Existence and uniqueness solution of a quasistatic electro-elastic antiplane contact problem with Tresca friction law

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Abstract. We study the antiplane frictional contact models for electro-elastic materials, both in quasistatic case. The material is assumed to be electro-elastic and the friction is modeled with Tresca's law and the foundation is assumed to be electrically conductive. First, we derive the classical variational formulation of the model which is given by a system coupling an evolutionary variational equality for the displacement field and a time-dependent variational equation for the potential field. Then we prove the existence and uniqueness of weak solution to the model.

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1 Introduction and preliminaries

Antiplane shear deformations are the simplest examples of deformations that solids can undergo, in antiplane shear of a cylindrical body, the displacement is parallel to the generators of the cylinder and is dependent of the axial coordinate [5, 6, 7, 10]. Piezoelectric materials for which the mechanical properties are elastic are called electro-elastic materials and those for which the mechanical properties are viscoelastic are called electro-viscoelastic materials. General models for electro-elastic materials can be found in [4, 8, 9]. Static frictional contact problems for electro-elastic materials and contact problems for electro-viscoelastic materials were considered in [2, 4, 8, 9]. In all these references, the foundation was assumed to be electrically insulated.

In the last years, a considerable attention has been paid to the analysis of antiplane shear deformations within the context of elasticity theory (see for example [1, 3, 10, 11] and the references therein). Processes of adhesion are important in industry where parts, usually nonmetallic, are glued together. Recently, composite materials reached prominence, since they are very strong and light, and therefore, of considerable importance in aviation and in the automotive industry. However,

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composite materials my undergo delamination under stress, in which different layers debond and move relative to each other.

In this paper, we study an antiplane contact problem for electro-elastic materials with Tresca friction law. We consider the case of antiplane shear deformation, i.e., the displacement is parallel to the generators of the cylinder and is dependent of the axial coordinate. We model the material with a homogeneous isotropic linear electro-elastic constitutive law and we neglect the inertial term in the equation of motion to obtain a quasistatic approximation of the process. Our interest is to describe a physical process in which both antiplane shear, contact, state of material with Tresca friction law and piezoelectric effect are involved, leading to a well posedness mathematical problem. In the variational formulation, this kind of problem leads to an integro-differential inequality. The main result which we provide concerns the existence and uniqueness of the weak solution to the model.

The rest of the paper is structured as follows. In Section 2, we describe the model of the frictional contact process between electro-elastic body and a conductive deformable foundation. In Section 3, we derive the variational formulation, it consists of a variational inequality for the displacement field coupled with a time-dependent variational equation for the electric potential. We state our main result, the existence and uniqueness of the weak solution to the model in Theorem 3.2. The proof of the theorem is provided in the end of section 4, where it is based on arguments of evolutionary inequalities.

2 The mathematical model

The physical setting is as follows: We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$, it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axes with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times (-\infty, +\infty)$, the cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b. On the other hand, we assume that the one-dimensional measure of Γ_1 and Γ_a, denoted meas Γ_1 and meas Γ_a, are positive. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and therefore the displacement field vanishes there, surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (-\infty, +\infty)$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b \times (-\infty, +\infty)$. The cylinder is in contact over $\Gamma_3 \times (-\infty, +\infty)$ with a conductive obstacle, so called foundation. The contact is frictional and is modeled with Tresca's law. Let:

$$\mathbf{f}_0 = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.1}$$

$$\mathbf{f}_2 = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times [0, T] \to \mathbb{R}, \tag{2.2}$$

$$q_0 = q_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.3}$$

$$q_2 = q_2(x_1, x_2, t) : \Gamma_b \times [0, T] \to \mathbb{R}. \tag{2.4}$$

The forces (2.1) and (2.2) and the electric charges (2.3), (2.4) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and to an electric potential field φ which are independent on x_3 and have the form

$$\mathbf{u} = (0, 0, u) \quad \text{with} \quad u = u(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.5}$$

$$\varphi = \varphi(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}. \tag{2.6}$$

Below in this paper, the indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable, also, a dot above represents the time derivative. We use $\mathcal{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or equivalently, the space of symmetric matrices of order 3, and "$\cdot$", $\|\cdot\|$ will represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and $\mathcal{S}^3$; we have :

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{1/2} \quad \text{for all} \quad \mathbf{u} = (u_i), \ \mathbf{v} = (v_i) \in \mathbb{R}^3$$

and

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{1/2} \quad \text{for all} \quad \boldsymbol{\sigma} = (\sigma_{ij}), \ \boldsymbol{\tau} = (\tau_{ij}) \in \mathcal{S}^3.$$

The infinitesimal strain tensor is denoted $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ and the stress field by $\boldsymbol{\sigma} = (\sigma_{ij})$. We also denote by $\mathbf{E}(\varphi) = (E_i(\varphi))$ the electric field and by $\mathbf{D} = (D_i)$ the electric displacement field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad E_i(\varphi) = -\varphi_{,i}\,.$$

The material's is modeled by the following electro-elastic constitutive law

$$\boldsymbol{\sigma} = \lambda (\operatorname{tr} \boldsymbol{\varepsilon}(\mathbf{u})) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{E}^* \mathbf{E}(\varphi), \tag{2.7}$$

$$\mathbf{D} = \mathcal{E} \boldsymbol{\varepsilon}(\mathbf{u}) + \beta \mathbf{E}(\varphi), \tag{2.8}$$

where λ and μ are the Lamé coefficients, tr $\boldsymbol{\varepsilon}(\mathbf{u}) = \varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, β is the electric permittivity constant, $\mathcal{E}$ represents the third-order piezoelectric tensor and $\mathcal{E}^*$ is its transpose. In the antiplane context (2.5) and (2.6), using the constitutive equations (2.7) and (2.8) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \boldsymbol{\sigma}_{13} \\ 0 & 0 & \boldsymbol{\sigma}_{23} \\ \boldsymbol{\sigma}_{31} & \boldsymbol{\sigma}_{32} & 0 \end{pmatrix}, \tag{2.9}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix} \tag{2.10}$$

where

$$\boldsymbol{\sigma}_{13} = \boldsymbol{\sigma}_{31} = \mu\partial_{x_1} u$$

and

$$\boldsymbol{\sigma}_{23} = \boldsymbol{\sigma}_{32} = \mu\partial_{x_2} u.$$

Let

$$\mathcal{E}\boldsymbol{\varepsilon} = \begin{pmatrix} e\left(\varepsilon_{13} + \varepsilon_{31}\right) \\ e\left(\varepsilon_{23} + \varepsilon_{32}\right) \\ e\varepsilon_{33} \end{pmatrix} \quad \forall \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathcal{S}^3, \tag{2.11}$$

where e is a piezoelectric coefficient. We also assume that the coefficients θ, μ, β and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\boldsymbol{\varepsilon} \cdot \mathbf{v} = \boldsymbol{\varepsilon} \cdot \mathcal{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon} \in \mathcal{S}^3$, $\mathbf{v} \in \mathbb{R}^3$, it follows from (2.11) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall \mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{2.12}$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0}, \quad D_{i,i} - q_0 = 0 \text{ in } \mathcal{B} \times (0,T),$$

where $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$ represents the divergence of the tensor field $\boldsymbol{\sigma}$. Taking into account (2.1), (2.3), (2.5), (2.6), (2.9) and (2.10), the equilibrium equations above are reduced to the following scalar equations

$$\operatorname{div}(\mu\nabla u) + \operatorname{div}(e\nabla\varphi) + f_0 = 0, \text{ in } \Omega \times (0,T), \tag{2.13}$$

$$\operatorname{div}(e\nabla u - \beta\nabla\varphi) = q_0. \tag{2.14}$$

Putting

$$\operatorname{div}\boldsymbol{\tau} = \tau_{1,1} + \tau_{1,2} \text{ in } \boldsymbol{\tau} = \left(\tau_1(x_1, x_2, t), \tau_2(x_1, x_2, t)\right)$$

and

$$\nabla v = (v_{,1}, v_{,2}), \quad \partial_\nu v = v_{,1}\,\nu_1 + v_{,2}\,\nu_2 \text{ for } v = v(x_1, x_2, t).$$

Now, we describe the boundary conditions. During the process, the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and the electric potential vanish on $\Gamma_1 \times (-\infty, +\infty)$; thus, (2.5) and (2.6) imply that

$$u = 0 \text{ on } \Gamma_1 \times (0,T), \tag{2.15}$$

$$\varphi = 0 \text{ on } \Gamma_a \times (0,T). \tag{2.16}$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty, +\infty)$ and

$$\boldsymbol{\nu} = (\nu_1, \nu_2, 0) \quad \text{with} \quad \nu_i = \nu_i\,(x_1, x_2) : \Gamma \to \mathbb{R}, \quad i = 1, 2. \tag{2.17}$$

For a vector $\mathbf{v}$, we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary, defined by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \tag{2.18}$$

respectively. In (2.18) and everywhere in this paper, "$\cdot$" represents the inner product on the space $\mathbb{R}^d$ ($d = 2, 3$). Moreover, for a given stress field σ, we denote by σ_ν and σ_τ the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \tag{2.19}$$

From (2.9), (2.10) and (2.17, we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0, 0, \mu\partial_\nu u + e\partial_\nu \varphi)\,, \quad \mathbf{D} \cdot \boldsymbol{\nu} = e\partial_\nu u - \beta\partial_\nu \varphi. \tag{2.20}$$

Taking into account (2.2), (2.4) and (2.20), the traction condition on $\Gamma_2 \times (-\infty, \infty)$ and the electric conditions on $\Gamma_b \times (-\infty, \infty)$ are

$$\mu\partial_\nu u + e\partial_\nu \varphi = f_2 \quad \text{on} \quad \Gamma_2 \times (0, T)\,, \tag{2.21}$$

$$e\partial_\nu u - \beta\partial_\nu \varphi = q_2 \quad \text{on} \quad \Gamma_b \times (0, T)\,. \tag{2.22}$$

For the description the frictional contact condition and the electric conditions on $\Gamma_3 \times (-\infty, +\infty)$. First, from (2.5) and (2.17), we infer that the normal displacement vanishes, $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (2.5) and (2.17)–(2.19), we conclude that

$$\mathbf{u}_\tau = (0, 0, u)\,, \quad \boldsymbol{\sigma}_\tau = (0, 0, \sigma_\tau) \tag{2.23}$$

where

$$\sigma_\tau = (0, 0, \mu\partial_\nu u + e\partial_\nu \varphi)\,.$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with Tresca's friction law, that is

$$\begin{cases} |\boldsymbol{\sigma}_\tau(t)| \leq g, \\ |\boldsymbol{\sigma}_\tau(t)| = -g\frac{\dot{\mathbf{u}}_\tau}{|\dot{\mathbf{u}}_\tau|} \quad \text{if} \quad \dot{\mathbf{u}}_\tau \neq 0 \quad \text{on} \quad \Gamma_3 \times [0, T]. \end{cases} \tag{2.24}$$

Here $g : \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{\mathbf{u}}_\tau$ represents the tangential velocity on the contact boundary (see [9, 8, 4] for details). Using now (2.23), it is straightforward to see that the friction law (2.24) implies

$$\begin{cases} |\mu\partial_\nu u + e\partial_\nu \varphi| \leq g, \\ |\mu\partial_\nu u + e\partial_\nu \varphi| = -g\frac{\dot{u}}{|\dot{u}|} \quad \text{if} \quad \dot{u} \neq 0 \quad \text{on} \quad \Gamma_3 \times [0, T]. \end{cases} \tag{2.25}$$

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D} \cdot \boldsymbol{\nu} = k\ (\varphi - \varphi_F) \quad \text{on} \quad \Gamma_3 \times (0, T)\,,$$

where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient. By using (2.20) and the previous equality, we obtain

$$e\partial_\nu u - \beta\partial_\nu\varphi = k\ (\varphi - \varphi_F) \quad \text{on} \quad \Gamma_3 \times (0, T)\,. \tag{2.26}$$

Finally, we prescribe the initial displacement,

$$u(0) = u_0 \quad \text{in} \quad \Omega, \tag{2.27}$$

where u_0 is a given function on Ω. We collect the above equations and conditions to obtain the following mathematical model which describes the antiplane shear of an electro-elastic cylinder in frictional contact with a conductive foundation.

Problem $\mathcal{P}$

Find the displacement field $u : \Omega \times [0, T] \to \mathbb{R}$ and the electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\operatorname{div}(\mu\nabla u) + \operatorname{div}(e\nabla\varphi) + f_0 = 0, \quad \text{in} \quad \Omega \times (0, T)\,, \tag{2.28}$$

$$\operatorname{div}(e\nabla u - \alpha\nabla\varphi) = q_0 \quad \text{in} \quad \Omega \times (0, T)\,, \tag{2.29}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times (0, T)\,, \tag{2.30}$$

$$\mu\partial_\nu u + e\partial_\nu\varphi = f_2 \quad \text{on} \quad \Gamma_2 \times (0, T)\,, \tag{2.31}$$

$$\begin{cases} |\mu\partial_\nu u + e\partial_\nu\varphi| \le g, \\ |\mu\partial_\nu u + e\partial_\nu\varphi| = -g\frac{\dot{u}}{|\dot{u}|} \quad \text{if} \quad \dot{u} \neq 0 \quad \text{on} \quad \Gamma_3 \times [0, T], \end{cases} \tag{2.32}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = q_2 \quad \text{on} \quad \Gamma_b \times (0, T)\,, \tag{2.33}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = k\,(\varphi - \varphi_F) \quad \text{on} \quad \Gamma_3 \times (0, T)\,, \tag{2.34}$$

$$u(0) = u_0 \quad \text{in} \quad \Omega. \tag{2.35}$$

Note that once the displacement field u and the electric potential φ which solve Problem $\mathcal{P}$ are known, then the stress tensor $\boldsymbol{\sigma}$ and the electric displacement field $\mathbf{D}$ can be obtained by using the constitutive laws (2.9) and (2.10), respectively.

3 Variational formulation and main result

We derive the variational formulation of the Problem $\mathcal{P}$. First, we introduce the Sobolev spaces

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}, \quad W = \{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_a\}$$

where, here and below, we write w for the trace γw of a function $w \in H^1(\Omega)$ on Γ. Since *meas* $\Gamma_1 > 0$ and $\operatorname{meas}\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u, v)_V = \int_\Omega \nabla u \cdot \nabla v \, dx \quad \forall u, v \in V, \quad (\varphi, \psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx \quad \forall \varphi, \psi \in W.$$

Moreover, the associated norms

$$\|v\|_V = \|\nabla v\|_{(L^2(\Omega)^2)} \quad \forall v \in V, \quad \|\psi\|_W = \|\nabla\psi\|_{(L^2(\Omega)^2)} \quad \forall \psi \in W \tag{3.1}$$

are equivalent on V and W, respectively, with the usual norm $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace theorem we deduce that there exist two positive constants $c_V > 0$ and $c_W > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \le c_V \|v\|_V \quad \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \le c_W \|\psi\|_W \quad \forall \psi \in W. \tag{3.2}$$

For a real Banach space $(X, \|\cdot\|_X)$, we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \le p \le \infty$, $k = 1, 2, \ldots$; we also denote by $C([0,T];X)$ the space of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X$$

and we use the standard notations for the Lebesgue space $L^2(0,T;X)$ as well as the Sobolev space $W^{1,2}(0,T;X)$. In particular, recall that the norm on the space $L^2(0,T;X)$ is given by the formula

$$\|u\|^2_{L^2(0,T;X)} = \int_0^T \|u(t)\|^2_X \, dt$$

and the norm on the space $W^2(0,T;X)$ is defined by the formula

$$\|u\|^2_{W^{1,2}(0,T;X)} = \int_0^T \|u(t)\|^2_X \, dt + \int_0^T \|\dot{u}(t)\|^2_X \, dt.$$

Finally, we use the notation $W^2(0,T)$ for the space $W^2(0,T;\mathbb{R})$ and the notation $\|\cdot\|_{W^2(0,T)}$ for the norm $\|\cdot\|_{W^2(0,T;\mathbb{R})}$. In the study of the Problem $\mathcal{P}$, we assume that the electric permittivity coefficient satisfy

$$\alpha \in L^\infty(\Omega) \text{ and there exists } \alpha^* > 0 \text{ such that } \alpha(\mathbf{x}) \ge \alpha^* \text{ a.e. } \mathbf{x} \in \Omega. \tag{3.3}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \text{ and } \mu(\mathbf{x}) > 0 \text{ a.e. } \mathbf{x} \in \Omega, \tag{3.4}$$

$$e \in L^{\infty}(\Omega). \tag{3.5}$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0 \in W^{1,2}\left(0,T;L^2(\Omega)\right), \quad f_2 \in W^{1,2}\left(0,T;L^2(\Gamma_2)\right), \tag{3.6}$$

$$q_0 \in L^2(\Omega), \quad q_2 \in L^2(\Gamma_b). \tag{3.7}$$

The electric conductivity coefficient and the friction bound function g satisfies the following properties

$$k \in L^{\infty}(\Gamma_3) \quad \text{and} \quad k(\mathbf{x}) \geq 0 \quad \text{a.e.} \quad \mathbf{x} \in \Gamma_3, \tag{3.8}$$

$$g \in L^{\infty}(\Gamma_3) \quad \text{and} \quad g(\mathbf{x}) \geq 0 \quad \text{a.e.} \quad \mathbf{x} \in \Gamma_3. \tag{3.9}$$

Finally, we assume that the electric potential of the foundation and the initial displacement are such that

$$\varphi_F \in L^2(\Gamma_3). \tag{3.10}$$

The initial data are chosen such that

$$u_0 \in \ V \tag{3.11}$$

and, moreover,

$$a_{\mu}(u_0,v) + a_e(\varphi_0,v) + j(v) \geq (f(0),v)_V \quad \forall v \in V, \tag{3.12}$$

where φ_0 is the unique element in W which satisfies the following properties :

$$a_{\beta}(\varphi_0,\psi)_V - a_e(\varphi_0,\psi) = (q(0),v)_W \quad \forall \psi \in W, \tag{3.13}$$

We define the functional $j : [0,T] \longrightarrow \mathbb{R}_+$ by the formula

$$j(v) = \int_{\Gamma_3} g|v|\, da \quad \forall v \in V. \tag{3.14}$$

Let us the mappings $f : [0,T] \rightarrow V$ and $q : [0,T] \rightarrow W$,given by the formulas

$$(f(t),v)_V = \int_{\Omega} f_0(t)v\, dx + \int_{\Gamma_2} f_2(t)v\, da, \tag{3.15}$$

$$(q(t),\psi)_W = \int_{\Omega} q_0(t)\psi\, dx - \int_{\Gamma_b} q_2(t)\psi\, da + \int_{\Gamma_3} k\, \varphi_F(t)\psi\, da, \tag{3.16}$$

for all $v \in V$, $\psi \in W$ and $t \in [0,T]$. The definition of f and q are based on Riesz's representation theorem; moreover, it follows from assumptions by (3.6)–(3.7), that the integrals above are well-defined and

$$f \in W^{1,2}(0,T;V), \tag{3.17}$$

$$q \in W^{1,2}(0,T;W). \tag{3.18}$$

Next, we define the bilinear forms $a_\mu : V \times V \to \mathbb{R}$, $a_e : V \times W \to \mathbb{R}$, $a_e : W \times V \to \mathbb{R}$, and $a_\alpha : W \times W \to \mathbb{R}$, by:

$$a_\mu(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v \, dx, \tag{3.19}$$

$$a_e(u,\varphi) = \int_\Omega e \nabla u \cdot \nabla \varphi \, dx = a_e(\varphi,u), \tag{3.20}$$

$$a_\alpha(\varphi,\psi) = \int_\Omega \beta \nabla \varphi \cdot \nabla \psi \, dx + \int_{\Gamma_3} k \, \varphi \psi \, dx, \tag{3.21}$$

for all u, $v \in V$, $\varphi, \psi \in W$. Assumptions (3.14)–(3.16) imply that the integrals above are well defined and, using (3.1) and (3.2), it follows that the forms a_μ, a_e and a_e are continuous; moreover, the forms a_μ and a_α are symmetric and, in addition, the form a_α is W-elliptic, since

$$a_\alpha(\psi,\psi) \geq \alpha^* \|\psi\|_W^2 \quad \forall \psi \in W. \tag{3.22}$$

the variational formulation of Problem is based on the following result.

Lemma 3.1. *If (u,φ) is a smooth solution to Problem $\mathcal{P}$, then $(u(t),\varphi(t)) \in X$ and*

$$\begin{aligned} & a_\mu(u(t), v - \dot{u}(t)) + a_e(\varphi(t), v - \dot{u}(t)) \\ & + j(v) - j(\dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V \quad \forall v \in V, \ t \in [0,T], \end{aligned} \tag{3.23}$$

$$a_\alpha(\varphi(t),\psi) - a_e(u(t),\psi) = (q(t),\psi)_W \quad \forall \psi \in W, \ t \in [0,T], \tag{3.24}$$

$$u(0) = u_0. \tag{3.25}$$

Proof. Let (u,φ) denote a smooth solution to Problem $\mathcal{P}$, we have $u(t) \in V$, $\dot{u}(t) \in V$ and $\varphi(t) \in W$ a.e. $t \in [0,T]$ and, from (2.28), (2.30) and (2.31), we get

$$\begin{aligned} & \int_\Omega \mu \nabla u(t) \cdot \nabla (v - \dot{u}(t)) \, dx + \int_\Omega e \nabla \varphi(t) \cdot \nabla (v - \dot{u}(t)) \, dx = \\ & \int_\Omega f_0(t) (v - \dot{u}(t)) \, dx + \int_{\Gamma_2} f_2(t) (v - \dot{u}(t)) \, da + \\ & \int_{\Gamma_3} (|\mu \partial_\nu u(t) + e \partial_\nu \varphi(t)|) (v - \dot{u}(t)) \, da, \quad \forall v \in V \ \ t \in (0,T), \end{aligned}$$

and from (2.29) and (2.33)–(2.34), we have

$$\begin{aligned} & \int_\Omega \alpha \nabla \varphi(t) \cdot \nabla \psi \, dx - \int_\Omega e \nabla u(t) \cdot \nabla \psi \, dx = \int_\Omega q_0(t) \psi \, dx - \int_{\Gamma_b} q_2(t) \psi \, da + \\ & \int_{\Gamma_3} k \, \varphi_F(t) \psi \, da, \quad \forall \psi \in W \ \ t \in (0,T). \end{aligned} \tag{3.26}$$

From (2.32) and (3.14), it follows that

$$\begin{aligned}(3.27)\qquad & a_\mu\left(u(t), v-\dot{u}(t)\right)+a_e\left(\varphi(t), v-\dot{u}(t)\right)-\int_{\Gamma_3}\left(\left|\mu \partial_\nu u(t)+e \partial_\nu \varphi(t)\right|\right) \\ & \left(v-\dot{u}(t)\right) da=\left(f(t), v-\dot{u}(t)\right)_V, \quad \forall v \in V,\ t \in[0, T].\end{aligned}$$

Keeping in mind (3.16) and (3.20)–(3.21), we find the second equality in Lemma 3.1, i.e.,

$$a_\alpha\left(\varphi(t), \psi\right)-a_e\left(u(t), \psi\right)=(q(t), \psi)_W \quad \forall \psi \in W,\ t \in[0, T]. \tag{3.28}$$

Using the frictional contact condition (2.32) and (3.14) on $\Gamma_3 \times(0, T)$, we deduce that for all $t \in[0, T]$

$$j(\dot{u}(t))=-\int_{\Gamma_3}\left(\left|\mu \partial_\nu u(t)+e \partial_\nu \varphi(t)\right|\right) \dot{u}(t)\, da, \tag{3.29}$$

it's very easy to see that

$$j(v) \geq-\int_{\Gamma_3}\left(\left|\mu \partial_\nu u(t)+e \partial_\nu \varphi(t)\right|\right) v\, da, \quad \forall v \in V. \tag{3.30}$$

The first inequality in Lemma 3.1 follows now from (3.27), (3.29) and (3.30). □

Now, the use of Lemma 3.1 and condition (3.25) gives the following variational Problem :

Problem $\mathcal{PV}$

Find a displacement field $u:[0, T] \rightarrow V$ and an electric potential field $\varphi:[0, T] \rightarrow W$ such that

$$\begin{aligned}(3.31)\qquad & a_\mu\left(u(t), v-\dot{u}(t)\right)+a_e\left(\varphi(t), v-\dot{u}(t)\right)+ \\ & j(v)-j(\dot{u}(t)) \geq\left(f(t), v-\dot{u}(t)\right)_V, \quad \forall v \in V,\ t \in[0, T],\end{aligned}$$

$$a_\alpha\left(\varphi(t), \psi\right)-a_e\left(u(t), \psi\right)=(q(t), \psi)_W, \quad \forall \psi \in W,\ t \in[0, T], \tag{3.32}$$

$$u(0)=u_0. \tag{3.33}$$

Our main existence and uniqueness result, which we state now and prove in the next section is the following :

Theorem 3.2. *Assume that (3.3)–(3.18) hold, then the variational problem $\mathcal{PV}$ admits a unique solution (u, φ) satisfying*

$$u \in W^{1,2}(0, T ; V), \quad \varphi \in W^{1,2}(0, T ; W). \tag{3.34}$$

We note that an element (u, φ) which solves Problem $\mathcal{PV}$ is called a weak solution of the antiplane contact Problem $\mathcal{PV}$. We conclude by Theorem 3.2 that the antiplane contact Problem $\mathcal{P}$ has a unique weak solution, provided that (3.3)–(3.18) hold.

4 Proof of Theorem 3.2

We start with the Proof of Theorem 3.2 which will be carried out in several steps. To this end, in the rest of this section we will assume that (3.3)–(3.18) hold. In the first step we will consider the following problem :

Lemma 4.1. *Let (u, φ) the solution of $\mathcal{PV}$ and it has the regularity expressed in (3.34). Then there exist a symmetric bilinear form and V-elliptic $a(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$ and there exist a function $\bar{f} \in W^{1,2}(0,T;V)$ such that*

$$a\left(u(t), v - \dot{u}(t)\right) + j(v) - j(\dot{u}(t)) \geq \left(\bar{f}(t), v - \dot{u}(t)\right)_V \ \forall v \in V, \ p.p. \ t \ \in [0,T], \tag{4.1}$$

$$u(0) = u_0. \tag{4.2}$$

Moreover, the initial data u_0 satisfies that

$$u_0 \in V, \tag{4.3}$$

and

$$a\left(u_0, v\right) + j(v) \geq \left(\bar{f}(0), v\right)_V \quad \forall v \in V. \tag{4.4}$$

Proof. We use the Riesz representation theorem to define the operators $B : W \longrightarrow W$ and $C : V \longrightarrow W$ by :

$$\left(B\varphi, \psi\right)_W = a_\beta\left(\varphi, \psi\right) \quad \forall \varphi, \psi \in W, \tag{4.5}$$

and

$$\left(Cv, \psi\right)_W = a_e\left(v, \varphi\right) \quad \forall \varphi, \psi \in W, \quad \forall v \in V. \tag{4.6}$$

From (4.5), it follows that the operator B satisfies the following points :

- B is an symmetric operator,
- B is a positive operator defined on W. By

  $$\left(B\varphi, \varphi\right)_W = a_\beta\left(\varphi, \varphi\right) = \int_\Omega \beta \nabla\varphi \nabla\varphi dx \geq \beta \left\|\nabla\varphi\right\|_W^2 > 0. \tag{4.7}$$

  Otherwise, from (4.6), it follows that the operator C satisfy the following point :

- C is a linear operator.

Using (4.5) and (4.6) in (3.32), we obtain

$$B\varphi(t) = Cu(t) + q(t) \ \text{ p.p. t } \ \in [0,T]. \tag{4.8}$$

Keeping in mind that the operator B is inversible, then equality (4.8) becomes

$$\varphi(t) = B^{-1}Cu(t) + B^{-1}q(t) \ \text{ p.p. t } \ \in [0,T] \tag{4.9}$$

where $B^{-1}: W \longrightarrow W$ represent the inverse operator of B. Using now (4.9) in (3.31) infer

$$a_\mu (u(t), v - \dot{u}(t)) + a_e \left(B^{-1}Cu(t) + B^{-1}q(t), v - \dot{u}(t)\right) + \\ j(v) - j(\dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V, \quad \forall v \in V,\ t \in [0,T]. \tag{4.10}$$

Now, the last inequality implies that

$$a_\mu (u(t), v - \dot{u}(t)) + a_e \left(B^{-1}Cu(t), v - \dot{u}(t)\right) + j(v) - j(\dot{u}(t)) \geq \\ (f(t), v - \dot{u}(t))_V - a_e \left(B^{-1}q(t), v - \dot{u}(t)\right), \quad \forall v \in V,\ t \in [0,T]. \tag{4.11}$$

Next, we define the bilinear forms $a(\cdot,\cdot): V \times V \to \mathbb{R}$ by :

$$a(u(t), v) = a_\mu (u(t), v) + a_e \left(B^{-1}Cu(t), v\right), \quad \forall u, v \in V, \tag{4.12}$$

and define the function $\bar{f}(\cdot): [0,T] \longrightarrow V$ by :

$$\left(\bar{f}(t), v\right)_V = (f(t), v)_V - a_e \left(B^{-1}q(t), v\right) \quad \forall v \in V. \tag{4.13}$$

Using the continuity of the operators B^{-1} and C, then the bilinear form $a(\cdot,\cdot)$ defined (4.12) is continuous on V.

Let $u, v \in V$ and let $B^{-1}Cu = w \in W$, $B^{-1}Cv = z \in W$, i.e. $Cu = Bw$ and $Cv = Bz$ respectively. Using now (4.5), thus

$$\int_\Omega \beta \nabla w \nabla \varphi dx = \int_\Omega e \nabla u \nabla \varphi dx \quad \forall \varphi \in W. \tag{4.14}$$

Similarly, from (4.6) we get

$$\int_\Omega \beta \nabla z \nabla \psi dx = \int_\Omega e \nabla v \nabla \psi dx \quad \forall \psi \in W. \tag{4.15}$$

Keeping in mind

$$a_e \left(B^{-1}Cu, v\right) = a_e (w, v) = \int_\Omega e \nabla w \nabla v dx \tag{4.16}$$

and from (4.14)–(4.15), it follows that

$$a_e (w, v) = a_e (u, z), \tag{4.17}$$

$$B^{-1}Cu = w \tag{4.18}$$

and

$$B^{-1}Cv = z. \tag{4.19}$$

We change v by ψ in (4.16) and we use (4.15) we deduce that

$$a_e \left(B^{-1}Cu, \psi\right) = \int_\Omega e \nabla w \nabla \psi dx = \int_\Omega \beta \nabla z \nabla w dx. \tag{4.20}$$

Using (4.17)–(4.19) for $w = \varphi$, then

$$a_e\left(B^{-1}Cu, v\right) = \int_\Omega \beta \nabla z \nabla u dx, \tag{4.21}$$

Hence,we obtain

$$a_e\left(B^{-1}Cu, v\right) = a_e\left(u, B^{-1}Cv\right). \tag{4.22}$$

Consequently, from (4.22) it follows that the bilinear form $a_e(\cdot,\cdot)$ is symmetric.

On the other hand, for all $u \in V$, we have

$$a_e\left(B^{-1}Cu, v\right) = a_e(w, v) = \int_\Omega e\nabla w \nabla v dx. \tag{4.23}$$

Now, from (4.14) and (4.21) with $\varphi = v$, we have

$$a_e\left(B^{-1}Cu, v\right) = \int_\Omega e\nabla w \nabla w dx \geq 0. \tag{4.24}$$

The last inequality gives that

$$a(u,u) = a_\mu(u,u) + a_e\left(B^{-1}Cu, u\right) \geq \mu^* \|u\|_V^2 \quad \forall u \in V.$$

Consequently, we can write

$$a(u,u) \geq \mu^* \|u\|_V^2 \quad \forall u \in V, \tag{4.25}$$

then the bilinear form $a(\cdot,\cdot)$ is V-elliptic. Finally, the regularity $f \in W^{1,2}(0,T;V)$ and $q \in W^{1,2}(0,T;W)$ combined with the definition of $\bar{f}(\cdot)$ in (4.13), implies that

$$\bar{f} \in W^{1,2}(0,T;V). \tag{4.26}$$

The inequality (4.11) combined with the equalities (4.12) and (4.13) prove that the bilinear form $a_e(\cdot,\cdot)$ satisfies (4.1) and (4.2).

Moreover, the use of (3.13), gives that

$$B\varphi_0 = Cu_0 + q(0). \tag{4.27}$$

Hence,

$$\varphi_0 = B^{-1}Cu_0 + B^{-1}q(0). \tag{4.28}$$

We combine (4.27) and (4.28), we infer

$$a_\mu(u_0, v) + a_e\left(B^{-1}Cu_0, v\right) + j(v) \geq (f(0), v)_V - a_e\left(B^{-1}q(0), v\right), \quad \forall v \in V. \tag{4.29}$$

Using (4.12) and (4.13), it follows that u_0 satisfy (4.3) and (4.4), which conclude the prof of lemma. □

In the second step, we state our main existence and uniqueness result.

Theorem 4.2. *Let X be a Hilbert space and assume that (4.3), (4.4), (4.12), (4.13), (3.14), (4.25) and (4.26) hold, and let j defined by (3.14) is a proper convex l.s.c. function. Then, there exists a unique solution u to the problems (4.1) and (4.2) which satisfies $u \in W^{1,2}(0,T;X)$.*

Proof. The proof of Theorem 4.2 will be carried out in several steps and is based on the study of a sequence of evolutionary variational inequalities. All details of proof are founded in ([10], p.77). □

Lemma 4.3. *There exists a unique function u solution to Problem (4.1) and (4.2) which satisfies $u \in W^{1,2}(0,T;V)$.*

Proof. Clearly, the existence and uniqueness of the solution u follows Prolem (4.1) and (4.2) result from Theorem 4.2 applied for $X = V$. Moreover, by Lemma 4.1, we conclude that the bilinear form defined by (4.12) is continuous, symmetric and V-elliptic; otherwise, the function $\bar{f}(\cdot)$ defined by (4.13) has the regularity $\bar{f} \in W^{1,2}(0,T;V)$, and the initial displacement u_0 satisfies (4.3) and (4.4). In addition, the functional j defined by (3.14) is a proper convex l.s.c. on V.

Under the state assumptions in Theorem 4.2,We conclude that, there exists a unique $u \in W^{1,2}(0,T;V)$. This conclude the proof of Lemma 4.3. □

Now, we have all the ingredients to prove Theorem 3.2.

Proof of Theorem 3.2. Let $u \in W^{1,2}(0,T;V)$ be the solution of Problems (4.1) and (4.2) obtained in Lemma 4.1, and let $\varphi : [0,T] \longrightarrow W$ be the electrical potential field defined by (4.8). Notice that regularity $u \in W^{1,2}(0,T;V)$ and $q \in W^{1,2}(0,T;W)$ imply that $\varphi \in W^{1,2}(0,T;W)$.

From (4.8) it follows that

$$(B\varphi,\psi)_W - (Cu,\psi)_W = (q(t),\psi)_W \quad \forall \psi \in W, \text{ p.p. t } \in [0,T], \tag{4.30}$$

and we by use of the definitions (4.5) and (4.6) of operators B and C, respectively, we deduce that the couple (u,φ) satisfies (4.2).

The similar arguments combined with the definition (4.5) of the bilinear form $a(\cdot,\cdot)$ and the definition (4.6) of the function $\bar{f}(\cdot)$ show that the couple (u,φ) satisfies (4.1); we conclude that (u,φ) is the solution of Problem $\mathcal{PV}$ with the regularity (3.34), which implies the existence part of Theorem 3.2.

The uniqueness of the solution of Problems (4.1) and (4.2) follows from Lemma 4.1 combined with (4.8), which achieves the proof of Theorem 3.2.

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