂U_1 and ∂U_2. Denote by $\widetilde{U} := \partial U_1 \times \partial U_2$. Taking $z, w \in U_1$ and $u_t(z), u_t(w) \in U_2$, by Cauchy integral's formula, we have

$$
\begin{aligned}
&|f_\alpha(t,z,u_t(z)) - f_\alpha(t,w,u_t(w))| \\
&= \Big|(\frac{1}{2\pi i})^2 \int\int_{\widetilde{U}} \frac{f_\alpha(t,\zeta,u_t(\zeta))}{(\zeta - z)(u_t(\zeta) - u_t(z))} d\zeta - \frac{f_\alpha(t,\zeta,u_t(\zeta))}{(\zeta - w)(u_t(\zeta) - u_t(w))} d\zeta \Big| \\
&= \Big|(\frac{1}{2\pi i})^2 \int\int_{\widetilde{U}} f_\alpha(t,\zeta,u_t(\zeta)) \\
&\times \frac{(\zeta - w)(u_t(\zeta) - u_t(w)) - (\zeta - z)(u_t(\zeta) - u_t(z))}{(\zeta - z)(u_t(\zeta) - u_t(z))(\zeta - w)(u_t(\zeta) - u_t(w))} d\zeta \Big| \\
&\le \frac{1}{(2\pi)^2} \int\int_{\widetilde{U}} |f_\alpha(t,\zeta,u_t(\zeta))| \\
&\times \frac{|(\zeta - w)(u_t(\zeta) - u_t(w)) - (\zeta - z)(u_t(\zeta) - u_t(z))|}{|(\zeta - z)(u_t(\zeta) - u_t(z))(\zeta - w)(u_t(\zeta) - u_t(w))|} |d\zeta| \\
&\le \frac{1}{\pi^2} \int\int_{\widetilde{U}} k(t) \frac{|u_t(z) - u_t(w)|}{(1-r)^2(1-R_1)^2} |d\zeta| \\
&\le \frac{k(t)|u_t(z) - u_t(w)|}{(1-r)^2(1-R_1)^2} \\
&:= \widetilde{k}(t)|u_t(z) - u_t(w)|,
\end{aligned}
$$

which completes the proof. $\square$

Theorem 4.4. *Let the assumptions of Theorem 4.3 hold. Then there exists a non-negative function* $\widehat{k} \in L^d([0,T],\mathbb{R})$, $d \in [1,\infty]$ *such that*

$$
|f_\alpha(t,z,u_t(z)) - f_\alpha(t,z,v_t(z))| \le \widehat{k}(t)|u_t(z) - v_t(z)| \tag{4.5}
$$

for almost every $t \in [0,T]$.

Proof. In the same manner of the Proof of Theorem 4.3, we obtain

$$
\begin{aligned}
&|f_\alpha(t,z,u_t(z)) - f_\alpha(t,z,v_t(z))| \\
&= \Big|(\frac{1}{2\pi i})^2 \int\int_{\widetilde{U}} \frac{f_\alpha(t,\zeta,u_t(\zeta))}{(\zeta - z)(u_t(\zeta) - u_t(z))} d\zeta - \frac{f_\alpha(t,\zeta,v_t(\zeta))}{(\zeta - z)(v_t(\zeta) - v_t(z))} d\zeta \Big| \\
&\le \frac{1}{(2\pi)^2} \int\int_{\widetilde{U}} k(t) \frac{|(v_t(\zeta) - v_t(z)) - (u_t(\zeta) - u_t(z))|}{|(\zeta - z)(u_t(\zeta) - u_t(z))(v_t(\zeta) - v_t(z))|} |d\zeta| \\
&\le \frac{1}{\pi^2} \int\int_{\widetilde{U}} k(t) \frac{|u_t(z) - v_t(z)|}{(1-r)(1-R_1)^2} |d\zeta| \le \frac{k(t)|u_t(z) - v_t(z)|}{(1-r)(1-R_1)^2} \\
&:= \widehat{k}(t)|u_t(z) - v_t(z)|,
\end{aligned}
$$

whence the claim follows. $\square$

Corollary 4.1. *Let the hypotheses of Theorem 4.4 hold. If* $\frac{\widehat{k}(t)}{\Gamma(\alpha+1)} < 1$, *then problem (3.1) has a unique solution.*

5 Conclusion

We conclude from the Schwarz-Pick lemma that a non-identity self-map ϕ of the unit disk can have at most one fixed point in U. If such a unique fixed point in U exists, it is called the Denjoy-Wolff point . The sequence of iterates $\{\phi_n\}$ of ϕ converges to it uniformly on the compact subsets of U whenever ϕ is not a disk automorphism. If ϕ has no fixed points in U, the Denjoy-Wolff theorem (see [1]) guarantees the existence of a unique point ω on the unit circle ∂U which is the attractive fixed point, that is, the sequence of iterates ϕ_n converges to ω uniformly on the compact subsets of U. Such ω is again called the Denjoy-Wolff point of ϕ. When $\omega \in \partial U$ is the Denjoy-Wolff point of ϕ, then $\phi'(\omega)$ is actually real-valued and $0 < \phi'(\omega) < 1$(see [20]). The holomorphic self-maps of the disk can be classified into three categories according to their behavior near the Denjoy-Wolff point: (a) elliptic: the ones with a fixed point inside the disk ; (b) hyperbolic: the ones with the Denjoy-Wolff point such that $\phi'(\omega) < 1$; (c) parabolic: the ones with the Denjoy-Wolff point such that $\phi'(\omega) = 1$. In addition, if ϕ is a univalent analytic function that maps the unit disk into itself, then the fixed point set of ϕ has capacity zero (see [3]).

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