On preserving the univalence integral operator

Aisha Ahmed Amer and Maslina Darus

Abstract. In this paper, the work deals with preserving the univalence of an integral operator defined by a generalized derivative operator. For this purpose, we introduce new generalized derivative operator and define an integral operator. It is shown that the univalence of an integral operator is preserved under certain sufficient conditions.

M.S.C. 2010: 30C45.
Key words: derivative operator; integral operator; univalence; sufficient conditions.

1 Introduction

Let A denote the class of functions f in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, given by the normalized power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U), \tag{1.1}$$

where a_k is a complex number. Let S be the subclass of A consisting of univalent functions. For functions f given by (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in U).$$

Let $(f * g)$ denote the Hadamard product (convolution) of f and g, defined by:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

where $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by:

$$(x)_k = \begin{cases} 1, & \text{for } k = 0,\ x \in \mathbb{C}\backslash\{0\}, \\ x(x+1)(x+2)...(x+k-1), & \text{for } k \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}$$

The authors in [2], have recently introduced a new generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$ as follows:

Applied Sciences, Vol.14, 2012, pp. 15-25.

Definition 1.1. Let $f \in A$, then the generalized derivative operator is given by

$$I^m(\lambda_1, \lambda_2, l, n)f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1)+l)^{m-1}}{(1+l)^{m-1}(1+\lambda_2(k-1))^m} c(n,k)a_k z^k, \tag{1.2}$$

where $n, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, and $\lambda_2 \geq \lambda_1 \geq 0, l \geq 0$, $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

By using the generalized derivative operator $I^m(\lambda_1, \lambda_2, l, n)f(z)$, we introduce the following integral operator:

Definition 1.2. Let $n, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$, and let $\alpha_i \in \mathbb{C}$, let also $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$.

We define the integral operator:

$$D_\beta^m(\lambda_1, \lambda_2, l, n)(f_1, ..., f_s)(z) : A^s \longrightarrow A,$$

(1.3)

$$D_\beta^m(\lambda_1, \lambda_2, l, n)(f_1, ..., f_s)(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{s} \left(\frac{I^m(\lambda_1, \lambda_2, l, n)f_i(t)}{t} \right)^{\alpha_i} dt \right)^{\frac{1}{\beta}},$$

where $I^m(\lambda_1, \lambda_2, l, n)$ is the generalized derivative operator.

Remark 1.3. (i) For $n, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$, and $\alpha_i \in \mathbb{C}$, let also $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, and let $I^1(\lambda_1, 0, l, 0)f_i(z) = f_i(z) \in S, (i \in 1, ..., s)$. We have the integral operator

$$D_\beta(f_1, ..., f_s)(z) = \left(\beta \int_0^z t^{\beta-1} \prod_{i=1}^{s} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt \right)^{\frac{1}{\beta}},$$

which was introduced in [3].

(ii) For $n, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \lambda_2 \geq \lambda_1 \geq 0, l \geq 0, c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$, and $\alpha_i \in \mathbb{C}$, let also $\beta = 1$, and $I^1(\lambda_1, 0, l, 0)f_i(z) = f_i(z) \in S, (i \in 1, ..., s)$. We have the integral operator

$$I(f_1, ..., f_s)(z) = \int_0^z \prod_{i=1}^{s} \left(\frac{f_i(t)}{t} \right)^{\alpha_i} dt,$$

which was introduced in [3].

(iii) For $n, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \lambda_2 \geq \lambda_1 \geq 0, l \geq 0, c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$, and $\alpha_i \in \mathbb{C}$, let also $\beta = 1$, and $I^{m+1}(\lambda_1, 0, 0, 0)f_i(z) = D^m f_i(z), (i \in 1, ..., n)$, where D^m is the generalized Salagean derivative operator. The integral operator is as follows:

$$I^m(f_1, ..., f_s)(z) = \int_0^z \prod_{i=1}^{s} \left(\frac{D^m f_i(t)}{t} \right)^{\alpha_i} dt,$$

which was studied in [4].

(iv) For $\alpha_1 = 1, \alpha_2 = \alpha_3 = ... = \alpha_s = 0$, let also $\beta = 1$, and $I^1(\lambda_1, 0, l, 0)f(z) = f(z) \in S$. We have Alexander integral operator

$$I(f)(z) = \int_0^z \frac{f(t)}{t} dt,$$

which was introduced in [1].
(v) For $\alpha_1 = \alpha \in [0,1], \alpha_2 = \alpha_3 = ... = \alpha_s = 0$ and $\alpha_i \in \mathbb{C}$, let also $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$. and $I^1(\lambda_1, 0, l, 0)f_1(z) = f_1(z) \in S, (i \in 1, ..., s)$. We have the integral operator

$$I(f)(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt,$$

which was introduced in [5].

To discuss our problems, we have to recall here the following results.

General Schwarz Lemma[6]. Let the function f be regular in the disk

$$U_R = \{z \in \mathbb{C} : |z| < R\},$$

with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then

$$|f(z)| \leq \frac{M}{R^m}|z|^m, (z \in U_R).$$

The equality can hold only if,

$$f(z) = e^{i\theta}(\frac{M}{R^m})z^m,$$

where θ is constant.

Lemma 1.1. *[7] Let $f \in A$, and α be a complex number with $\Re(\alpha) > 0$. If f satisfies*

$$\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

then for all $z \in U$, the integral operator

$$F_\alpha(z) = \left\{\alpha \int_0^z t^{\alpha-1} f'(t) dt\right\}^{\frac{1}{\alpha}},$$

is in the class S.

Lemma 1.2. *[8] Let $f \in A$, and α be a complex number with $\Re(\alpha) > 0$. If f satisfies*

$$\frac{1 - |z|^{2\Re(\alpha)}}{\Re(\alpha)} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1,$$

then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \geq \Re(\alpha)$, the integral operator

$$F_\beta(z) = \left\{\beta \int_0^z t^{\beta-1} f'(t) dt\right\}^{\frac{1}{\beta}},$$

is in the class S.

Lemma 1.3. *[9] Let $f \in A$ and $\beta, c \in \mathbb{C}$ where $\Re(\beta) > 0$ and $(|c| \leq 1, \quad c \neq -1)$. If*

$$\left| c|z|^{2\beta} + (1 - |z|^2\beta)\frac{zf''(z)}{\beta f'(z)} \right| \leq 1,$$

for all $z \in U$, then the function

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)dt\right]^{\frac{1}{\beta}},$$

is analytic and univalent in U.

2 Main results

Theorem 2.1. *Let $\alpha_1, ..., \alpha_s, \beta \in \mathbb{C}$ and each of the functions $f_i \in A (i \in 1, ..., s)$. If*

$$\left| \frac{z(I^m(\lambda_1, \lambda_2, l, n)f_i(z))'}{I^m(\lambda_1, \lambda_2, l, n)f_i(z)} - 1 \right| \leq 1, \qquad (z \in U),$$

for some $n, m \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, and $\lambda_2 \geq \lambda_1 \geq 0$, $l \geq 0$, $c(n, k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$, and

$$\Re(\beta) \geq \sum_{i=1}^{s} |\alpha_i| > 0,$$

then the operator $D_\beta^m(\lambda_1, \lambda_2, l, n)(f_1, ..., f_s)(z)$ defined by (1.3) is in the class S.

Proof. Since $f_i \in A \quad (i \in 1, ..., s)$. by (1.2), we get

$$\frac{I^m(\lambda_1, \lambda_2, l, n)f_i(z)}{z} = 1 + \sum_{k=2}^{\infty} \frac{(1 + \lambda_1(k-1) + l)^{m-1}}{(1+l)^{m-1}(1 + \lambda_2(k-1))^m} c(n, k) a_k z^{k-1},$$

and

$$\frac{I^m(\lambda_1, \lambda_2, l, n)f_i(z)}{z} \neq 0,$$

for all $z \in U$. □

Let us define

$$h(z) = \int_0^z \prod_{i=1}^{s} \left(\frac{I^m(\lambda_1, \lambda_2, l, n)f_i(z)}{t}\right)^{\alpha_i} dt,$$

so that, obviously,

$$h'(z) = \left(\frac{I^m(\lambda_1, \lambda_2, l, n)f_i(z)}{z}\right)^{\alpha_1} ... \left(\frac{I^m(\lambda_1, \lambda_2, l, n)f_i(z)}{z}\right)^{\alpha_s},$$

for all $z \in U$. This equality implies that

$$lnh'(z) = \alpha_1 ln\frac{I^m(\lambda_1, \lambda_2, l, n)f_i(z)}{z} + ... + \alpha_s ln\frac{I^m(\lambda_1, \lambda_2, l, n)f_i(z)}{z}.$$

By differentiating the above equality, we get

$$\frac{h''(z)}{h'(z)} = \sum_{i=1}^{s} \alpha_i \left[\frac{(I^m(\lambda_1, \lambda_2, l, n) f_i(z))'}{I^m(\lambda_1, \lambda_2, l, n)} - \frac{1}{z} \right].$$

Hence, we obtain from this equality that

$$\left| \frac{zh''(z)}{h'(z)} \right| \le \sum_{i=1}^{s} |\alpha_i| \left| \frac{z(I^m(\lambda_1, \lambda_2, l, n) f_i(z))'}{I^m(\lambda_1, \lambda_2, l, n) f_i(z)} - 1 \right|.$$

So by the conditions of Theorem 2.1, we find

$$\frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \left| \frac{zh''(z)}{h'(z)} \right| \le \frac{1 - |z|^{2\Re(\beta)}}{\Re(\beta)} \sum_{i=1}^{s} |\alpha_i| \left| \frac{z(I^m(\lambda_1, \lambda_2, l, n) f_i(z))'}{I^m(\lambda_1, \lambda_2, l, n) f_i(z)} - 1 \right|$$

$$\le \frac{1}{\Re(\beta)} \sum_{i=1}^{s} |\alpha_i| \le 1.$$

Finally, applying Lemma 1.1 for the function $h(z)$, we prove that $D_\beta^m(\lambda_1, \lambda_2, l, n)(f_1, ..., f_s)(z) \in S$.

Remark 2.1. If we set $\beta = 1$ and $\lambda_2 = l = n = 0$ in Theorem 2.1, then we have Theorem 2.3 in [4].

Corollary 2.2. *Let $\alpha_i > 0, \beta \in \mathbb{C}$, and each of the functions $f_i \in A(i \in \{1, ..., s\})$. If*

$$\left| \frac{z(I^m(\lambda_1, \lambda_2, l, n) f_i(z))'}{I^m(\lambda_1, \lambda_2, l, n) f_i(z)} - 1 \right| \le 1, \qquad (z \in U),$$

and

$$\Re(\beta) \ge \sum_{i=1}^{s} \alpha_i,$$

then the integral operator $D_\beta^m(\lambda_1, \lambda_2, l, n)(f_1, ..., f_s)(z)$ defined by (1.3) is in the univalent function class S.

Remark 2.2. If we set $\beta = 1$ and $\lambda_2 = l = n = 0$ in Theorem 2.2, then we have Corollary 2.5 in [4].

Theorem 2.3. *Let $M_i \ge 1$, and suppose that each of the functions $f_i \in A(i \in \{1, ..., s\})$, satisfies the inequality*

$$\left| \frac{z^2(I^m(\lambda_1, \lambda_2, l, n) f_i(z))'}{(I^m(\lambda_1, \lambda_2, l, n) f_i(z))^2} - 1 \right| \le 1, \qquad (z \in U),$$

also let $\alpha_i, \alpha \in \mathbb{C}$ with

$$\Re(\alpha) \ge \sum_{i=1}^{s} |\alpha_i| (2M_i + 1) > 0.$$

If

$$|I^m(\lambda_1, \lambda_2, l, n) f_i(z)| \le M_i \quad (z \in U),$$

then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \ge \Re(\alpha)$, the integral operator $D_\beta^m(\lambda_1, \lambda_2, l, n)(f_1, ..., f_s)(z)$ defined by (1.3) is in the univalent function class S.

Proof. We know from the proof of Theorem 2.1 that

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{s} |\alpha_i| \left|\frac{z(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{I^m(\lambda_1,\lambda_2,l,n)f_i(z)} - 1\right|.$$

So, by the imposing the conditions, we find

$$\frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\sum_{i=1}^{s}|\alpha_i|\left(\left|\frac{z(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{I^m(\lambda_1,\lambda_2,l,n)f_i(z)}\right|+1\right)$$

$$\leq \frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\sum_{i=1}^{s}|\alpha_i|\left(\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2}\right|\left|\frac{I^m(\lambda_1,\lambda_2,l,n)f_i(z)}{z}\right|+1\right)$$

$$\leq \frac{1-|z|^{2\Re(\alpha)}}{\Re(\alpha)}\sum_{i=1}^{s}|\alpha_i|\left(\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2-1}\right|M_i+M_i+1\right)$$

$$\leq \frac{1}{\Re(\alpha)}\sum_{i=1}^{s}|\alpha_i|(2M_i+1) \leq 1.$$

Then, by applying Lemma 1.2 for the function $h(z)$, we finally conclude that $D^m_\beta(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z) \in S$. □

Corollary 2.4. *Let $M_i \geq 1, \alpha_i > 0$ and suppose that each of the functions $f_i \in A(i \in \{1,...,s\})$, satisfies the inequality*

$$\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha \in \mathbb{C}$ with

$$\Re(\alpha) \geq \sum_{i=1}^{s}\alpha_i(2M_i+1) > 0.$$

If

$$|I^m(\lambda_1,\lambda_2,l,n)f_i(z)| \leq M_i \quad (z \in U),$$

then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \geq \Re(\alpha)$, the integral operator $D^m_\beta(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z)$ defined by (1.3) is in the univalent function class S.

Corollary 2.5. *Let $M \geq 1$ and suppose that each of the functions $f_i \in A(i \in \{1,...,s\})$, satisfies the inequality*

$$\left|\frac{z^2(I^{m+1}(\lambda_1,0,0,0)f_i(z))'}{(I^{m+1}(\lambda_1,0,0,0)f_i(z))^2} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha_i, \alpha \in \mathbb{C}$ with

$$\Re(\alpha) \geq (2M+1)\sum_{i=1}^{s}|\alpha_i| > 0.$$

If

$$|I^{m+1}(\lambda_1,0,0,0)f_i(z)| \leq M \quad (z \in U),$$

then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \geq \Re(\alpha)$, the integral operator $D^m_\beta(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z)$ given by (1.3) is in the univalent function class S.

Proof. In Theorem 2.3, we consider $M_1 = ... = M_s = M$. $\square$

Corollary 2.6. *Suppose that each of the functions $f_i \in A(i \in \{1, ..., s\})$. satisfies the inequality*

$$\left|\frac{z^2(I^m(\lambda_1, \lambda_2, l, n)f_i(z))'}{(I^m(\lambda_1, \lambda_2, l, n)f_i(z))^2} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha_i, \alpha \in \mathbb{C}$ with

$$\Re(\alpha) \geq 3\sum_{i=1}^{s} |\alpha_i| > 0.$$

If

$$|I^m(\lambda_1, \lambda_2, l, n)f_i(z)| \leq 1 \quad (z \in U),$$

then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \geq \Re(\alpha)$, the integral operator $D^m_\beta(\lambda_1, \lambda_2, l, n)(f_1, ..., f_s)(z)$ defined by (1.3) is in the univalent function class S.

Proof. In Corollary2.5, we consider $M = 1$. $\square$

Remark 2.3. In Corollary2.6, if we set

$\beta = 1$ and $\lambda_2 = l = n = 0$, then we have Theorem 2.6 in [4],

$\beta = 1$ and $\lambda_2 = l = n = 0$, and $\alpha_i > 0$, then we have Corollary 2.8 in [4].

Theorem 2.7. *Suppose that each of the functions $f_i \in A(i \in \{1, ..., s\})$, satisfies the inequality*

$$\left|\frac{z(I^m(\lambda_1, \lambda_2, l, n)f_i(z))'}{I^m(\lambda_1, \lambda_2, l, n)f_i(z)} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha_i, \alpha \in \mathbb{C}$ with

$$\Re(\alpha) \geq \sum_{i=1}^{s} |\alpha_i| > 0.$$

And let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\Re(\beta)}\sum_{i=1}^{s} |\alpha_i|.$$

Then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \geq \Re(\alpha)$, the integral operator $D^m_\beta(\lambda_1, \lambda_2, l, n)(f_1, ..., f_s)(z)$ defined by (1.3) is in the univalent function class S.

Proof. We know from the proof of Theorem 2.1 that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{s} \alpha_i \left[\frac{z(I^m(\lambda_1, \lambda_2, l, n)f_i(z))'}{I^m(\lambda_1, \lambda_2, l, n)f_i(z)} - 1\right].$$

So we get

$$\left|c|z|^{2\beta} + (1 - |z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right|$$

$$= \left|c|z|^{2\beta} + (1 - |z|^{2\beta})\frac{1}{\beta}\sum_{i=1}^{s} \alpha_i \left[\frac{z(I^m(\lambda_1, \lambda_2, l, n)f_i(z))'}{I^m(\lambda_1, \lambda_2, l, n)f_i(z)} - 1\right]\right|$$

$$\leq |c| + \left|\frac{1-|z|^{2\beta}}{\beta}\right| \sum_{i=1}^{s} |\alpha_i| \left|\frac{z(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{I^m(\lambda_1,\lambda_2,l,n)f_i(z)} - 1\right|$$

$$\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{s}|\alpha_i| \leq |c| + \frac{1}{\Re(\beta)}\sum_{i=1}^{s}|\alpha_i| \leq 1.$$

By applying Lemma 1.3 for the function $h(z)$, we infer $D^m_\beta(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z) \in S$. □

Corollary 2.8. *Suppose that each of the functions $f_i \in A(i \in \{1,...,s\})$, satisfies the inequality*

$$\left|\frac{z(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{I^m(\lambda_1,\lambda_2,l,n)f_i(z)} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha_i, \alpha \in \mathbb{C}$ with

$$\Re(\alpha) \geq \sum_{i=1}^{s} \alpha_i.$$

And let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\Re(\beta)}\sum_{i=1}^{s}\alpha_i.$$

Then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \geq \Re(\alpha)$, the integral operator $D^m_\beta(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z)$ defined by (1.3) is in the univalent function class S.

Theorem 2.9. *Let $M_i \geq 1$, and suppose that each of the functions $f_i \in A(i \in \{1,...,s\})$, satisfies the inequality*

$$\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha_i, \beta \in \mathbb{C}$ with

$$\Re(\beta) \geq \sum_{i=1}^{s}|\alpha_i|(2M_i+1) > 0.$$

Let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\Re(\beta)}\sum_{i=1}^{s}|\alpha_i|,$$

and

$$|I^m(\lambda_1,\lambda_2,l,n)f_i(z)| \leq M_i \quad (z \in U),$$

then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \geq \Re(\alpha)$, the integral operator $D^m_\beta(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z)$ defined by (1.3) is in the univalent function class S.

Proof. We know from the proof of Theorem 2.1 that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{s}\alpha_i\left[\frac{z(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{I^m(\lambda_1,\lambda_2,l,n)f_i(z)} - 1\right].$$

So we get

$$\left|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right|$$

$$= \left|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{1}{\beta}\sum_{i=1}^{s}\alpha_i\left[\frac{z(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{I^m(\lambda_1,\lambda_2,l,n)f_i(z)} - 1\right]\right|$$

$$\leq |c| + \left|\frac{1-|z|^{2\beta}}{\beta}\right|\sum_{i=1}^{s}|\alpha_i|\left|\frac{z(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{I^m(\lambda_1,\lambda_2,l,n)f_i(z)} + 1\right|$$

$$\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{s}|\alpha_i|\left(\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2}\right|\left|\frac{I^m(\lambda_1,\lambda_2,l,n)f_i(z)}{z}\right| + 1\right)$$

$$\leq |c| + \frac{1}{\Re(\beta)}\sum_{i=1}^{s}|\alpha_i|\left(\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2 - 1}\right|M_i + M_i + 1\right)$$

$$\leq |c| + \frac{1}{\Re(\beta)}\sum_{i=1}^{s}|\alpha_i|(2M_i+1) \leq 1.$$

By applying Lemma 1.3 for the function $h(z)$, we prove that
$D^m_\beta(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z) \in S$. □

Corollary 2.10. *Let $M_i \geq 1, \alpha_i > 0$ and suppose that each of the functions $f_i \in A(i \in \{1,...,s\})$, satisfies the inequality*

$$\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha \in \mathbb{C}$ with

$$\Re(\alpha) \geq \sum_{i=1}^{s}\alpha_i(2M_i+1) > 0.$$

If Let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{\Re(\beta)}\sum_{i=1}^{s}\alpha_i(2M_i+1),$$

and

$$|I^m(\lambda_1,\lambda_2,l,n)f_i(z)| \leq M_i \quad (z \in U),$$

then for any complex number $\beta \in \mathbb{C}$ where $\Re(\beta) \geq \Re(\alpha)$, the integral operator $D^m_\beta(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z)$ defined by (1.3) is in the univalent function class S.

Corollary 2.11. *Let $M \geq 1$ and suppose that each of the functions $f_i \in A(i \in \{1,...,s\})$. satisfies the inequality*

$$\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha_i, \alpha \in \mathbb{C}$ *with*

$$\Re(\alpha) \geq (2M+1)\sum_{i=1}^{s}|\alpha_i| > 0.$$

If Let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{2M+1}{\Re(\beta)}\sum_{i=1}^{s}|\alpha_i|,$$

and

$$|I^m(\lambda_1,\lambda_2,l,n)f_i(z)| \leq M \quad (z \in U),$$

then for any complex number $\beta \in \mathbb{C}$ *where* $\Re(\beta) \geq \Re(\alpha)$, *the integral operator* $D_\beta^m(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z)$ *defined by (1.3) is in the univalent function class S.*

Proof. In Theorem 2.9, we consider $M_1 = ... = M_s = M$. □

Corollary 2.12. *Suppose that each of the functions* $f_i \in A(i \in \{1,...,s\})$, *satisfies the inequality*

$$\left|\frac{z^2(I^m(\lambda_1,\lambda_2,l,n)f_i(z))'}{(I^m(\lambda_1,\lambda_2,l,n)f_i(z))^2} - 1\right| \leq 1, \qquad (z \in U),$$

also let $\alpha_i, \alpha \in \mathbb{C}$ *with*

$$\Re(\alpha) \geq 3\sum_{i=1}^{s}|\alpha_i| > 0.$$

If Let $c \in \mathbb{C}$ *be such that*

$$|c| \leq 1 - \frac{3}{\Re(\beta)}\sum_{i=1}^{s}|\alpha_i|,$$

and

$$|I^m(\lambda_1,\lambda_2,l,n)f_i(z)| \leq 1 \quad (z \in U),$$

then for any complex number $\beta \in \mathbb{C}$ *where* $\Re(\beta) \geq \Re(\alpha)$, *the integral operator* $D_\beta^m(\lambda_1,\lambda_2,l,n)(f_1,...,f_s)(z)$ *defined by (1.3) is in the univalent function class S.*

Proof. In Corollary2.11, we consider $M = 1$. □

Acknowledgements. The work presented here was partially supported by UKM-ST-06-FRGS0244-2010.

References

[1] I. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. 17 (1915), 12-22.

[2] A.A. Amer and M. Darus, *On some properties for new generalized derivative operator*, Jordan Journal of Mathematics and Statistics (JJMS) 4, 2 (2011), 91-101.

[3] D. Breaz and N. Breaz, *Two integral operators*, Studia Univ. Babes-Bolyai Math. 47, 3 (2002), 13-19.

[4] S. Bulut, *Sufficient conditions for univalence of an integral operator defined by Al-Oboudi differential operator*, J. Inequal. Appl. 2008, Art. ID 957-042, 1-5.

[5] S. S. Miller, P. T. Mocanu and M. O. Reade, *Starlike integral operators*, Pacific J. Math. 79, 1 (1978), 157-168.

[6] Z. Nehari, *Conformal Mappings*, McGraw-Hill Book Comp., New York, 1952; Dover. Publ. Inc., 1975.

[7] N. Pascu, *On a univalence criterion*, II, Itinerant Seminar on Functional Equations, Approximation and Convexity (Cluj-Napoca, 1985), Preprint, vol. 85, Universitatea Babes-Bolyai, Cluj-Napoca, 1985, 153-154.

[8] N. Pascu, *An improvement of Beckers univalence criterion*, Proceedings of the Commemorative Session Simion Stoilow, Brasov, 1987, 43-48.

[9] V. Pescar, *A new generalization of Ahlforss and Beckers criterion of univalence*, Bull. Malaysian Math. Soc. 2, 19 (1996), 53-54.

Authors' address:

Aisha Ahmed Amer , *Maslina Darus
School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600 Selangor D. Ehsan, Malaysia.
E-mail: maslina@ukm.my *(corresponding author)*