A note on Morita equivalence of twisted crossed products

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Abstract. In this paper we prove that strongly Morita equivalent twisted actions of a locally compact group on C^*-algebras have strongly Morita equivalent twisted crossed products [4],[31]. We also present an elementary proof that every twisted C^*-dynamical system is Morita equivalent to an ordinary system [20] and we remind the notion of Morita equivalence of twisted inverse semigroup actions introduced by N. Sieben in [31] and a theorem that states that Morita equivalent actions have Morita equivalent twisted crossed products [31].

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1 Introduction and preliminaries

The notion of strong Morita equivalence of C^*-algebras was introduced by Rieffel in his study of induced representations of C^*-algebras [29]. Roughly speaking, two C^*-algebras A and B are said to be strongly Morita equivalent if there is a full Hilbert B-module X such that the algebra $\mathcal{K}(X)$ of compact operators is isomorphic to A. Strong Morita equivalence is an equivalence relation on C^*-algebras and played an important role in the study of transformation group C^*-algebras.

Strong Morita equivalence of crossed products by actions was discussed in [13], [20], [14], [9], [34]. Let A and B be C^*-algebras which are strongly Morita equivalent via a Banach A, B-imprimitivity bimodule X and let α and β be actions of a locally compact group G on A and B. It was shown that if there is a strongly continuous map $\tau\colon G \to Iso(X)$ which is compatible with α and β, then the crossed products $A \times_\alpha G$ and $B \times_\beta G$ are strongly Morita equivalent. In [4] Bui generalized this result to the cases of twisted actions and Green's twisted actions, introduced the notions of strong Morita equivalence of twisted actions and of Green's twisted actions and in each case showed that the associated crossed products C^*-algebras are strongly Morita equivalent.

Twisted actions were introduced by Busby and Smith [7], who constructed a twisted Banach $*$-algebra $L^1(A, G, \alpha, u)$ whose enveloping C^*-algebra is called *the*

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twisted-crossed-product of A by G under the action α relative to the multiplier u denoted by $A \times_{\alpha,u} G$. Packer and Raeburn studied in [24] the twisted crossed products of C^*-algebras by twisted actions of locally compact groups. They established that every twisted action is stably exterior equivalent to an ordinary action and hence every twisted crossed product is stably isomorphic to an ordinary crossed product. The twisted crossed product $A \times_{\alpha,u} G$ was defined [24], [2], [21], [26] as a C^*-algebra whose representation theory is the same as the covariant representation theory of (A, G, α, u) on Hilbert space. In [25], Packer and Raeburn deduced a structure theorem for twisted crossed products of C^*-bundles. They also showed that $A \times_{\alpha,u} G$ has a universal property that any covariant representation (π, v) of (A, G, α, u) in a multiplier algebra $M(C)$ gives rise to a homomorphism of $A \times_{\alpha,u} G$ into $M(C)$. In fact, this property gives a more satisfactory characterization of the twisted crossed product: it is a triple (B, i_A, i_G), consisting of a C^*-algebra B and a covariant homomorphism (i_A, i_G) of (A, G, α, u) into $M(B)$ [10], [11], [12].

Echterhoff [14] showed that every twisted action in the original sense of Green [17] is Morita equivalent to an ordinary action. The notion of Morita equivalent actions, introduced by Combes [9], is a useful generalization of the notion of stably exterior equivalence. In [14] it is proved that Morita equivalent twisted actions give rise to Morita equivalent twisted crossed products and almost all important properties of twisted actions are invariant under Morita equivalence.

In [3], Bui discussed strong Morita equivalence of crossed products by coactions and twisted crossed products by coactions. Twisted coactions were introduced by Phillips and Raeburn in [28]. If the group is abelian then a twisted coaction is effectively the same as a Green's twisted action [16]. Bui [3] introduced a natural notion of strong Morita equivalence of twisted coactions which is sufficient to ensure strong Morita equivalence of the associated twisted crossed product C^*-algebras.

Bui [3] proved that Morita equivalent systems (A, δ_A), (B, δ_B) have Morita equivalent crossed products, by showing that, if X is an $A - B$ imprimitivity bimodule with a compatible coaction δ_X of G, then $X \otimes_B (B \times_{\delta_B} G)$ is an $A \times_{\delta_A} G - B \times_{\delta_B} G$ imprimitivity bimodule [18]. Echterhoff and Raeburn [15] obtained a symmetric version of this result and also gave a relatively short proof of Bui's main theorem, which is the corresponding Morita equivalence for the twisted systems of [28].

In [5], Bui presented a new proof for Theorem 3.3, [3] based on the notion of crossed products of Hilbert C^*-modules introduced in [6]. Crossed products of Hilbert C^*-modules in [6] were defined as subspaces of adjointable operators between Hilbert C^*-modules.

In [1] the main result is related to the concept of strong Morita equivalence [29], that is, considering two crossed products by Hilbert C^*-bimodules $A \times_X \mathbb{Z}$ and $B \times_Y \mathbb{Z}$ in which the underlying algebras A and B are known to be strongly Morita equivalent to each other, under an imprimitivity bimodule M, are also strongly Morita equivalent. This is a generalization obtained independently by Curto, Muhly and Williams [13] on the one hand and Combes [9] on the other, in which a necessary condition is given for two strongly Morita equivalent C^*-algebras to remain strongly Morita equivalent, after one takes their crossed products by a locally compact group.

In this section we remind some basic definitions and notations about Busby-Smith twisted actions, twisted crossed products and strong Morita equivalence of twisted crossed products.

Let A be a C^*-algebra and let G be a locally compact group. We denote by $\mathrm{Aut}(A)$ the automorphism group of A and by $UM(A)$ the group of unitary elements in the multiplier algebra $M(A)$ of A. If $u \in UM(A)$, then $\mathrm{Ad}u$ is the inner automorphism $a \longmapsto uau^*$.

Definition 1.1. ([7], [24]) A *Busby-Smith twisted action* of G on A is a pair (α, u) of maps $\alpha\colon G \to A$, $u\colon G \times G \to UM(A)$ satisfying :

(a) u is strictly Borel measurable and for each $a \in A, s \longmapsto \alpha_s(a)$ is Borel measurable;

(b) $\alpha_e = id, u(e,s) = u(s,e) = 1$ for all $s \in G$;

(c) $\alpha_s \circ \alpha_t = \mathrm{Ad}u(s,t) \circ \alpha_{st}$ for all $s, t \in G$;

(d) $\alpha_r(u(s,t))u(r,st) = u(r,s)u(rs,t)$ for all $r, s, t \in G$.

The quadruple (A, G, α, u) is called a (*Busby-Smith*)(*separable*) *twisted dynamical system.*

If the cocycle u is trivial (i.e. identically 1) we say that (A, G, α, u) is an *ordinary dynamical system* and write (A, G, α) for short.

Definition 1.2. ([7], [24]) A *covariant representation* of a twisted dynamical system (A, G, α, u) is a pair (π, U) consisting of a non-degenerate representation π of A on a Hilbert space H and a Borel measurable map $U\colon G \to U(H)$ such that

(a) $U_sU_t = \pi(u(s,t))U_{st}$ for all $s, t \in G$;

(b) $\pi(\alpha_s(a)) = U_s\pi(a)U_s^*$ for all $a \in A, \in G$.

Definition 1.3. ([7], [24]) Let (A, G, α, u) be a twisted dynamical system. A *twisted crossed product* for (A, G, α, u) is a C^*-algebra B together with a non-degenerate homomorphism $i_A\colon A \to (B)$ and a strictly Borel map $i_G\colon G \to UM(B)$ satisfying :

(a) (i_A, i_G) is covariant in the sense that for $a \in A,\ s, t \in G$ we have

$$i_A(\alpha_s(a)) = i_G(s)i_A(a)i_G(s)^*,$$
$$i_G(s)i_G(t) = i_A(u(s,t))i_G(st);$$

(b) for any covariant representation (π, U) of (A, G, α, u) on a Hilbert space H, there is a non-degenerate representation $\pi \times U$ of B on H such that $\pi = (\pi \times U) \circ i_A$ and $U = (\pi \times U) \circ i_G$ ($\pi \times U$ is called the integrated form of (π, U));

(c) the set $\{i_A \times i_G(z);\ \ z \in L^1(G, A)\}$ is a dense subspace of B, where $i_A \times i_G(z)$ denotes the strictly defined Bochner integral $\int i_A(z(s))i_G(s)ds$.

We denote the twisted crossed product by $A \times_{\alpha,u} G$.

The set $B_c(G, A)$ of (equivalence classes of) bounded measurable functions from G into A with compact support is a $*$-algebra with the convolution and the involution defined by

$$(f * g)(y) = \int f(x)[\alpha_x(g(x^{-1}y))]u(x, x^{-1}y)dx,$$
$$f^*(y) = \Delta_G(y)^{-1}u(y, y^{-1})^*[\alpha_y(f(y^{-1}))^*].$$

We denote this $*$-algebra by $B_c(A, G, \alpha, u)$ or $\mathcal{A}_c$ for short and we view it as a dense $*$-subalgebra of $A \times_{\alpha,u} G$.

Definition 1.4. [29] a) Let B be a pre-C^*-algebra. A *right B-rigged space* is a right B-module X, which is a pre-B-Hilbert space (with compatible multiplication by complex numbers on B and X), with preinner product conjugate linear in the first variable such that

$$\langle x, yb\rangle_B = \langle x, y\rangle_B b$$

for all $x, y \in X$ and $b \in B$, which implies that

$$\langle xb, y\rangle_B = b^*\langle x, y\rangle_B$$

and such that the range of $\langle , \rangle_B$ generates a dense subalgebra of B. *Left B-rigged spaces* are defined similarly except that we require that B acts on the left of X, that the preinner product be conjugate linear in the second variable and that

$$\langle bx, y\rangle_B = b\langle x, y\rangle_B.$$

b) Let E and B be pre-C^*-algebras. An *$E - B$-imprimitivity bimodule* is a left-E-right-B-bimodule, X, which is equipped with an E-valued and B-valued preinner product with respect to which X is a left E-rigged space and a right B-rigged space such that

(1) $\langle x, y\rangle_E z = x\langle y, z\rangle_B$ for all $x, y, z \in X$;
(2) $\langle ex, ex\rangle_B \leq \|e\|^2\langle x, x\rangle_B$ for all $x \in X$ and $e \in E$;
(3) $\langle xb, xb\rangle_E \leq \|b\|^2\langle x, x\rangle_E$ for all $x \in X$ and $b \in B$.

c) Let $\widetilde{X}$ denote the additive group X with the conjugate operations of E, B and the complex numbers and the corresponding preinner products. When an element $x \in X$ is viewed as an element of $\widetilde{X}$, we write it as $\widetilde{x}$. Then these conjugate operations and preinner products on $\widetilde{X}$ are defined by

$$b\widetilde{x} = \widetilde{xb^*}, \ \ \widetilde{x}e = \widetilde{e^*x}$$

$$\langle\widetilde{x}, y\rangle_B = \langle x, y\rangle_B, \ \ \langle\widetilde{x}, y\rangle_E = \langle x, y\rangle_E,$$

for $x, y \in X$, $b \in B, e \in E$. We we also let $\widetilde{x}$ denote the corresponding element of $\mathrm{Hom}_B(X, B)$ defined by $\widetilde{x}(y) = \langle\widetilde{x}, y\rangle_B$. We call $\widetilde{X}$ the *dual* of the imprimitivity bimodule X.

Definition 1.5. [4] Let (A, G, α, u) and (B, G, β, v) be separable twisted dynamical systems. Suppose that X is a Banach A, B-imprimitivity bimodule. Let $\mathrm{Iso}(X)$ denote the set of all bijective linear isometries of X. An *$(\alpha, u), (\beta, v)$-compatible action of G on X* is a map $\tau\colon G \to \mathrm{Iso}(X)$ satisfying the following conditions :

(i) for each $x \in X$, the map $s \longmapsto \tau_s(x)$ from G into X is Borel;
(ii) ${}_A\langle\tau_s(x)|\tau_s(y)\rangle = \alpha_s({}_A\langle x|y\rangle)$, $\forall x, y \in X, \forall s \in G$,
$\langle\tau_s(y)|\tau_s(x)\rangle_B = \beta_s(\langle y|x\rangle_B)$, $\forall x, y \in X, \forall s \in G$;
(iii) $\tau_e(x) = x$, $\forall x \in X$,
$\tau_r(\tau_s(x)) = u(r, s)\tau_{rs}(x)v(r, s)^*$, $\forall x \in X, \forall\forall r, s \in G$.

The twisted actions (α, u) and (β, v) are called *strongly Morita equivalent* by means of the imprimitivity system (X, τ).

The relation of strongly Morita equivalence is an equivalence relation ([4]). We assume that (α, u) is strongly Morita equivalent to (β, v) by means of (X, τ). The map $\widetilde{\tau}_s(\widetilde{x}) = \widetilde{\tau_s(x)}$ for all $s \in G$ and $\widetilde{x} \in \widetilde{X}$ is a $(\beta, v), (\alpha, u)$-compatible action of G on $\widetilde{X}$.

2 Morita equivalence of twisted crossed products

In this section we prove that the associated twisted crossed products C^*-algebras of strong Morita equivalence of twisted actions and of Green's twisted actions are strongly Morita equivalent [4], [31]. We also present an elementary proof that every twisted C^*-dynamical system is Morita equivalent to an ordinary system [20].

Theorem 2.1. *[4] Let (A, G, α, u) and (B, G, β, v) be separable twisted dynamical systems. If the twisted actions (α, u) and (β, v) are strongly Morita equivalent by means of an imprimitivity system (X, τ), then $B_c(G, X)$ is a $B_c(A, G, \alpha, u)$, $B_c(B, G, \beta, v)$-imprimitivity bimodule. Therefore the twisted crossed products $A \times_{\alpha,u} G$ and $B \times_{\beta,v} G$ are strongly Morita equivalent.*

The proof of the theorem will result from the following three lemmas.

We denote $\mathcal{H}_c = B_c(G, X)$, $\mathcal{Y}_c = B_c(G, \widetilde{X})$.

Lemma 2.2. *[4] Let $T\colon \mathcal{H}_c \to \mathcal{Y}_c$ be defined by*

$$(T\xi)(s) = \Delta_G(s^{-1})[\tau_s(\xi(s^{-1}))v(s, s^{-1})]\tilde{}, \;\; \forall \xi \in \mathcal{H}_c, \forall s \in G.$$

Then T is conjugate linear and

(i) $T(f \cdot \xi) = (T\xi) \cdot f^*, \;\; \forall \xi \in \mathcal{H}_c, \forall f \in \mathcal{A}_c;$

$$T(\xi \cdot g) = g^* \cdot (T\xi), \;\; \forall \xi \in \mathcal{H}_c, \forall g \in \mathcal{B}_c$$

(ii) $\langle T\xi | T\eta \rangle_{\mathcal{A}_c} = {}_{\mathcal{A}_c}\langle \xi | \eta \rangle, \;\; \forall \xi, \eta \in \mathcal{H}_c$

$${}_{\mathcal{B}_c}\langle T\eta | T\xi \rangle = \langle \eta | \xi \rangle_{\mathcal{B}_c}, \;\; \forall \xi, \eta \in \mathcal{H}_c$$

Also T is bijective and its inverse T^{-1} is given by

$$(T^{-1}\phi)(s) = \Delta_G(s^{-1})[\widetilde{\tau}_s(\phi(s^{-1}))u(s, s^{-1})]\tilde{}, \;\; \forall \phi \in \mathcal{Y}_c, \forall s \in G.$$

Lemma 2.3. *[4] The following statements hold:*

(i) For each $\xi \in \mathcal{H}_c$, $\langle \xi | \xi \rangle_{\mathcal{B}_c}$ is a positive element in $\mathcal{B}$.

(ii) The linear span of the range of $\langle \cdot | \cdot \rangle_{\mathcal{B}_c}$ is dense in $\mathcal{B}_c$.

(iii) For any $f \in \mathcal{A}_c$ and $\xi \in \mathcal{H}_c$, we have in $\mathcal{B}$

$$\langle f \cdot \xi | f \cdot \xi \rangle_{\mathcal{B}_c} \leq \|f\|^2_{\mathcal{A}} \langle \xi | \xi \rangle_{\mathcal{B}_c}.$$

Proof. (i) Let $(\pi, L, \mathcal{H})$ be a covariant representation of the twisted system (B, G, β, v) such that the integrated form $(\pi \times L, \mathcal{H})$ is faithful. For any $\eta, \eta' \in \mathcal{H}_c$ and $h, h' \in \mathcal{H}$, we have

$$\langle(\pi \times L)(\langle \eta | \eta' \rangle_{\mathcal{B}_c})h | h'\rangle = \int\int \langle \pi(\langle \eta(t) | \eta'(s)\rangle_B) L_s h | L_t h'\rangle ds dt. \tag{2.1}$$

Let $\eta = \sum_{i=1}^{p} \lambda_i \odot x_i \in B_c(G) \odot X$. By [33], Lemma IV.3.2, the matrix $(\langle x_i | x_j\rangle_B)$ is a positive element of $M_p(B)$ and therefore there is a matrix $(b_{ij}) \in M_p(B)$ such that

$$\langle x_i | x_j \rangle_B = \sum_{m=1}^{p} b_{mi}^* b_{mj}, \ \ \forall i, j = 1, \ldots, p.$$

We then obtain

$$\langle \eta(t) | \eta(s) \rangle_B = \sum_{m=1}^{p} \left(\sum_{i=1}^{p} \lambda_i(t) b_{mi} \right)^* \left(\sum_{j=1}^{p} \lambda_j(s) b_{mj} \right) \tag{2.2}$$

It now follows from (1) and (2) that for any $h \in \mathcal{H}$

$$\langle(\pi \times L)(\langle \eta | \eta \rangle_{\mathcal{B}_c})h | h\rangle = \sum_{m=1}^{p} \left\| \int \left(\sum_{j=1}^{p} \lambda_j(s) b_{mj} \right) L_s h ds \right\|.$$

Therefore $\langle \eta(t) | \eta(s) \rangle_{\mathcal{B}_c}$ is a positive element of $\mathcal{B}$. Since

$$\|\langle \xi | \xi \rangle_{\mathcal{B}_c} - \langle \eta | \eta \rangle_{\mathcal{B}_c}\|_{\mathcal{B}} \leq \|\xi\|_1 \|\xi - \eta\|_1 + \|\xi - \eta\|_1 \|\eta\|_1,$$

we deduce that $\|\langle \xi | \xi \rangle_{\mathcal{B}_c}$ is also a positive element of $\mathcal{B}$.

(ii) Suppose that the linear span I_c of the range of $\langle \cdot | \cdot \rangle_{\mathcal{B}_c}$ is not dense in $\mathcal{B}_c$. Let I be the closure of I_c and let $(\pi, L, \mathcal{H})$ be a covariant representation of (B, G, β, v) such that $\ker(\pi \times L) = I$ and $\pi \times L \neq 0$. Since the linear span of elements $\langle x | x' \rangle_B$ with $x, x' \in X$ is dense in B, there are $\lambda \odot \langle x | x' \rangle_B \in B_c(G) \odot B$ and $h \in \mathcal{H}$ such that $\|(\pi \times L)(\lambda \odot \langle x | x' \rangle_B)h\|^2 \neq 0$. Put $\eta = \lambda \odot x \langle x | x' \rangle_B$ and $\xi = \lambda \odot x'$. We have

$$\|(\pi \times L)(\lambda \odot \langle x | x' \rangle_B)h\|^2 = \langle (\pi \times L)(\langle \eta | \xi \rangle_{\mathcal{B}_c})h | h\rangle$$

Hence $(\pi \times L)(\langle \eta | \xi \rangle_{\mathcal{B}_c}) \neq 0$. This is a contradiction.

(iii) Let ω be a state of $\mathcal{B}$. Put

$$\langle \eta | \eta' \rangle_\omega = \omega(\langle \eta | \eta' \rangle_{\mathcal{B}_c}), \ \ \forall \eta, \eta' \in \mathcal{H}_c.$$

Let $N_\omega = \{\eta \in \mathcal{H}_c ; \langle \eta | \eta \rangle_\omega\}$, let $q_\omega \colon \mathcal{H}_c \to \mathcal{H}_c / N_\omega$ be the quotient map and let $\mathcal{H}_\omega$ be the Hilbert space obtained by completing the space $\mathcal{H}_c / N_\omega$. The linear map $q_\omega \colon \mathcal{H}_c \to \mathcal{H}_\omega$ is bounded with respect to the L^1-norm on $\mathcal{H}_c$. For any $a \in A, s \in G$ and $\eta \in \mathcal{H}_c$, we put

$$(l_A(a)\eta)(t) = a\eta(t), \ \ (l_G(s)\eta)(t) = \tau_s(\eta(s^{-1}t))v(s, s^{-1}t)$$

and define

$$\pi(a)(q_\omega(\eta)) = q_\omega(l_A(a)\eta), \ \ L_s(q_\omega(\eta)) = q_\omega(l_G(s)\eta).$$

Then $(\pi, L, \mathcal{H}_\omega)$ is a covariant representation of (A, G, α, u) . Observe that

$$f \cdot \xi = \int l_A(f(s))l_G(s)\xi ds$$

$$q_\omega(f \cdot \xi = (\pi \times L)(f)q_\omega(\xi).$$

It then follows that

$$\omega(\langle f{\cdot}\xi|f{\cdot}\xi\rangle_{\mathcal{B}_c}) = \|q_\omega(f{\cdot}\xi)\|^2 = \|(\pi{\times}L)(f)q_\omega(\xi)\|^2 \le \|f\|^2_{\mathcal{A}}\|q_\omega(\xi)\|^2 = \|f\|^2_{\mathcal{A}}\omega(\langle\xi|\xi\rangle_{\mathcal{B}_c})$$

Since this is true for all states of $\mathcal{B}$, the inequality in (iii) holds. □

Lemma 2.4. *The following statements are true:*

(i) For each $\xi \in \mathcal{H}_c$, ${}_{\mathcal{A}_c}\langle\xi|\xi\rangle$ is a positive element in $\mathcal{A}$.

(ii) The linear span of the range of ${}_{\mathcal{A}_c}\langle\cdot|\cdot\rangle$ is dense in $\mathcal{A}_c$.

(iii) For any $g \in \mathcal{B}_c$ and $\xi \in \mathcal{H}_c$, we have

$$ {}_{\mathcal{A}_c}\langle\xi \cdot g|\xi \cdot g\rangle \le \|g\|^2_{\mathcal{B}} \ {}_{\mathcal{A}_c}\langle\xi|\xi\rangle$$

in $\mathcal{A}$.

Proof. We apply first Lemma 2.3 to $\mathcal{Y}_c$ in place of $\mathcal{H}_c$ and $\langle\cdot|\cdot\rangle_{\mathcal{A}_c}$ in place of $\langle\cdot|\cdot\rangle_{\mathcal{B}_c}$. Then we use Lemma 2.2 to get the desired results. □

Definition 2.1. [20] Let (B, G, β, v) and (A, G, α, u) be twisted systems and X is a $B - A$ equivalence bimodule. Let $\mathrm{Aut}(X)$ denote the set of bicontinuous linear bijections ϕ of X which satisfy the ternary homomorphism identity $\phi(x \cdot \langle y, z\rangle_A) = \phi(x) \cdot \langle\phi(y), \phi(z)\rangle_A$.(The analogous identity using B-valued inner products is equivalent.) We say that (B, G, β, v) and (A, G, α, u) are *Morita equivalent* if there is a strongly Borel map $\gamma\colon G \to \mathrm{Aut}(X)$ such that for $s, t \in G$ and $x, y \in X$:

1. $\alpha_s(\langle x, y\rangle_A) = \langle\gamma_s(x), \gamma_s(y)\rangle_A$;
2. $\beta_s({}_B\langle x, y\rangle) =_B \langle\gamma_s(x), \gamma_s(y)\rangle$;
3. $\gamma_s \circ \circ\gamma_t(x) = v(s, t) \cdot \gamma_{st}(x) \cdot u(s, t)^*$.

We write $(B, G, \beta, v) \sim_{X,\gamma} (A, G, \alpha, u)$ and call (X, γ) a *system of imprimitivity* implementing the equivalence.

Theorem 2.5. *[20] Let (A, G, α, u) be a twisted dynamical system and let $\mathcal{K}$ denote the compact operators on $\mathcal{H} = L^2(G)$. Then there is an ordinary action β of G on $A \otimes \mathcal{K}$ and a map $\delta\colon G \to Aut(A \otimes \mathcal{H})$ such that*
$(A \otimes \mathcal{K}, G, \beta)$ is Morita equivalent with (A, G, α, u).

Proof. By Packer-Raeburn stabilization trick (Theorem 3.4, [24]), we have a Borel map $w\colon G \to \mathcal{UM}(A\otimes\mathcal{K})$ which implements an exterior equivalence between an ordinary action $(\beta, 1)$ of G on $A\otimes\mathcal{K}$ and $(\alpha\otimes \mathrm{id}_{\mathcal{K}}, u\otimes 1)$.

Let $A\otimes\mathcal{H}$ have the canonical $A\otimes\mathcal{K}-A$ equivalence bimodule structure; so $A\otimes\mathcal{H}$ is the completion of the algebraic tensor product $A\odot\mathcal{H}$ with respect to the norm induced by the A-valued inner product $\langle a\otimes\xi, b\otimes\eta\rangle_A = \langle \eta,\xi\rangle_{\mathcal{H}} a^*b$. For $s\in G$, the rule $a\otimes\xi \longmapsto \alpha_s(a)\otimes\xi$ defines an automorphism of $A\odot\mathcal{H}$ which satisfies condition 1 in Definition 2.1 for this inner product, so is isometric with respect to the induced norm and thus extends to a map $\alpha_s\otimes\mathrm{id}_{\mathcal{H}}$ of $A\otimes\mathcal{H}$ into itself. Then for $x\in A\otimes\mathcal{H}$, the map $s\longmapsto \alpha_s\otimes\mathrm{id}_{\mathcal{H}}(x)$ is Borel, using the fact that $s\longmapsto\alpha_s(a)$ is Borel for $a\in A$, together with a routine density argument.

Now define $\delta_s\colon A\otimes\mathcal{H}\to A\otimes\mathcal{H}$ by $\delta_s(x) = w_s^*\cdot\alpha_s\otimes\mathrm{id}_{\mathcal{H}}(x)$.

Then straightforward calculations on elementary tensors in $A\otimes\mathcal{H}$ verify that each δ_s satisfies the ternary homomorphism identity and the map $s\longmapsto\delta_s$ satisfies conditions 1-3 in Definition 2.1. For example, for any $s,t\in G$ and $a\otimes\xi\in A\otimes\mathcal{H}$ we have

$$\begin{aligned}
\delta_s\circ\delta_t(a\otimes\xi) &= w_s^*\cdot\alpha_s\otimes\mathrm{id}_{\mathcal{H}}(w_s^*\cdot\alpha_s\otimes\mathrm{id}_{\mathcal{H}}(a\otimes\xi))\\
&= w_s^*\alpha_s\otimes\mathrm{id}_{\mathcal{K}}(w_t)^*\cdot\alpha_s\circ\alpha_t(a)\otimes\xi\\
&= (w_s^*\alpha_s\otimes\mathrm{id}_{\mathcal{K}}(w_t)^*u(s,t)\otimes 1)\cdot\alpha_{st}\otimes\mathrm{id}_{\mathcal{H}}(a\otimes\xi)\cdot u(s,t)^*\\
&= w_{st}^*\cdot\alpha_{st}\otimes\mathrm{id}_{\mathcal{H}}(a\otimes\xi)\cdot u(s,t)^* = \delta_{st}(a\otimes\xi)\cdot u(s,t)^*
\end{aligned}$$

In particular, condition 1 (or 2) implies each δ_s is isometric and therefore bicontinuous since each δ_s is invertible. Thus each δ_s belongs to $\mathrm{Aut}(A\otimes\mathcal{H})$.

It remains to show that the map $s\longmapsto\delta_s$ is strongly Borel. Fix $x\in A$ and let $\{e_i\}$ be a countable approximate identity for $A\otimes\mathcal{K}$). Thus each of the maps $s\longmapsto w_s^*e_i$ and $s\longmapsto\alpha_s\otimes\mathrm{id}_{\mathcal{H}}(x)$ are Borel, so that $s\longmapsto w_s^*e_i\cdot\alpha_s\otimes\mathrm{id}_{\mathcal{H}}(x)$ is Borel for each i. Since $s\longmapsto\delta_s$ is the pointwise limit of these Borel maps, it too is Borel and the theorem follows. $\square$

3 Morita equivalence of twisted crossed products by inverse semigroup

In this section we remind the notion of Morita equivalence of twisted inverse semigroup actions introduced by N. Sieben in [31] and a theorem that states that Morita equivalent actions have Morita equivalent twisted crossed products.

Definition 3.1. [30] If ${}_AX_B$ is an imprimitivity bimodule then there is a bijective correspondence, called the **Rieffel correspondence**, between closed subbimodules of X and closed ideals of A. If I is a closed ideal of A then $I\cdot X$ is a closed subbimodule of X. Similarly $X\cdot J$ is a closed subbimodule of X if J is a closed ideal of B. On the other hand, if Y is a closed subbimodule of X then ${}_IY_J$ is an imprimitivity bimodule, where I is the closed span of ${}_A\langle Y,Y\rangle$ and J is the closed span of $\langle Y,Y\rangle_B$. We call ${}_IY_J$ an **imprimitivity subbimodule** of X.

Definition 3.2. [31] A *partial automorphism* of the imprimitivity bimodule ${}_AX_B$ is an isomorphism between two imprimitivity subbimodules of X.

We denote the set of partial automorphisms by $\mathrm{PAut}(X)$.

Definition 3.3. [32] Let A be a C^*-algebra and let S be a unital inverse semigroup with idempotent semilattice E and unit e. A *Busby-Smith twisted action of S on A* is a pair (β, v), where for all $s \in S$, $\beta_s\colon A_{s^*} \to A_s$ is a partial automorphism, that is, an isomorphism between closed ideals of A and for all $s, t \in S$, $v_{s,t}$ is a unitary multiplier of A_{st}, such that for all $r, s, t \in S$ we have :

1. $A_e = A$;
2. $\beta_s \beta_t = \mathrm{Ad} v_{s,t} \circ \beta_{st}$;
3. $v_{s,t} = 1_{M(A_{st})}$ if s or t is an idempotent;
4. $\beta_r(a v_{s,t}) v_{r,st} = \beta_r(a) v_{r,s} v_{rs,t}$ for all $a \in A_{r^*} A_{st}$

Definition 3.4. [31] The Busby-Smith twisted actions (A, S, α, u) and (B, S, β, v) are *Morita equivalent* if there is an imprimitivity bimodule ${}_A X_B$ and a map $s \longmapsto (\alpha_s, \phi_s, \beta_s) : S \longrightarrow \mathrm{PAut}(X)$ such that $\phi_s\colon X_{s^*} \to X_s$, where $X_s := A_s \cdot X = X \cdot B_s$ and for all $s, t \in S$ we have

$$\phi_s \phi_t = u_{s,t} \cdot \phi_{st}(\cdot) \cdot v^*_{s,t}.$$

We say that (X, ϕ) is a Morita equivalence between (α, u) and (β, v) and we write $(A, S, \alpha, u) \sim_{X,\phi} (B, S, \beta, v)$.

Remark 3.5. [31] Morita equivalence of Busby-Smith twisted actions is an equivalence relation.

Remark 3.6. [31] If the actions (A, S, α, u) and (B, S, β, v) are Morita equivalent, then the C^*-algebras A and B are Morita equivalent.

The crossed product $A \times_{\alpha,u} S$ of a Busby-Smith twisted action (A, S, α, u) is the Hausdorff completion of the Banach $*$-algebra

$$L_\alpha = \left\{ x \in l^1(S, A) : \ x(s) \in A_s \ \text{ for all } \ s \in S \right\}$$

with operations $(x * y)(s) \sum_{rt=s} \alpha_r(\alpha_{r^{-1}}(x(r)) y(t)) u_{r,t}$ and $x^*(s) = u_{s,s^{(*}} \alpha_s(x(s^*)^*)$ in the C^*-seminorm $\| \cdot \|_\alpha$ defined by

$$\| \cdot \|_\alpha = \sup \left\{ \|(\pi \times V)(x))\| : \ (\pi, V) \ \text{ is a covariant representation of } \ (A, S, \alpha, u) \right\}.$$

Theorem 3.1. *[31] If (A, S, α, u) and (B, S, β, v) are Morita equivalent actions, then the crossed products $A \times_{\alpha,u} S$ and $B \times_{\beta,v} S$ are also Morita equivalent.*

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