On the properties of Cartesian powers of coset groups and polyadic groups of matrices

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Abstract. For any $\ell \geq 3$, $k \geq 2$ and any permutation $\sigma \in S_k$ on the Cartesian power A^k of a group A which admits a normal subgroup B, so that the factor group A/B be cyclic and having its order a divisor of $\ell - 1$, we define the ℓ-ary group $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$, endowed with the ℓ-ary operation $[\ \ ]_{\ell,\sigma,k}$. We study the properties of this ℓ-ary operation on Cartesian powers of conjugate group classes of the group A associated to its subgroup B.

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1 Introduction

It is known that n-ary algebraic systems have numerous applications to various fields, like: theory of automata ([9]), theory of quantum groups ([11]), Geometry ([14]), Physics ([15]), Cryptology ([10]), etc. Within this framework, our paper aims to derive specific properties of certain normal subgroups, factor groups of Cartesian powers of groups and on polyadic groups of matrices.

We remind first several notions from the theory of n-ary groups, which we shall use throughout the paper. According to V. Dërnte ([2]), we have the following

Definition 1.1. A universal algebra $\langle A, [\ \ ]\rangle$ with an n-ary ($n \geq 2$) operation $[\ \ ] : A^n \to A$ is called *n-ary group*, if the following conditions are fulfilled:

a) the n-ary operation $[\ \ ]$ on the set A is associative, i.e.,

$$[[a_1 \dots a_n]a_{n+1} \dots a_{2n-1}] = [a_1 \dots a_i[a_{i+1} \dots a_{i+n}]a_{i+n+1} \dots a_{2n-1}],$$

for all $i = 1, 2, \dots, n-1$ and all $a_1, a_2, \dots, a_{2n-1} \in A$;

b) each of the equations

$$[a_1 \dots a_{i-1}x_i a_{i+1} \dots a_n] = b, i = 1, 2, \dots, n$$

can be uniquely solved in A for x_i, for all $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b \in A$.

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It is known (E. Post, [12]), that the requirement of unique solution in Dërnte's definition may be relaxed to require only solvability, and that the number of equations can be reduced from n to two ($i = \overline{1, n}$), and for $n \geq 3$ - just to one equation, where i is arbitrary and fixed in $\{2, \ldots, n-1\}$.

We say that an n-ary group $\langle A, [\ \ ]\rangle$ is said to be *abelian,* if ([2, 12]):

$$[a_1 a_2 \ldots a_n] = [a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)}],$$

for all $a_1, a_2, \ldots, a_n \in A$ and any permutation σ of the set $\{1, \ldots, n\}$, and *semi-abelian,* if

$$[a a_1 \ldots a_{n-2} b] = [b a_1 \ldots a_{n-2} a],$$

for all $a, a_1, \ldots, a_{n-2}, b \in A$.

Definition 1.2. An n-ary subgroup $\langle B, [\ \ ]\rangle$ of an n-ary group $\langle A, [\ \ ]\rangle$ is said to be *invariant* in A ([2, 12]), if

$$[x \underbrace{B \ \ldots \ B}_{n-1}] = [Bx \underbrace{B \ \ldots \ B}_{n-2}] = \ldots = [\underbrace{B \ \ldots \ B}_{n-2} xB] = [\underbrace{B \ \ldots \ B}_{n-1} x]$$

for any $x \in A$. If

$$[x \underbrace{B \ \ldots \ B}_{n-1}] = [\underbrace{B \ \ldots \ B}_{n-1} x]$$

for any $x \in A$, then $\langle B, [\ \ ]\rangle$ is said to be *semi-invariant* in $\langle A, [\ \ ]\rangle$ ([2, 12]).

According to E.Post ([12]), and to [13], we have the following

Definition 1.3. The group A is said to be *covering* for the n-ary group $\langle H, [\ \ ]\rangle$, if it is generated by the set H, and the binary operation on the group A and the n-ary operation $[\ \ ]$, are related by the condition

$$[x_1 x_2 \ldots x_n] = x_1 x_2 \ldots x_n,$$

for any $x_1, x_2, \ldots, x_n \in H$. The set

$$B = \{a_1 \ldots a_{n-1} | a_1, \ldots, a_{n-1} \in H\}$$

is a normal subgroup of A, whose factor group A/B is cyclic, and has its order a divisor of $n-1$. The group B is called *associated* to the n-ary group $\langle H, [\ \ ]\rangle$.

The converse Post Theorem on conjugate classes ([12, 13]) asserts that, if the factor group A/B of the group A relative to its normal subgroup B is cyclic, admitting as generator an element aB and has its order a divisor of $n-1$, then $\langle aB, [\ \ ]\rangle$ is an n-ary group with the n-ary operation

$$[a_1 a_2 \ldots a_n] = a_1 a_2 \ldots a_n.$$

The covering group for $\langle aB, [\ \ ]\rangle$ is A, and the corresponding group is the subgroup B.

2 Preliminary results

Definition 2.1. [**3**] Let A be a groupoid, $k \geq 2$, $\ell \geq 2$, and let σ be a permutation from S_k. We first define on A^k a binary operation

$$\mathbf{x} \overset{\sigma}{\circ} \mathbf{y} = (x_1, x_2, \ldots, x_k) \overset{\sigma}{\circ} (y_1, y_2, \ldots, y_k) = (x_1 y_{\sigma(1)}, x_2 y_{\sigma(2)}, \ldots, x_k y_{\sigma(k)}),$$

and then an ℓ-ary operation,

$$[\mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_\ell]_{\ell,\sigma,k} = \mathbf{x}_1 \overset{\sigma}{\circ} (\mathbf{x}_2 \overset{\sigma}{\circ} (\ldots (\mathbf{x}_{\ell-2} \overset{\sigma}{\circ} (\mathbf{x}_{\ell-1} \overset{\sigma}{\circ} \mathbf{x}_\ell)) \ldots)).$$

It is clear that the operation $[\ \ ]_{2,\sigma,k}$ coincides with the operation $\overset{\sigma}{\circ}$.

Remark 2.2. It is easy to remark that, if $\sigma = (12\ldots k)$, then the operation $\overset{\sigma}{\circ}$ coincides with the operation

$$\mathbf{x} \circ \mathbf{y} = (x_1, x_2, \ldots, x_k) \circ (y_1, y_2, \ldots, y_k) = (x_1 y_2, x_2 y_3, \ldots, x_{k-1} y_k, x_k y_1)$$

from [4, Definition 2.2.3], and the operation $[\ \ ]_{\ell,\sigma,k}$ - with the operation $[\ \ ]_{\ell,k}$ from the same definition. The operations $\circ$ and $[\ \ ]_{\ell,k}$ were first defined in [5], like the operation $[\ \ ]_{\ell,\sigma,k}$ for the cases of semigroups of A. We remark that the operation $[\ \ ]_{n,n-1}$ is analogous to an n-ary operation, which E. Post constructed on the set of all n-ary permutations in [12].

Theorem 2.1. [**3**] *Let A be a semigroup,*

$$\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ik}) \in A^k, \quad i = 1, 2, \ldots, \ell.$$

Then $[\mathbf{x}_1 \mathbf{x}_2 \ldots \mathbf{x}_\ell]_{\ell,\sigma,k} = (y_1, y_2, \ldots, y_k)$, *where*

$$y_j = x_{1j} x_{2\sigma(j)} \ \ldots \ x_{(\ell-1)\sigma^{\ell-2}(j)} x_{\ell\sigma^{\ell-1}(j)}, \quad j = 1, 2, \ldots, k.$$

Theorem 2.2. [**4**]

a) If A is a group and σ is a permutation from S_k, which satisfies the condition $\sigma^\ell = \sigma$, then $\langle A^k, [\ \ ]_{\ell,\sigma,k} \rangle$ is an ℓ-ary group.

b) If the semigroup A contains the unity 1, and permutation $\sigma \in S_k$ satisfies the condition $\sigma^\ell = \sigma$, then the ℓ-ary semigroup $\langle A^k, [\ \ ]_{\ell,\sigma,k} \rangle$ is semi-Abelian if and only if the semigroup A is Abelian.

c) Let the semigroup A contain more than one element, and let σ be a non-identity permutation from S_k. Then the ℓ-ary groupoid $\langle A^k, [\ \ ]_{\ell,\sigma,k} \rangle$ contains no unity.

Lemma 2.3. *Let σ be a permutation from S_k, which satisfies the condition $\sigma^\ell = \sigma$, let B be a subgroup of the group A, and let $\mathbf{x} = (x_1, \ldots, x_k) \in A^k$. Then:*

a) $[\underbrace{B^k \ldots B^k}_{i-1} \ \mathbf{x} \ \underbrace{B^k \ldots B^k}_{\ell-i}]_{\ell,\sigma,k} = B x_{\sigma^{i-1}(1)} B \ \times \ldots \times B x_{\sigma^{i-1}(k)} B$, *for any* $i = 2, \ldots, \ell - 1$*;*

b) $[\mathbf{x}\underbrace{B^k\ldots B^k}_{\ell-1}]_{\ell,\sigma,k} = x_1B\times\ldots\times x_kB$*;*

c) $[\underbrace{B^k\ldots B^k}_{\ell-1}\mathbf{x}]_{\ell,\sigma,k} = Bx_1\times\ldots\times Bx_k$.

Proof. a) Since

$$\begin{aligned}
&[\underbrace{B^k\ldots B^k}_{i-1}\ \mathbf{x}\ \underbrace{B^k\ldots B^k}_{\ell-i}]_{\ell,\sigma,k} =\\
&= \{[\mathbf{h}_1\ldots\mathbf{h}_{i-1}\mathbf{x}\mathbf{h}_{i+1}\ldots\mathbf{h}_\ell]_{\ell,\sigma,k}|\mathbf{h}_1,\ldots\mathbf{h}_{i-1},\mathbf{h}_{i+1},\ldots,\mathbf{h}_\ell\in B^k\} =\\
&= \{[(b_{11},\ldots,b_{1k})\ldots(b_{(i-1)1},\ldots,b_{(i-1)k})(x_1,\ldots,x_k)(b_{(i+1)1},\ldots,b_{(i+1)k})\ldots\\
&\quad\ldots(b_{\ell1},\ldots,b_{\ell k})]_{\ell,\sigma,k}|b_{ij}\in B\} =\\
&= \{(b_{11}b_{2\sigma(1)}\ldots b_{(i-1)\sigma^{i-2}(1)}x_{\sigma^{i-1}(1)}b_{(i+1)\sigma^i(1)}\ldots b_{\ell\sigma^{\ell-1}(k)},\ldots\\
&\quad\ldots,b_{1k}b_{2\sigma(k)}\ldots b_{(i-1)\sigma^{i-2}(k)}x_{\sigma^{i-1}(k)}b_{(i+1)\sigma^i(k)}\ldots b_{\ell\sigma^{\ell-1}(k)})|b_{ij}\in B\} =\\
&= Bx_{\sigma^{i-1}(1)}B\times\ \ldots\times Bx_{\sigma^{i-1}(k)}B,
\end{aligned}$$

then a) holds true.
b) Is proved similarly to a), using the definition of the coset and the operations $[\ \ ]_{\ell,\sigma,k}$.
c) First, using the definitions of the coset and of the operations $[\ \ ]_{\ell,\sigma,k}$ we get the equality

$$[\underbrace{B^k\ldots B^k}_{\ell-1}\mathbf{x}]_{\ell,\sigma,k} = Bx_{\sigma^{\ell-1}(1)}\times\ldots\times Bx_{\sigma^{\ell-1}(k)},$$

whence, in view of the identity of $\sigma^{\ell-1}$, the claim follows. □

Considering $\ell = n$, $k = n-1$, and $\sigma = (12\ldots n-1)$ in Lemma 2.3, we get

Corollary 2.4. *Let $n\geq 3$, B be a subgroup of group A, and let $\mathbf{x} = (x_1,\ldots,x_{n-1})\in A^{n-1}$. Then:*

a) for any $i = 2,\ldots,n-1$, we have

$$\begin{aligned}
&[\underbrace{B^{n-1}\ldots B^{n-1}}_{i-1}\ \mathbf{x}\ \underbrace{B^{n-1}\ldots B^{n-1}}_{n-i}]_{n,n-1} =\\
&= Bx_iB\ \times\ldots\times Bx_{n-1}B\times Bx_1B\times\ldots\times Bx_{i-1}B;
\end{aligned}$$

b) $[\mathbf{x}\underbrace{B^{n-1}\ldots B^{n-1}}_{n-1}]_{n,\,n-1} = x_1B\times\ldots\times x_{n-1}B$*;*

c) $[\underbrace{B^{n-1}\ldots B^{n-1}}_{n-1}\mathbf{x}]_{n,\,n-1} = Bx_1\times\ldots\times Bx_{n-1}$.

According to Theorem 2.2 - item a), if A is a group and σ is a permutation from S_k which satisfies the condition $\sigma^l = \sigma$, then $\langle A^k,[\ \ ]_{\ell,\sigma,k}\rangle$ is an ℓ-ary group. It is clear, that if B is a subgroup of the group A, and if the permutation $\sigma\in S_k$ satisfies the condition $\sigma^\ell = \sigma$, then $\langle B^k,[\ \ ]_{\ell,\sigma,k}\rangle$ is an ℓ-ary subgroup of the ℓ-ary group $\langle A^k,[\ \ ]_{\ell,\sigma,k}\rangle$.

Proposition 2.5. *Let σ be a permutation from S_k, which satisfies the condition $\sigma^\ell = \sigma$, and let B be a subgroup of the group A. Then:*

a) The n-ary subgroup $\langle B^k, [\ \ ]_{\ell,\sigma,k}\rangle$ is semi-invariant in the ℓ-ary group $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$ if and only if the subgroup B is normal in the group A;

b) If $B \neq A$, and σ is not the identity permutation, then $\langle B^k, [\ \ ]_{\ell,\sigma,k}\rangle$ is not invariant in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$.

Proof. a) $\Rightarrow$. The semi-invariance of $\langle B^k, [\ \ ]_{\ell,\sigma,k}\rangle$ in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$ means that

$$[\mathbf{x}\underbrace{B^k \ldots B^k}_{\ell-1}]_{\ell,\sigma,k} = [\underbrace{B^k \ldots B^k}_{\ell-1}\mathbf{x}]_{\ell,\sigma,k} \tag{2.1}$$

for any $\mathbf{x} = (x_1, \ldots, x_k) \in A^k$. Then, from Lemma 2.3, we have

$$x_1B \times \ldots \times x_kB = \ Bx_1 \times \ldots \times \ Bx_k, \tag{2.2}$$

whence $x_1B = Bx_1$ for any $x_1 \in B$, which means the normality of B in A.

$\Leftarrow$. From the normality of B in A, it follows

$$x_1B = Bx_1, \ldots, x_kB = Bx_k$$

for any $x_1, \ldots, x_k \in A$, which shows that (2.2) holds true. Then, according to Lemma 2.3, we infer (2.1), which means that $\langle B^k, [\ \ ]_{\ell,\sigma,k}\rangle$ is semi-invariant in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$.

b) If σ is not the identical permutation, then $\sigma(j) \neq j$ for some $j \in \{1, \ldots, k\}$, and since $B \neq A$, then we can find an element $u \in A$, different from the unity e of the group A, so that $uB \neq B$. We choose in A^k an element $\mathbf{x} = (x_1, \ldots, x_k)$, so that $x_j = u, x_{\sigma(j)} = e$, and all the other components can be arbitrary elements from A. The condition $\sigma(j) \neq j$ ensures that such a choice exists. For the case $\sigma(j) = j$, this choice might not be achieved, since $u \neq e$. If we assume the invariance of $\langle B^k, [\ \ ]_{\ell,\sigma,k}\rangle$ in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$, then

$$[\mathbf{x}\underbrace{B^k \ldots B^k}_{\ell-1}]_{\ell,\sigma,k} = [B^k\mathbf{x}\ \underbrace{B^k \ldots B^k}_{\ell-2}]_{\ell,\sigma,k}$$

for the chosen element $\mathbf{x}$. Since the invariance of the n-ary subgroup in the n-ary group entails the semi-invariance in this n-ary group, then the assumption about the invariance of $\langle B^k, [\ \ ]_{\ell,\sigma,k}\rangle$ in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$ yields, considering a), the normality of B in A. Applying to the left hand side of the last equality the assertion b) from Lemma 2.3, and to the right one-the assertion a) for $i = 2$, and using the normality of B in A, we get

$$x_1B \times \ldots \times x_jB \times \ldots \times x_kB = x_{\sigma(1)}B \times \ldots \times x_{\sigma(j)}B \times \ldots \times x_{\sigma(k)}B.$$

Hence, $x_jB = x_{\sigma(j)}B$, and using the conditions $x_j = u$, $x_{\sigma(j)} = e$, we infer $uB = B$, which conflicts to the choice $uB \neq B$. Then $\langle B^k, [\ \ ]_{\ell,\sigma,k}$ is not invariant in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$, and the claim is proved. □

Assuming in Proposition 2.5 that $\ell = n, k = n-1$ and $\sigma = (12 \ldots n-1)$, we get

Corollary 2.6. *Let $n \geq 3$ and let B be a subgroup of the group A. Then:*

a) The n-ary subgroup $\langle B^{n-1}, [\ ]_{n,n-1}\rangle$ of the n-ary group $\langle A^{n-1}, [\ ]_{n,n-1}\rangle$ is semi-invariant in it if and only if the subgroup B is normal in the group A;

b) If $B \neq A$, then $\langle B^{n-1}, [\ ]_{n,n-1}\rangle$ is not invariant in $\langle A^{n-1}, [\ ]_{n,n-1}\rangle$.

If σ is a non-identical permutation from S_k which satisfies the condition $\sigma^\ell = \sigma$, then according to Proposition 2.5, $\langle \{e\}^k, [\ ]_{\ell,\sigma,k}\rangle$ contains one element and is semi-invariant, but not an invariant ℓ-ary subgroup of the ℓ-ary group $\langle A^k, [\ ]_{\ell,\sigma,k}\rangle$. Since the unity of the ℓ-ary group represents an invariant ℓ-ary subgroup, then the element $\{\underbrace{e, \dots, e}_{k}\}$ is not the unity of $\langle A^k, [\ ]_{\ell,\sigma,k}\rangle$. In fact, according to item c) of Theorem 2.2, the ℓ-ary group $\langle A^k, [\ ]_{\ell,\sigma,k}\rangle$ has no unity at all.

3 Main results

According to Proposition 2.5, if the permutation $\sigma \in S_k$ satisfies the condition $\sigma^\ell = \sigma$ and B is a normal subgroup of the group A, then the Cartesian power B^k is a semi-invariant ℓ-ary subgroup of the ℓ-ary group $\langle A^k, [\ ]_{\ell,\sigma,k}\rangle$. In fact, if we require that the factor group A/B be cyclic, and if its order divides $\ell - 1$, then not only the Cartesian power B^k, but also the k-th Cartesian power of any element of A/B is a semi-invariant ℓ-ary subgroup of $\langle A^k, [\ ]_{\ell,\sigma,k}\rangle$.

Theorem 3.1. *Let A be a group, B be a proper normal subgroup, let the factor group A/B be cyclic and have the order a divisor of $\ell - 1$, and let the permutation σ from S_k satisfy the condition $\sigma^\ell = \sigma$. Then for any element H of the factor group A/B, the Cartesian power H^k is closed relative to the ℓ-ary operation $[\ ]_{\ell,\sigma,k}$, and the universal algebra $\langle H^k, [\ ]_{\ell,\sigma,k}\rangle$ is a semi-invariant ℓ-ary subgroup of the ℓ-ary group $\langle A^k, [\ ]_{\ell,\sigma,k}\rangle$. Moreover, if σ is a non-identity permutation, then $\langle H^k, [\ ]_{\ell,\sigma,k}\rangle$ is not invariant in $\langle A^k, [\ ]_{\ell,\sigma,k}\rangle$.*

Proof. Let the factor group A/B be generated by the coset aB, i.e., $A/B = \{B, aB, \dots, a^{t-1}B\}$, where t divides $\ell - 1$. We shall consider, for convenience, that $H = a^s B$ for some $s = 0, 1, \dots, t-1$. Since $(aB)^t = a^t B = B$, then $a^t \in B$, whence considering that t divides $\ell - 1$, it follows that $a^{\ell-1} \in B$. Now, if

$$\mathbf{h}_i = (h_{i1}, \dots, h_{ik}) = (a^s b_{i1}, \dots, a^s b_{ik}), \quad i = 1, \dots, \ell$$

are arbitrary elements from H^k, then using the normality of B in A, we shall have

$$[\mathbf{h}_1, \dots, \mathbf{h}_\ell]_{\ell,\sigma,k} = (y_1, \dots, y_k),$$

where

$$y_j = a^s b_{1j} a^s b_{2\sigma(j)} \dots a^s b_{(\ell-1)\sigma^{\ell-2}(j)} a^s b_{\ell j} = a^{sl} b_j$$

for some $b_j \in B$. But then, since $a^{l-1} \in B$, we have $y_j = a^{sl} b_j = a^s (a^{l-1})^s b_j = a^s b'_j$ for some $b'_j \in B$. Hence, $y_j \in H$ for any $j = 1, \dots, k$, and then

$$[\mathbf{h}_1, \dots, \mathbf{h}_\ell]_{\ell,\sigma,k} \in H^k,$$

which proves that the set H^k is closed relative to the ℓ-ary operation $[\ \ ]_{\ell,\sigma,k}$.

We shall examine now in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$ the equation

$$[\mathbf{x}\mathbf{h}_2 \dots \mathbf{h}_\ell]_{\ell,\sigma,k} = \mathbf{g}, \tag{3.1}$$

where

$$\mathbf{g} = (g_1, \dots, g_k) = (a^s c_1, \dots, a^s c_k) \in H^k, (c_1, \dots, c_k) \in B.$$

The elements $\mathbf{h}_2, \dots, \mathbf{h}_\ell$ were defined above, and they belong to H^k. Since $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$ is an ℓ-ary group, then equation (3.1) admits the solution

$$\mathbf{x} = (a_1, \dots, a_k) \in A^k.$$

Substituting this solution into (3.1), and making equal the j-th order components from left and right sides of the obtained equation, we get

$$a_j a^s b_{2\sigma(j)} \dots a^s b_{(l-1)\sigma^{l-2}(j)} a^s b_{\ell j} = a^s c_j.$$

Then due to the normality of B in A and of the condition $a^{\ell-1} \in B$, the left side of the last equality gets the form $a_j a^{(\ell-1)s} d = a_j b$ for some $d, b \in B$, and the equality itself becomes $a_j b = a^s c_j$. But then $a_j = a^s c_j b^{-1}$, where $c_j b^{-1} \in B$. Hence, $a_j \in a^s B = H$, i.e. $\mathbf{x} = (a_1, \dots, a_k) \in H^k$. This shows that equation (3.1) may admit solutions in $\langle H^k, [\ \ ]_{\ell,\sigma,k}\rangle$. A similar proof can be provided for the solvability in $\langle H^k, [\ \ ]_{\ell,\sigma,k}\rangle$ of the equation

$$[\mathbf{h}_1, \dots, \mathbf{h}_{\ell-1}\mathbf{y}]_{\ell,\sigma,k} = \mathbf{g}$$

for any $\mathbf{h}_1, \dots, \mathbf{h}_{\ell-1}, \mathbf{g} \in H^k$. In this way, according to the Post criterion ([12]), $\langle H^k, [\ \ ]_{\ell,\sigma,k}\rangle$ is an ℓ-ary subgroup of $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$.

If $\mathbf{x} = (x_1, \dots, x_k)$ is an arbitrary element from A^k, then, using the normality of B in A and the condition $a^{\ell-1} \in B$, we get

$$\begin{aligned}
&[\mathbf{x}\underbrace{H^k \dots H^k}_{\ell-1}]_{\ell,\sigma,k} = \{[\mathbf{x}\mathbf{h}_2 \dots \mathbf{h}_\ell]_{\ell,\sigma,k} | \mathbf{h}_2, \dots, \mathbf{h}_\ell \in H^k\} = \\
&= \{[(x_1, \dots, x_k)(a^s b_{21}, \dots, a^s b_{2k}) \dots (a^s b_{\ell 1}, \dots, a^s b_{\ell k})]_{\ell,\sigma,k} | b_{ij} \in B\} = \\
&= \{(x_1 a^s b_{2\sigma(1)} \dots a^s b_{(\ell-1)\sigma^{\ell-2}(1)} a^s b_{\ell 1}, \dots, x_k a^s b_{2\sigma(k)} \dots \\
&\qquad \dots a^s b_{(\ell-1)\sigma^{l-2}(k)} a^s b_{\ell k}) | b_{ij} \in B\} = \\
&= \{(x_1 a^{(l-1)s} d_1, \dots, x_k a^{(l-1)s} d_k) | d_1, \dots, d_k \in B\} = \\
&= \{(x_1 b_1, \dots, x_k b_k) | b_1, \dots, b_k \in B\} = x_1 B \times \dots \times x_k B,
\end{aligned}$$

i.e.,

$$[\mathbf{x}\underbrace{H^k \dots H^k}_{\ell-1}]_{\ell,\sigma,k} = x_1 B \times \dots \times x_k B. \tag{3.2}$$

Analogously, it can be proved the equality

$$[\underbrace{H^k \dots H^k}_{\ell-1}\mathbf{x}]_{\ell,\sigma,k} = B x_1 \times \dots \times B x_k. \tag{3.3}$$

From the normality of B in A it follows the equality between the right hand sides of (3.2) and (3.3), and hence the equality of their left sides as well, for any $\mathbf{x} \in A^k$, which implies the semi-invariance of $\langle H^k, [\ \ ]_{\ell,\sigma,k}\rangle$ in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$.

If σ is a non-identical permutation, then $\sigma(j) \neq j$ for some $j \in \{1, \dots, k\}$, and since $B \neq A$, then we can find an element $u \in A$, different from the unity e of the group A, so that $uB \neq B$. We choose in A^k the element $\mathbf{x} = (x_1, \dots, x_k)$, which satisfies

$$x_j = u, x_{\sigma(j)} = (a^{-1})^{(\ell-1)s}, \tag{3.4}$$

where all the remaining components are arbitrary from A. The condition $\sigma(j) \neq j$ allows such a choice. If $\sigma(j) = j$, the choice might not have been possible - e.g., for $u \neq (a^{-1})^{(\ell-1)s}$. If we assume the invariance of $\langle H^k, [\ \ ]_{\ell,\sigma,k}\rangle$ in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$, then

$$[\mathbf{x}\underbrace{H \dots H}_{\ell-1}]_{\ell,\sigma,k} = [H\mathbf{x}\underbrace{H \dots H}_{\ell-2}]_{\ell,\sigma,k}$$

for the chosen $x \in A^k$. We apply (3.2) to the left hand side of the obtained equality and in the right side we use the normality of B in A, and perform a calculation similar to the one used for obtaining (3.2). As a result, we get

$$\begin{aligned} &x_1B \times \dots \times x_jB \times \dots \times x_kB = \\ &= a^s x_{\sigma(1)} a^{(\ell-2)s} B \times \dots \times a^s x_{\sigma(j)} a^{(\ell-2)s} B \times \dots \times a^s x_{\sigma(k)} a^{(\ell-2)s} B. \end{aligned}$$

Hence, $x_jB = a^s x_{\sigma(j)} a^{(\ell-2)s} B$, which, in view of (3.4), leads to $uB = B$, which contradicts with $uB \neq B$. We conclude that $\langle H^k, [\ \ ]_{\ell,\sigma,k}\rangle$ is not invariant in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$. □

Corollary 3.2. *Let A be a group, let B be a proper normal subgroup, let the factor group A/B be cyclic, generated by the coset aB and having the order divisor of $\ell - 1$. Then the Cartesian power $(aB)^k$ is closed relative to the ℓ-ary operation $[\ \ ]_{\ell,\sigma,k}$, and the universal algebra $\langle (aB)^k, [\ \ ]_{\ell,\sigma,k}\rangle$ is a semi-invariant ℓ-ary subgroup of the ℓ-ary group $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$. Moreover, if σ is a non-identity permutation, then $\langle (aB)^k, [\ \ ]_{\ell,\sigma,k}\rangle$ is not invariant in $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$.*

Setting $\ell = n$, $k = n - 1$, and $\sigma = (12 \dots n - 1)$ in Theorem 3.1, we get

Corollary 3.3. [**7**] *Let $n \geq 3$, let A be a group, let B be a proper normal subgroup of A, let the factor group A/B be cyclic and having its order a divisor of $n - 1$. Then for any element H of the factor group A/B, the Cartesian power H^{n-1} is closed relative to the n-ary operation $[\ \ ]_{n,n-1}$, and the universal algebra $\langle H^{n-1}, [\ \ ]_{n,n-1}\rangle$ is a semi-invariant, but not an invariant n-ary subgroup of the n-ary group $\langle A^{n-1}, [\ \ ]_{n,n-1}\rangle$.*

Any semi-invariant n-ary subgroup $\langle B, [\ \ ]\rangle$ of the n-ary group $\langle A, [\ \ ]\rangle$ defines on it a congruence ρ_B, whose classes coincide with the elements of the n-ary factor group $\langle A/B, [\ \ ]\rangle$ (e.g., see [6, Prop. 7.4]). The following Theorem establishes the link between the congruences of the n-ary group $\langle A^{n-1}, [\ \ ]_{n,n-1}\rangle$, which are defined by the semi-invariant n-ary subgroups, which are constructed by means of different elements of the factor group A/B from Theorem 3.1.

Theorem 3.4. *Let H be an arbitrary element of the factor group A/B from Theorem 3.1. Then:*

a) $\langle A^k/H^k, [\ \ ]_{\ell,\sigma,k}\rangle = \langle A^k/B^k, [\ \ ]_{\ell,\sigma,k}\rangle = \langle (A/B)^k, [\ \ ]_{\ell,\sigma,k}\rangle$*;*

b) $\rho_{H^k} = \rho_{B^k}$*.*

Proof. a) Replacing $H = B$ in (3.2), we get

$$[\mathbf{x}\underbrace{B^k \dots B^k}_{\ell-1}]_{\ell,\sigma,k} = x_1B \times \dots \times x_kB, \tag{3.5}$$

where $\mathbf{x} = (x_1, \dots, x_k) \in A^k$, whence using (3.2), we infer

$$[\mathbf{x}\underbrace{H^k \dots H^k}_{\ell-1}]_{\ell,\sigma,k} = [\mathbf{x}\underbrace{B^k \dots B^k}_{\ell-1}]_{\ell,\sigma,k}$$

for any $\mathbf{x} \in A^k$. Therefore the ℓ-ary factor groups $\langle A^k/H^k, [\ \ ]_{\ell,\sigma,k}\rangle$ and $\langle A^k/B^k, [\ \ ]_{\ell,\sigma,k}\rangle$ coincide. If $[\mathbf{x}\underbrace{H^k \dots H^k}_{\ell-1}]_{\ell,\sigma,k}$ is an arbitrary coset from $\langle A^k/B^k, [\ \ ]_{\ell,\sigma,k}\rangle$, then, taking into consideration (3.2), we get $[\mathbf{x}\underbrace{H^k \dots H^k}_{\ell-1}]_{\ell,\sigma,k} \in (A/B)^k$, i.e., it holds the inclusion $A^k/H^k \subseteq (A/B)^k$.

If $x_1B \times \dots \times x_kB$ is an arbitrary element from $(A/B)^k$, then again, using (3.2), we get $x_1B \times \dots \times x_kB \in A^k/B^k$, and we infer $(A/B)^k \subseteq A^k/B^k$. From the proved inclusions we obtain that the ℓ-ary factor groups $\langle A^k/H^k, [\ \ ]_{\ell,\sigma,k}\rangle$ and $\langle (A/B)^k, [\ \ ]_{\ell,\sigma,k}\rangle$ coincide.

b) From [7, Prop. 7.4], we have

$$\langle A^k/\rho_{B^k}, [\ \ ]_{\ell,\sigma,k}\rangle = \langle A^k/B^k, [\ \ ]_{\ell,\sigma,k}\rangle, \langle A^k/\rho_{H^k}, [\ \ ]_{\ell,\sigma,k}\rangle = \langle A^k/H^k, [\ \ ]_{\ell,\sigma,k}\rangle,$$

and, using a), we get

$$\langle A^k/\rho_{H^k}, [\ \ ]_{\ell,\sigma,k}\rangle = \langle A^k/\rho_{B^k}, [\ \ ]_{\ell,\sigma,k}\rangle,$$

which shows that the congruences ρ_{H^k} and ρ_{B^k} coincide. □

Setting $\ell = n$, $k = n-1$ and $\sigma = (12 \dots n-1)$ in Theorem 3.4, we get

Corollary 3.5. [**6**] *Let H be an arbitrary coset from Corollary 3.3. Then:*

a) $\langle A^{n-1}/H^{n-1}, [\ \ ]_{n,n-1}\rangle = \langle A^{n-1}/B^{n-1}, [\ \ ]_{n,n-1}\rangle = \langle (A/B)^{n-1}, [\ \ ]_{n,n-1}\rangle$*;*

b) $\rho_{H^{n-1}} = \rho_{B^{n-1}}$*.*

Theorem 3.4 and Corollary 3.5 can be rephrased in the following different, more concrete way:

Theorem 3.6. *Let the permutation $\sigma \in S_k$ satisfy the condition $\sigma^\ell = \sigma$, let A be a group, let B be a proper normal subgroup of A, let the factor group A/B be cyclic, generated by the element aB and having the order t, divisor of $\ell - 1$: $A/B = \{B, aB, \dots, a^{t-1}B\}$. Then:*

a) $\langle A^k/B^k, [\ \ ]_{\ell,\sigma,k}\rangle = \langle A^k/(aB)^k, [\ \ ]_{\ell,\sigma,k}\rangle = \ldots = \langle A^k/(a^{t-1}B)^k, [\,]_{\ell,\sigma,k}\rangle = \langle (A/B)^k, [\ \ ]_{\ell,\sigma,k}\rangle$;

b) $\rho_{B^k} = \rho_{(aB)^k} = \ldots = \rho_{(a^{t-1}B)^k}$.

Corollary 3.7. [**6**] *Let $n \geq 3$, let A be a group, let B be a proper normal subgroup of A, let the factor group A/B be cyclic, generated by the element aB and having its order t a divisor of $n-1$: $A/B = \{B, aB, \ldots, a^{t-1}B\}$. Then:*

a) $\langle A^{n-1}/B^{n-1}, [\ \ ]_{n,n-1}\rangle = \langle A^{n-1}/(aB)^{n-1}, [\ \ ]_{n,n-1}\rangle = \ldots$

$$= \langle A^{n-1}/(a^{t-1}B)^{n-1}, [\,]_{n,n-1}\rangle = \langle (A/B)^{n-1}, [\ \ ]_{n,n-1}\rangle;$$

b) $\rho_{B^{n-1}} = \rho_{(aB)^{n-1}} = \ldots = \rho_{(a^{t-1}B)^{n-1}}$.

4 Polyadic matrices

The ordered set $(A_1, A_2, \ldots, A_{m-1})$ of matrices of the same order n over the field of complex numbers $\mathbb{C}$ was called by E. Post as being *m-ary* (or *polyadic*) *matrix* over $\mathbb{C}$ ([12]). On the set of all m-ary matrices whose determinants of all the matrix components are nonzero, E. Post defined the m-ary operation

$$\begin{aligned}[A_1 \ldots A_m] &= [(A_{11}, \ldots, A_{1(m-1)}) \ldots (A_{m1}, \ldots, A_{m(m-1)})] = \\ &= (Y_1, \ldots, Y_{m-1}),\end{aligned} \tag{4.1}$$

where

$$Y_j = A_{1j}A_{2(j+1)} \ldots A_{(n-j)(n-1)}A_{(n-j+1)1} \ldots A_{(n-1)(j-1)}A_{nj}, j = 1, \ldots, m-1.$$

E. Post showed that this set, together with the m-ary operation (4.1) is an m-ary group, which he called *m-ary linear group.* Operation (4.1) coincides with the operation $[\ \ ]_{\ell,\sigma,k}$ for $\ell = m$, $k = m-1$, and $\sigma = (12 \ldots m-1)$, i.e., with the operation $[\ \ ]_{m,m-1}$.

We shall examine the ordered sets of matrices of same order over an arbitrary field. The family of all the ordered sets $\mathbf{A} = (A_1, \ldots, A_k)$ of matrices of the same order n over the field F, the whose determinant of each component A_j is different from the zero of the field F, will be denoted $GL(n, k, F)$. The elements of this family, according to E. Post, will be called k-component polyadic matrices over F.

It is clear that the family $GL(n, k, F)$ coincides with the k-th Cartesian power of the full linear group $GL(n, F) : GL(n, k, F) = (GL(n, F))^k$. Therefore, putting $A = GL(n, F)$ in item a) of Theorem 2.2, we get

Proposition 4.1. *If the permutation σ from S_k satisfies the condition $\sigma^\ell = \sigma$, then the family $GL(n, k, F)$ is closed relative to the ℓ-ary operation $[\ \ ]_{\ell,\sigma,k}$, and the universal algebra $\langle GL(n, k, F), [\ \ ]_{\ell,\sigma,k}\rangle$ is an ℓ-ary group.*

Since $GL(n, m-1, \mathbb{C}) = (GL(n, \mathbb{C}))^{m-1}$, the operation (4.1), as shown before, coincides with the operation $[\ \ ]_{m,m-1}$, and hence from Proposition 4.1 it follows Post's mentioned result.

Corollary 4.2. [**12**] *The set $GL(n, m-1, \mathbb{C})$ is closed relative to the m-ary operation $[\ ]_{m,m-1}$, and the universal algebra $\langle GL(n, m-1, \mathbb{C}), [\ ]_{m,m-1}\rangle$ is an m-ary group.*

In the set $GL(n, k, F)$ we pay a special attention to the subset $SL(n, k, F)$ of all the k-component polyadic matrices, where the determinant of each component is equal to the unity of the field F. Since $SL(n, k, F) = (SL(n, F))^k$, then, applying again item a) of Theorem 2.2, we get

Proposition 4.3. *If the permutation $\sigma \in S_k$ satisfies the condition $\sigma^\ell = \sigma$, then the set $SL(n, k, F)$ is closed relative to the ℓ-ary operation $[\ ]_{\ell,\sigma,k}$, and the universal algebra $\langle SL(n, k, F), [\ ]_{\ell,\sigma,k}\rangle$ is an ℓ-ary subgroup of the ℓ-ary group $\langle GL(n, k, F), [\ ]_{\ell,\sigma,k}\rangle$.*

The polyadic group $\langle SL(n, k, F), [\ ]_{\ell,\sigma,k}\rangle$, similarly to the binary case, can be naturally called *the special polyadic linear group.*

It is clear that for $k = 1$ and $\ell = 2$, the 1-component matrices are the usual matrices, and that the polyadic groups $GL(n, 1, F)$ and $SL(n, 1, F)$ respectively coincide with the full linear group $GL(n, F)$ and the special linear group $SL(n, F)$.

We shall further use the standard notations: F_q or $GF(q)$ - for the Galois field, i.e., a finite field with $q = p^\alpha$ elements, where p is a prime number; $GL(n, q)$ - for the full linear group over the field $GF(q)$, i.e., the group of all the invertible matrices of order n over $GF(q)$; $SL(n, q)$ - for the special linear group of order n over the field $GF(q)$, i.e., the subgroup of all the matrices from $GL(n, q)$ whose determinant is equal to the unity of the field $GF(q)$.

Theorem 4.4. *Let p be a prime number, $q = p^\alpha$, $n \geq 2$, $k \geq 2$, $\ell \geq 3$; let $q-1$ divide $\ell - 1$, and let the permutation $\sigma \in S_k$ satisfy the condition $\sigma^\ell = \sigma$. Then:*

a) $\langle GL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$ and $\langle SL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$ are non-semi-Abelian ℓ-ary groups;

b) if σ is a non-identity permutation, then the ℓ-ary groups $\langle GL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$ and $\langle SL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$ contain no unity;

c) the k-th Cartesian power of each element $H_0 = SL(n, q), H_1, \ldots, H_{q-2}$ of the factor group $GL(n, q)/SL(n, q)$ is closed relative to the ℓ-ary operation $[\ ]_{\ell,\sigma,k}$, and the universal algebras

$$\langle H_0^k, [\ ]_{\ell,\sigma,k}\rangle, \langle H_1^k, [\ ]_{\ell,\sigma,k}\rangle, \ldots, \langle H_{q-2}^k, [\ ]_{\ell,\sigma,k}\rangle \tag{4.2}$$

are semi-invariant ℓ-ary subgroups in $\langle GL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$; in particular, $\langle SL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$ is a semi-invariant ℓ-ary subgroup of $\langle GL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$;

d) if σ is a non-identity permutation, then all the semi-invariant ℓ-ary subgroups (4.2) are not invariant ℓ-ary subgroups of $\langle GL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$; in particular, $\langle SL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$ is not invariant in $\langle GL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$;

e) all the semi-invariant ℓ-ary subgroups (4.2) define on $\langle GL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle$ the same congruence $\rho = \rho_{H_0^k} = \rho_{H_1^k} = \ldots = \rho_{H_{q-2}^k}$;

f) the following ℓ-ary factor groups coincide

$$\langle GL(n, k, F_q)/SL(n, k, F_q), [\ ]_{\ell,\sigma,k}\rangle, \langle GL(n, k, F_q)/H_1^k, [\ ]_{\ell,\sigma,k}\rangle, \ldots$$
$$\langle GL(n, k, F_q)/H_{q-2}^k, [\ ]_{\ell,\sigma,k}\rangle, \langle (GL(n, q)/SL(n, q))^k, [\ ]_{\ell,\sigma,k}\rangle.$$

Proof. For the brevity of notation, we write $A = GL(n,q)$, and $B = SL(n,q)$. Then

$$A^k = (GL(n,q))^k = GL(n,k,F_q), \quad B^k = (SL(n,q))^k = SL(n,k,F_q).$$

a) Propositions 4.1 and 4.2 show that $\langle A^k, [\ \ ]_{\ell,\sigma,k}\rangle$ and $\langle B^k, [\ \ ]_{\ell,\sigma,k}\rangle$ are l-ary groups. Their non-Abelianity follows from the non-Abelianity of the groups A and B and from item b) of Theorem 2.2.
b) Follows from item c) of Theorem 2.2;
c) and d) Follow from Theorem 3.1, since the factor group $A/B = GL(n,q)/SL(n,q)$ is isomorphic to the multiplicative group F_q^* of the field F_q, which is cyclic and has its order $q-1$. Moreover, $q-1$ divides $\ell - 1$;
e) Follows from the assertion b) of Theorem 3.4;
f) Follows from the assertion a) of Theorem 3.4. □

Setting $\ell = q$, $k = q-1$, and $\sigma = (12 \ldots q-1)$ in Theorem 4.4, we get

Corollary 4.5. [**6**] *Let p be a prime number, $q = p^\alpha$, $q \geq 3$, and $n \geq 2$. Then:*

a) $\langle GL(n,q-1,F_q), [\ \ ]_{q,q-1}\rangle$ and $\langle SL(n,q-1,F_q), [\ \ ]_{q,q-1}\rangle$ are non-semi-Abelian q-ary groups with empty center, and hence, without units;

b) any element H of the factor group $GL(n,q)/SL(n,q)$ is closed relative to the q-ary operation $[\ \ ]_{q,q-1}$, and the universal algebra $\langle H^{q-1}, [\ \ ]_{q,q-1}\rangle$ is semi-invariant, but a non-invariant q-ary subgroup in $\langle GL(n,q-1,F_q), [\ \ ]_{q,q-1}\rangle$; in particular, $\langle SL(n,q-1,F_q), [\ \ ]_{q,q-1}\rangle$ is a semi-invariant, but not an invariant q-ary subgroup of $\langle GL(n,q-1,F_q), [\ \ ]_{q,q-1}\rangle$;

c) the semi-invariant q-ary subgroups $\langle H^{q-1}, [\ \ ]_{q,q-1}\rangle$, and $\langle SL(n,q-1,F_q), [\ \ ]_{q,q-1}\rangle$ from b) define on $\langle GL(n,q-1,F_q), [\ \ ]_{q,q-1}\rangle$ the same congruence: $\rho_{H^{q-1}} = \rho_{SL(n,q-1,F_q)}$;

d) for any element H of the group $GL(n,q)/SL(n,q)$, the following q-ary factor groups coincide:

$$\langle GL(n,q-1,F_q)/H^{q-1}, [\ \ ]_{q,q-1}\rangle,$$
$$\langle GL(n,q-1,F_q)/SL(n,q-1,F_q), [\ \ ]_{q,q-1}\rangle,$$
$$\langle (GL(n,q)/SL(n,q))^{q-1}, [\ \ ]_{q,q-1}\rangle.$$

5 Particular cases. Applications

The n-ary operations considered in the paper - as well as the polyadic matrix structures, for the trivial case $n = 1$, are tightly related to the Berwald-Moor, Chernov and Bogoslovski multilinear forms defined on the Cartesian powers of the field of real numbers, which are used in Relativity Theory ([8]).

In particular, for a group G, the induced $n-$ary operation $\mu_{n,m} = [\,\cdot\,, \ldots, \,\cdot\,]_{n,m} : (G^m)^n \to G^m$, given by

$$\mu_{n,m}(x_1, \ldots x_n) \stackrel{\text{not}}{=} [x_1, \ldots x_n]_{n,m} \stackrel{\text{def}}{=} (p_1, \ldots p_m),$$

for all $x_k = (x_{k1}, \ldots, x_{km}) \in G^m$, $k \in \overline{1,n}$, where

$$p_k = \prod_{j=1}^{n} x_{j\tau(j,k)}, \ \tau(j,k) = mod_m(j+k-2)+1, \quad k \in \overline{1,m}.$$

provides, in the particular case of positive reals ($\mathbb{R}_+^* = (0,\infty), \ \cdot$), by means of the mapping $\theta : G^m \to G$,

$$\theta(p) = p_1 + \ldots + p_m, \forall p = (p_1, \ldots, p_m) \in G, \tag{5.1}$$

the positive $n-$multilinear composition $\theta \circ \mu_{n,m} : (G^m)^n \to G$. Then, denoting by σ the cycle $(1 \ldots m) \in \sigma_m$ (the roll-left operator), this leads by extension to $V = \mathbb{R}^m \supset G^m$ to a tensor $\mathcal{A}^g \in \mathcal{T}_n^0(V) = \otimes^n V^*$ whose coefficients are

$$\mathcal{A}^g_{i_1 \ldots i_n} = \begin{cases} 1, & \text{if } \exists j \in \overline{1,m}, \ i_k = \sigma^j(mod_m(k-1)+1), \ \forall k = \overline{1,n}, \\ 0, & \text{the rest.} \end{cases} \tag{5.2}$$

As notable particular cases, one gets the generating multilinear tensors ([8]):

a) the Bogoslovsky tensor $\mathcal{A}^g_B = \theta \circ \mu_{n,n-1}$ on $\mathbb{R}^{n-1}$, which generates the $m-$root Finsler norm on $(\mathbb{R}_+^*)^{n-1}$:

$$F_B(y) = \mathcal{A}^g_B(y, \ldots, y) = \left[y^1 \cdot \ldots \cdot y^{n-1} \cdot (y^1 + \ldots + y^{n-1}) \right]^{1/n};$$

b) the Berwald-Moor tensor $\mathcal{A}^g_{rBM} = \theta \circ \mu_{n,n}$ on $\mathbb{R}^n$, which generates the $m-$root Finsler norm on $(\mathbb{R}_+^*)^{n-1}$:

$$F_{BM}(y) = \mathcal{A}^g_{BM}(y, \ldots, y) = \sqrt[n]{y^1 \cdot \ldots \cdot y^n};$$

c) the Chernov tensor $\mathcal{A}^g_C = \theta \circ \mu_{n-1,n}$ on $\mathbb{R}^n$ which generates the $m-$root Finsler norm on $(\mathbb{R}_+^*)^n$:

$$F_C(y) = \mathcal{A}^g_C(y, \ldots, y) = \left(\sum_{k=1}^{n} y^1 \cdot \ldots \cdot \hat{y}^k \cdot \ldots \cdot y^n \right)^{1/(n-1)},$$

where the symbol "hat" denotes absence of the corresponding factor.

The algebraic properties of these tensors represent a proficient subject of recent research, especially due to the existing interrelation between the properties of their attached algebras and the Finsler geometry which lies beyond the related physical models ([1]).

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