Left invariant vector fields of a class of top spaces

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Abstract. The notion of top space as a generalization of Lie group is defined in [1] and the case with finite number of identities was studied in [2, 3, 4]. In this paper we investigate top spaces whose right/left translations are diffeomorphisms. These top spaces - called right/left top spaces may have an infinite number of identities. A characterization of right top spaces is given in Section 2, where we prove that left/right invariant vector fields of this kind of top spaces form a Lie algebra, whose dimension is equal to the rank of the left shift mapping l_g, for any g from the top space. In Section 3, a relation between the Lie algebra of a top space and the Lie algebra of a spatial class of Lie groups is presented. In Section 4 the quotient space of right top spaces and the generalized action of a top space on a manifold are defined and discussed.

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1 Introduction

Lie groups were initially introduced as a tool to solve or simplify ordinary and partial differential equations, and found numerous applications in Physics. The notion of top space as a generalization of Lie group was considered in [1]. We recall first its definition.

Definition 1.1. A top space T is a smooth manifold admitting an operation called multiplication, subject to the following set of rules:

- $(xy)z = x(yz)$ for all $x, y, z \in T$;
- for each $x \in T$ there exists a unique $z \in T$ such that $xz = zx = x$ (we denote z by $e(x)$);
- for each $x \in T$ there exists $y \in T$ such that $xy = yx = e(x)$ (we denote y by x^{-1});
- the mapping $m_1 : T \to T$ is defined by $m_1(x) = x^{-1}$ and the mapping $m_2 : T \times T \to T$ is defined by $m_2(x, y) = xy$ are smooth maps;
- $e(xy) = e(x)e(y)$ for all $x, y \in T$.

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We denote the left translation and the right translation associated to an element $g \in T$, the maps $l_g : T \to T$ and $r_g : T \to T$, respectively defined by

$$l_g(p) = gp; \qquad r_g(p) = pg, \qquad \forall p \in T.$$

A vector field X on a top space T is called a *left invariant vector field* if $(l_g)_*(X(p)) = X(l_g(p))$, where $p, g \in T$. Right invariant vector fields are similarly defined by substituting r_g with l_g.

Note that left and right translations in Lie groups are diffeomprphisms, while top spaces don't generally have this property. Suppose that T is a top space such that $Tt = T$, for some $t \in T$, where $Tt = \{gt : g \in T\}$. The following theorem implies that r_t is a diffeomorphism for all $t \in T$. In particular, $r_{e(t)} = id$, for all $t \in T$.

Theorem 1.1. *[1] If $Tt \cap Tg \neq \varnothing$, then $Tt = Tg$, where $t, g \in T$. In particular, $Tt = Te(t)$.*

Definition 1.2. A top space T is called a right (left) top space if $Tt = T$ ($tT = T$), for some $t \in T$.

There exist right(left) top spaces with an infinite number of identities.

Example 1.3. [1] The n- dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ with the product

$$((a_1, a_2, ..., a_n) + \mathbb{Z}^n, (b_1, b_2, ..., b_n) + \mathbb{Z}^n) = (a_1 + b_1, a_2 + b_2, ..., a_{n-1} + b_{n-1}, a_n) + \mathbb{Z}^n$$

is a top space. We have $e((a_1, a_2, ..., a_n) + \mathbb{Z}^n) = (0, 0, ..., 0, a_n) + \mathbb{Z}^n$, and hence T^n has an infinite number of identities. Moreover, $T^n a = T^n$ for all $a \in T^n$.

Example 1.4. Suppose that G is a Lie group and that two smooth manifolds Λ and I are given. If $p : I \times \Lambda \to G$ is a smooth mapping, then $\Lambda \times G \times I$ with the product $(\lambda, g, i)(\lambda_1, g_1, i_1) = (\lambda, gp(i, \lambda_1)g_1, i_1)$ is a top space, which is called *Rees matrix top space*, and which is denoted by $M(G, I, \Lambda, p)$ [1]. Suppose that I is an one point set; then $\Lambda \times G \times I \simeq \Lambda \times G$ is a right top space, which we call *right Rees matrix*, denoted by $M(G, \Lambda, p)$. Further, we have $e((\lambda, g)) = (\lambda, p(\lambda)^{-1})$ and consequently $card(e(\Lambda \times G)) = card(\Lambda)$.

2 Lie algebra of right top spaces

It was proved that the set of left invariant vector fields of a top space T with a finite number of identities form a Lie algebra [1]. The following theorem shows that the same statement holds for right top spaces. Its proof is similar to the one from [5, Proposition 3.7].

Theorem 2.1. *The set of left invariant vector fields on the right top space T is a Lie algebra under the Lie bracket operation.*

Proof. We only need to show that a left invariant vector field is smooth. Let X be a left invariant vector field and fix $t \in T$. X is determined by $X(e(t))$, since $Te(t) = T$. If $f \in C^\infty(T)$ then

$$Xf(g) = Xf(ge(t)) = (l_g)_* X(e(t))f = X(e(t))(f \circ l_g). \tag{2.1}$$

Let Y be a smooth vector field on T such that $Y(e(t)) = X(e(t))$. We define the following smooth functions on T:

$$i^1_{e(t)} : T \to T \times T, \quad i^1_{e(t)}(h) = (h, e(t))$$
$$i^2_g : T \to T \times T, \quad i^2_g(h) = (g, h).$$

We note that $(0, Y)$ is a smooth vector field on $T \times T$. Consequently $[(0, Y)(f \circ m_2)] \circ i^1_{e(t)}$ is a smooth function. Then we have

$$\begin{aligned}[(0, Y)(f \circ m_2)] \circ i^1_{e(t)}(g) &= (0, Y)(g, e(t))(f \circ m_2) \\ &= 0(g)(f \circ m_2 \circ i^1_{e(t)}) + Y(e(t))(f \circ m_2 \circ i^2_g) \\ &= X(e(t))(f \circ m_2 \circ i^2_g) = X(e(t))(f \circ l_g).\end{aligned}$$

Hence (2.1) implies that Xf is a smooth function, and consequently X is smooth. □

Theorem 2.2. *Let T be a right top space. Then:*

- *rank(l_t) is constant for all $t \in T$;*
- *rank(l_t) =rank($l_{t'}$) for all $t, t' \in T$.*

Proof. Suppose that $\mathrm{rank}(l_t) = k$ at g and $\mathrm{rank}(l_{t'}) = k'$ at g'. Let $X_i(g)$, $i = 1, 2, ..., k$ be independent tangent vectors such that their images under $(l_t)_*$ remain independent. We note that $(l_p)_* X_i(g)$, $i = 1, 2, ..., k$, for all $p \in T$ are independent, since if $\sum a_i (l_p)_*(X_i(g)) = 0$ and then $(l_{tp^{-1}})_* \sum a_i (l_p)_*(X_i(g)) = \sum a_i (l_{te(p)})_* X_i(g) = \sum a_i (l_t)_* X_i(g) = 0$, and consequently $a_i = 0$ for $i = 1, 2, ..., k$. Hence $(l_{g'g^{-1}})_* X_i(g) \in T_{g'}(T)$, $i = 1, ..., k$, are independent. If $\sum a_i (l_{t'})_*((l_{g'g^{-1}})_* X_i(g)) = 0$, then $\sum a_i (l_{tt'^{-1}})_*(l_{t'})_*((l_{g'g^{-1}})_* X_i(g)) = \sum a_i (l_{tg'g^{-1}})_* X_i(g) = 0$, and consequently $a_i = 0$ for $i = 1, 2, ..., k$. Therefore we have $k \leq k'$. In a similar way it can be proved that $k' \leq k$. Hence $k = k'$. □

Theorem 2.3. *Let T be a right top space with its Lie algebra τ. Then* $\dim(\tau) =$ *rank($l_{e(t)}$), for all $t \in T$.*

Proof. Let $\alpha : \tau \to T_{e(t)}(T)$ be the map defined by $\alpha(X) = X(e(t))$, for some $t \in T$. We show that α is an one to one homorphism of Lie algebras and that $img(\alpha) = img((l_{e(t)})_*)$. But α is injective, for if $\alpha(X) = \alpha(Y)$, then $X(e(t)) = Y(e(t))$. Since X and Y are left invariant vector fields $X(g) = (l_g)_* X(e(t)) = (l_g)_* Y(e(t)) = Y(g)$ for all $g \in T$ then consequently $X = Y$. Let $X(e(t)) \in T_{e(t)}(T)$. If there exists a left invariant vector field $Y \in \tau$ such that $\alpha(Y) = X(e(t))$, then $Y(e(t)) = (l_{e(t)})_* Y(e(t)) = X(e(t))$, and consequently $X(e(t)) \in img((l_{e(t)})_*)$. Conversely, if $X(e(t)) \in img((l_{e(t)})_*)$, then there exists $Y(e(t)) \in T_{e(t)}(T)$ such that $(l_{e(t)})_* Y(e(t)) = X(e(t))$. Hence $(l_{e(t)})_* X(e(t)) = X(e(t))$.

Now let X be the left invariant vector field which is determined by $X(e(t))$. Then $\alpha(X) = X(e(t))$. It is trivial to check that α is linear and preserves the bracket operation. Hence $\alpha : \tau \to img((l_{e(t)})_*$ is an isomorphism and $\dim(\tau) = rank(l_{e(t)})$, which concludes the proof. □

The following lemmas give a characterization of right top spaces.

Lemma 2.4. *Let T be a right top space. Then $e(T)$ is embedded in T.*

Proof. If $t,\ g \in e(T)$, then $(l_t|_{gT})^{-1}(t) = g$. Since $T = \cup_{g\in e(T)} gT$, we have $l_t^{-1}(t) = \cup_{g\in e(T)}(l_t|_{gT})^{-1}(t) = e(T)$. Moreover, by theorem 2.2 $rank(l_t)$ is constant and consequently $e(T)$ is embedded in T. □

Lemma 2.5. *The function $e : T \to T$ is a smooth map and has constant rank.*

Proof. Since $e = m_2 \circ (i \times m_1)$, then e is a smooth map. T is a right top space, and hence $e(p) = e(p)e(k) = e(pk) = e \circ r_k(p)$, for every $p, k \in T$ and $e_*(p) = e_*(pk) \circ (r_k)_*(p)$. Since r_k is a diffeomorphism, $rank(e)(p) = rank(e)(pk)$ and consequently $rank(e)$ is constant on $e^{-1}(e(p)) = pT$. Lemma 2.4 implies that $e : T \to e(T)$ is a smooth map. Hence $rank(e)(p) \leq \dim(e(T))$, for all $p \in T$. In addition, $e : e(T) \to e(T)$ is the identity map and consequently $rank(e)(e(p)) = \dim(e(T))$, for all $p \in T$. Hence e has constant rank on T. □

Lemma 2.6. *Let T be a right top space. Then (tT, i) is embedded in T, for $t \in T$. Moreover, tT is a Lie group.*

Proof. Since $tT = e^{-1}(e(t))$ and the function e has constant rank, it is an embedded submanifold of T. Therefore the maps $m_2 \circ (i \times i) : tT \times tT \to tT$ and $m_1 \circ i : tT \to tT$ are smooth and tT is a Lie group. □

Theorem 2.7. *Let T be a right top space. Then T is isomorphic with a right Rees matrix.*

Proof. Let $t \in e(T)$ and let the map $p : e(T) \to tT$ be defined by $p(s) = t$, for every $s \in e(T)$. We prove that the right Rees matrix $M(tT, e(T), p)$ is isomorphic with T. Let $\alpha : T \to e(T) \times tT$ be the map defined by $\alpha(g) = (e(g), tg)$, for every $g \in T$. α is injective and surjective and $\alpha^{-1} : e(T) \times tT \to T$ is given by $\alpha^{-1}(s, tg) = stg$. α and α^{-1} are smooth. In addition $\alpha(gg') = \alpha(g)\alpha(g')$, hence α is an isomorphism of top spaces. □

3 One parameter subgroups of a right top space

We need to first introduce the definition of top space covering projection:

Definition 3.1. [2] Let T be a top space and let G be a topological group. Then a covering projection $P : T \to G$ is called a top space covering projection, if P satisfies the following conditions:

- $P(t) = e$, for all $t \in e(T)$, where e is identity element;
- $P(t_1t_2) = P(t_1)P(t_2)$, for all $t_1, t_2 \in T$.

Theorem 3.1. *[3] If T is a top space with a finite number of identities, then there exists a topological group G and a top space covering $P : T \to G$.*

Let T be a top space with finite number of identities. Since $e^{-1}(e(t))$, for every $t \in T$, it is open and it is a Lie group. Theorem 2.3 of [3] implies that these Lie groups are diffeomorphic. In addition, the proof of theorem 3.1 implies that $P : T \to$

$e^{-1}(e(t_0))$, $P(t) = e(t_0)te(t_0)$ is a top space covering, for every $t \in T$. Hence the following theorem shows that the Lie algebra of a top space T with a finite number of identities and the Lie algebra of the Lie group $e^{-1}(e(t_0))$ are isomorphic.

Theorem 3.2. *[2] If T is a top space with $card(e(T)) < \infty$, G is a Lie group and $P : T \to G$ a top space covering projection, then there exists a one-to-one correspondence between left invariant vector fields of G and left invariant vector fields of T. Moreover the Lie algebra created by the Left invariant vector fields of T is isomorphic to the Lie algebra of G.*

In the remainder of this section we show that the Lie algebra of a right top space T and $tT = e^{-1}(e(t))$ are isomorphic as well.

Definition 3.2. [2] Suppose that T is a top space. A curve $\varphi : \mathbb{R} :\to T$ is called *a one parameter subgroup* of T if it satisfies the condition $\varphi(t_1 + t_2) = \varphi(t_1)\varphi(t_2)$, for all $t_1, t_2 \in \mathbb{R}$.

Theorem 3.3. *Suppose that T is a right top space then there is a correspondence between one parameter subgroups of T and its left invariant vector fields.*

Proof. Let $\varphi : \mathbb{R} \to T$ be a one parameter subgroup. We have $\varphi(s) = \varphi(0 + s) = \varphi(0)\varphi(s) = \varphi(s)\varphi(0)$. Hence $e(\varphi(s)) = \varphi(0)$, for all $s \in \mathbb{R}$. Consequently, $\varphi(\mathbb{R}) \subseteq \varphi(0)T$ and φ is a one parameter subgroup of the Lie group $\varphi(0)T$. Hence there exists a left invariant vector field corresponding to φ.

Conversely, suppose that X is a left invariant vector field on T; then X defines a local one parameter group action $\sigma : W \subseteq \mathbb{R} \times T \to T$, such that $\sigma(0, g) = g$ and $(\sigma_g)_*(\frac{d}{dt}|_0) = X(g)$. We define $\varphi : (-\delta, \delta) \to T$ by $\varphi(t) = \sigma(t, \varphi(0))$, where $\varphi(0) \in e(T)$, and $(-\delta, \delta)$ is such that $(-\delta, \delta) \times \varphi(0) \subseteq W$. We show that if $t, s, t+s \in (-\delta, \delta)$.

Then $\varphi(t + s) = \varphi(t)\varphi(s)$. If the parameter s is fixed and $\overline{\sigma}(t, \varphi(s)) = \varphi(s)\varphi(t)$, then we have $\overline{\sigma}(0, \varphi(s)) = \varphi(s)\varphi(0) = \varphi(s)$, since $\varphi(0) \in e(T)$ and T is a right top space. In addition, $\overline{\sigma}$ satisfies the same differential equation for σ:

$$\begin{aligned} \tfrac{d}{dt}(\overline{\sigma}(t, \varphi(s))) &= \tfrac{d}{dt}(\varphi(s)\varphi(t)) = l_{\varphi(s)*}\varphi_*(\tfrac{d}{dt}|_t) = l_{\varphi(s)*}(X(\varphi(t))) \\ &= X(\varphi(s)\varphi(t)) = X(\overline{\sigma}(t, \varphi(s)). \end{aligned}$$

By the uniqueness theorem of ordinary differential equation, we conclude that:

$$\varphi(t + s) = \sigma(t + s, \varphi(0)) = \sigma(t, \sigma(s, \varphi(0))) = \overline{\sigma}(t, \varphi(s)) = \varphi(s)\varphi(t).$$

This implies that $\varphi((-\delta, \delta)) \subseteq \varphi(0)T$. Now it suffices to prove that $(-\delta, \delta) = R$. But $\varphi : (-\delta, \delta) \to \varphi(0)T$ is smooth, since $\varphi(0)T$ is imbedded in T. Hence $X(\varphi(s)) \in T_{\varphi(s)}\varphi(0)T$. Consequently, $X|\varphi(0)T$ is a smooth left invariant vector field on the Lie group $\varphi(0)T$ and we have $(-\delta, \delta) = \mathbb{R}$. □

Theorem 3.4. *Let T be a right top space. Then there exists a correspondence between one parameter subgroups of tT, for every $t \in e(T)$ and one parameter subgroups of T.*

Proof. Let $t \in e(T)$. Lemma 2.6 implies that tT is Lie group and $l_t : T \to tT$ is a smooth map. Then $l_t(g) = t$, for $g \in e(T)$ and $l_t(t_1t_2) = tt_1t_2 = tt_1tt_2 = l_t(t_1)l_t(t_2)$. Hence if α is a one parameter subgroup of T, then $l_t \circ \alpha$ is a one parameter subgroup of tT. In addition if α is a one parameter subgroup of tT then $l_t^{-1}(\alpha) = \bigcup_{g \in e(T)} g\alpha$. For every $g \in e(T)$, $g\alpha$, is a one parameter subgroup of T. □

Theorem 3.5. *Let T be a right top space. Then there is a one to one correspondence between left invariant vector fields of T and left invariant vector fields of tT, for $t \in e(T)$. Moreover, the Lie algebra created by the left invariant vector fields of T is isomorphic to the Lie algebra of tT.*

Proof. If X is a left invariant vector field of T then by theorem 3.3 there is a one parameter subgroup of T correspondence to X that $\varphi(0) = t$. Theorem 3.4 implies that $t\varphi$ is a one parameter subgroup of tT. Since tT is a Lie group, then there exists exactly one left invariant vector field in correspondence to $t\varphi$. Conversely, if X is a left invariant vector field on tT, then its one parameter group is contained in T and by theorem 3.3 there exists a left invariant vector field correspondence to this one parameter subgroup. □

4 Quotient space of right top spaces

We begin this section with the definition of a sub-top space.

Definition 4.1. A couple (N, φ) is a sub - top space of the top space T if:

- N is a top space;
- (N, φ) is a submanifold of T;
- $\varphi : N \to T$ is a homomorphism.

For a sub - top space N of the top space T, we set $N_a = N \cap e^{-1}(e(a))$ and $\Gamma_N = \{a \in T | N_a \neq \varnothing\}$. Note that a sub - top space N of a right top space T is a right top space too.

Lemma 4.1. *Let (N, φ) be a sub - top space of the right top space T. Then Γ_N is a sub - top space of T.*

Proof. $e(N)$ is an embedded submanifold of N by lemma 2.4. Since T is a right top space we have $T \simeq M(tT, e(T), p)$ (see the proof of theorem 2.7 for details). Under this isomorphism Γ_N and the top space $M(tT, e(N), p)$ are in one to one correspondence. $(M(tT, e(N), p), (\varphi \circ i) \times i)$ is a submanifold of $M(tT, e(T), p)$ and $(\varphi \circ i) \times i$ is a homomorphism. Hence $(M(tT, e(N), p), (\varphi \circ i) \times i)$ is a sub - top space of $M(tT, e(T), p)$ and consequently Γ_N is a sub - top space of T. □

Remark 4.2. We recall that $T/N = \cup_{x \in \Gamma(N)} xN_x$ [4]. Let T be a right top space then $xN_x = xN$, for if $n \in N$ and $x \in \Gamma_N$, $xn = xe(x)n$. Since T is a right top space $e(x)n \in e^{-1}(e(x))$. In addition, $e(x)n \in N$, since $x \in \Gamma_N$. Hence $xn = xe(x)N \in xN_x$. The converse is trivial.

Theorem 4.2. *Let (N, φ) be a closed sub - top space of the right top space T. Then T/N has a manifold structure such that:*

- *π is smooth.*
- *There exist local smooth sections of T/N in Γ_N, that is, if $tN \in T/N$ then there are a neighborhood W of tN and a smooth map $\tau : W \to \Gamma_N$ such that $\pi \circ \tau = id$.*

Proof. Theorem 2.7 implies that $T \simeq^{\alpha} M(nT, e(T), p)$ and $N \simeq M(nN, e(N), p)$, for some $n \in e(N)$. In addition by using Lemma 4.1, $\alpha|_{\Gamma_N} : \Gamma_N \to M(nT, e(N), p)$ is an isomorphism. Note that $nN = N_n$ and $nT = T_n$. N_n is a closed subgroup of the Lie group T_n, since N is closed by assumption. Hence by theorem 3.58 of [5], T_n/N_n has a manifold structure such that $\pi_0 : T_n \to T_n/N_n$ is smooth and for every $gN_n \in T_n/N_n$ there exists a local section i.e, there are a neighborhood W_0 of gN_n and a map $\tau_0 : W_0 \to T_n$ such that $\pi_0 \circ \tau_0 = id$. A coset of $M(T_n, e(T), p)/M(T_n, e(N), p)$ is of form (m, gN_n), where $m \in e(N)$ and $g \in T_n$. Hence $M(T_n, e(T), p)/M(N_n, e(N), p) = e(N) \times T_n/N_n$ and consequently it has a manifold structure. Under this manifold structure $\pi_1 = id \times \pi_0$ is smooth. In addition if $(m, gN_n) \in M(T_n, e(T), p)/M(N_n, e(N), p)$ then we define $\tau_1 = (id, \tau_0)$ on $W_1 = e(N) \times W_0$. Now it is clear that $\tau_1 \circ \pi_1 = id$.
Let $\beta : T/N \to M(T_n, e(T), p)/M(N_n, e(N), p)$ be the map defined by $\beta(xN_x) = (e(x), nxN_n)$. We prove that β is a one to one correspondence. It is well defined, for if $xN_x = yN_y$ then $e(x) = e(y)$. Remark 4.2 implies that $xN = yN$ hence $nxN = nyN$ and consequently $\beta(xN_x) = \beta(yN_y)$. It is one to one, since if $\beta(xN_x) = \beta(yN_y)$ then $e(x) = e(y)$ and $nxN = nyN$. Consequently $xN = e(x)nxN = e(x)nyN = yN$. In addition β is on to, sine if $(m, gN_n) \in M(T_n, e(T), p)/M(N_n, e(N), p)$ then $\beta(mgN) = (m, ngN_n)$. Now β induces a manifold structure on T/N. Note that $\pi_1 \circ \alpha = \beta \circ \pi$. Hence π is a smooth map. In addition if one define $\tau : \beta^{-1}(W_1) \to T/N$ by $\tau = \alpha^{-1} \circ \tau_1 \circ \beta$. Then $\pi \circ \tau = id$ and the proof is complete. □

Definition 4.3. A generalized left action of a top space T on a manifold M is a smooth map $\lambda : T \times M \to M$ such that:

- for each t_1 and $t_2 \in T$, $\lambda(t_1, \lambda(t_2, m)) = \lambda(t_1 t_2, m)$;
- for each $m \in M$ and $t \in T$, $\lambda(e(t), m) = m$.

Definition 4.4. Let λ be an action on the top space T. Then $H(m) = \{t \in T : \lambda(t, m) = m\}$ is called *the stabilizer* of λ.

Remark 4.5. Let λ be a generalized left action of the top space T on the manifold M. One can define for each $m \in M$ and $t \in T$ two maps which are related to λ:

- $\lambda_m : T \to M, \lambda_m(s) = \lambda(s, m)$ for every $s \in T$;
- $\lambda^t : M \to M$, $\lambda^t(n) = \lambda(t, n)$ for every $n \in M$.

Note that $\lambda^t \circ \lambda^{t^{-1}} = \lambda^{t^{-1}} \circ \lambda^t = id$, and consequently λ^t is a diffeomorphism for every $t \in T$. Moreover, $H(m) = \lambda_m^{-1}(m)$. A generalized action λ is transitive if for every $n, m \in M$ there is $t \in T$ such that $\lambda(t, n) = m$.

Theorem 4.3. *Let T be a right top space and λ be a left action of T on the manifold M. Then $H(m)$ is a sub - top space of T.*

Proof. Since T is a right top space, $se(t) = s$, for every $s, t \in T$. Hence $\lambda(e(s), \lambda(s, m)) = \lambda(s, \lambda(e(t), m))$ and we have $\lambda^{e(s)} \circ \lambda_m(s) = \lambda^s \circ \lambda_m(e(t))$. Consequently $(\lambda^{e(s)})_*(\lambda_m(s)) \circ (\lambda_m)_*(s) = (\lambda^s)_*(\lambda_m(e(t)) \circ (\lambda_m)_*(e(t))$. Since $\lambda^{e(s)}$ and λ^s are diffeomorphisms $rank(\lambda_m)$ in $e(t)$ and s is equal. Hence rank(λ_m) is constant and consequently $H(m) = \lambda_m^{-1}(m)$ is an embedded submanifold. In addition it is a generalized group and consequently it is a sub - top space. □

Theorem 4.4. *[5] Let $\eta : G \times M \to M$ be a transitive action of the Lie group G on the manifold M on the left. Let $m_0 \in M$, and let H be the stabilizer of m_0. Define a mapping $\beta : G/H \to M$ by $\beta(gH) = \eta(g, m_0)$. Then β is a diffeomorphism.*

Theorem 4.5. *Let T be a right top space and $\theta : T \times M \to M$ be a transitive left action. If H is the stabilizer of m_0, then there exists a diffeomorphism between T/H and $e(H) \times M$.*

Proof. Since H is closed, theorem 4.2 implies that T/H is diffeomorphic with $e(H) \times T_n/H_h$, for $h \in e(H)$. We define $\theta_0 : T_h \times M \to M$, $\theta_0(g, m) = \theta(g, m)$, for every $g \in T_h$ and $m \in M$. θ_0 is a transitive left action, for if m, $m_0 \in M$ there is $t \in T$ that $\theta(t, m) = m_0$. But the second condition of definition 4.2 implies that $\theta(ht, m) = m_0$. Hence $\theta_0(ht, m) = \theta(ht, m) = m_0$. H_h is the stabilizer of m_0 for the action θ_0. Now theorem 4.4 implies that $\beta(e(h), gH_h) = (e(h), \theta_0(g, m_0))$ is a diffeomorphism. □

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