On n-ary operations and their applications

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Abstract. The authors extend previous results regarding the properties of certain n-ary operations provided by special structures (groupoid, semigroup) and investigate properties, as semi-ciclicity, ciclicity, semi-commutativity and commutativity. A special attention is paid to the relation to the associated Post group structure, which is shown to shed useful information of the primary n-ary operation.

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1 Introduction

The present work is devoted to n-ary operations on Cartesian powers, extending several prior developments on the subject ([17],[18],[11],[12],[14],[13]). In [17], for given arbitrary integers $\ell \geqslant 2$, $k \geqslant 2$, $m \geqslant 1$ and $n \geqslant 3$, there were defined and studied the ℓ-ary, respectively the n-ary operations $[\ \ ]_{l,k}$ and $[\ \ ]_{n,m,m(n-1)}$. Further, the work [18] contains examples and details on the properties of these operations, which are essentially relevant for the use of multidimensional spaces in geometry and physics. In [11] and [12], for the same ℓ and k as above, and for an arbitrary permutation $\sigma \in S_k$ we defined another ℓ-ary operation $[\ \ ]_{l,\sigma,k}$, which in the case $\sigma = (1\ 2\ \ldots\ k)$ coincides with $[\ \ ]_{l,k}$, namely $[\ \ ]_{l,k} = [\ \ ]_{l,(1\ 2\ \ldots\ k),k}$.

In this study, which naturally continues these works, we define and study another n-ary operation $[\ \ ]_{l,\sigma,m,mk}$, which includes as particular cases the previously studied operations:

$$[\ \ ]_{l,k} = [\ \ ]_{l,(1\ 2\ \ldots\ k),1,k}, \quad (\ \ )_{l,\sigma,k} = [\ \ ]_{l,\sigma,1,k},$$
$$[\ \ ]_{n,m,m(n-1)} = [\ \ ]_{n,(1\ 2\ \ldots\ n-1),m,m(n-1)}.$$

Most of the notions which are used hereafter, were introduced and described in [18], and numerous particular cases were addressed in [10],[22],[23],[15],[16],[14].

2 The ℓ-ary operation $[\ \ ]_{l,k}$

Consider a groupoid A, and the integers $k \geqslant 2$, $l \geqslant 2$. We first define on A^k the binary operation

$$\mathbf{x} \circ \mathbf{y} = (x_1, x_2, \ldots, x_k) \circ (y_1, y_2, \ldots, y_k) = (x_1y_2, x_2y_3, \ldots, x_{k-1}y_k, x_ky_1),$$

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and further, the l-ary operation

$$[\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_l]_{l,k} = \mathbf{x}_1 \circ (\mathbf{x}_2 \circ (\ldots (\mathbf{x}_{l-2} \circ (\mathbf{x}_{l-1} \circ \mathbf{x}_l))\ldots)). \tag{2.1}$$

It is obvious that the operation $[\ \ ]_{2,k}$ coincides with "$\circ$".

Example 2.1. The defined on $\mathbb{R}^2$ operations $[\ \ ]_{3,2}$ (ternary) and $[\ \ ]_{4,2}$ (quaternary), have the form[1]:

$$\begin{cases} [(x_1,x_2)(y_1,y_2)(z_1,z_2)]_{3,2} = (x_1y_2z_1, x_2y_1z_2); \\ [(x_1,x_2)(y_1,y_2)(z_1,z_2)(u_1,u_2)]_{4,2} = (x_1y_2z_1u_2, x_2y_1z_2u_1). \end{cases}$$

The ternary operation $[\ \ ]_{3,3}$, defined on $\mathbb{R}^3$, where $\mathbb{R}$ is groupoid with the usual operation of multiplication, has the form

$$[(x_1,x_2,x_3)(y_1,y_2,y_3)(z_1,z_2,z_3)]_{3,3} = (x_1y_2z_3, x_2y_3z_1, x_3y_1z_2).$$

It can be proved straightforward that the ternary operation $[\ \ ]_{3,2}$ is associative, while the 4-ary operation $[\ \ ]_{4,2}$ and the ternary one $[\ \ ]_{3,3}$ are non-associative. We shall further use the following

Lemma 2.2. [17]. *Let A be a semigroup, and for $n \geqslant 3$, let $[\ \ ]_{n,n-1}$ be an n-ary operation defined on A^{n-1} by (2.1) for $k = n-1$, $l = n$. If $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{i(n-1)}) \in A^{n-1}$, $i = 1, \ldots, n$, then*

$$[\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n]_{n,n-1} = (y_1, y_2, \ldots, y_{n-1}),$$

where, for $j = 1, \ldots, n-1$, we have

$$y_j = x_{1j}x_{2(j+1)}\ldots x_{(n-j)(n-1)}x_{(n-j+1)1}\ldots x_{(n-1)(j-1)}x_{nj}.$$

As direct consequence, we infer:

$$\begin{aligned} [\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n]_{n,n-1} = (&x_{11}x_{22}\ldots x_{(n-1)(n-1)}x_{n1}, \\ &x_{12}x_{23}\ldots x_{(n-2)(n-1)}x_{(n-1)1}x_{n2}, \ldots \\ &x_{1(n-2)}x_{2(n-1)}x_{31}\ldots x_{n(n-2)}, x_{1(n-1)}x_{21}\ldots x_{n(n-1)}). \end{aligned} \tag{2.2}$$

This Lemma shows that if A is a semigroup, then the n-ary operation $[\ \ ]_{n,n-1}$ defined on A^{n-1} in a similar way to the n-ary operation introduced by Post over the set of all n-permutations ([22], [23], [16]): he showed that, relative to this operation, the set of all $n-$permutations becomes an n-ary group; in particular this operation is associative. Then it is likely that (2.2) is associative as well. Indeed, we have the following result:

Theorem 2.3. [18, 11]. *The operation $[\ \ ]_{s(n-1)+1,n-1}$, $(s \geqslant 1)$, defined on the Cartesian power A^{n-1} of the semigroup A is associative. In particular, the n-ary operation $[\ \ ]_{n,n-1}$ is associative as well.*

We note that the components from the right side of (2.2) can be written, as shows the following

[1] Here, $\mathbb{R}$ is regarded as groupoid with the usual operation of multiplication of reals.

Proposition 2.4. [11]. *Let A be a semigroup, and let $[\ \ ]_{n,n-1}$ be the n-ary operation ($n \geqslant 3$) which is defined on A^{n-1} by (2.1), considered for $k = n-1$, $l = n$. Let $\alpha = (1\,2\ldots n-1)$ be a cyclic permutation from S_{n-1}, and let $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{i(n-1)}) \in A^{n-1}$, $i = 1, \ldots, n$. Then*

$$
\begin{aligned}
[\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n]_{n,n-1} = \ & (x_{11}x_{2\alpha(1)}\ldots x_{n\alpha^{n-1}(1)}, \ldots \\
& \ldots, x_{1(n-1)}x_{2\alpha(n-1)}\ldots x_{n\alpha^{n-1}(n-1)}).
\end{aligned} \tag{2.3}
$$

Remark. Since α^{n-1} is the identity permutation, then $\alpha^{n-1}(j) = j$ for all $j \in \{1, 2, \ldots, n-1\}$. Hence the relation (2.3) can be written as:

$$
\begin{aligned}
[\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n]_{n,n-1} = (x_{11}x_{2\alpha(1)}\ldots x_{(n-1)\alpha^{n-2}(1)}x_{n1}, \ldots, \\
\ldots x_{1(n-1)}x_{2\alpha(n-1)}\ldots x_{(n-1)\alpha^{n-2}(n-1)}x_{n(n-1)}).
\end{aligned} \tag{2.4}
$$

Let $\langle A, *\rangle$ be a groupoid, and let $k \geqslant 2$, $i \in \{1, \ldots, k-1\}$. We define a transformation f_i of the set A^k as

$$f_i : (a_1, a_2, \ldots, a_{k-1}, a_k) \to (a_{i+1}, \ldots, a_k, a_1, \ldots, a_i).$$

In particular, we have $f_1 : (a_1, a_2, \ldots, a_{k-1}, a_k) \to (a_2, \ldots, a_{k-1}, a_k, a_1)$.

The properties of such mappings are described by the following

Lemma 2.5. [11]. *The following assertions hold true:*

1) $f_i = f_1^i$ for all $i \in \{2, \ldots, k-1\}$;

2) f_1^k is the identity transformation;

3) for all $i \in \{1, \ldots, k-1\}$, the transformation f_i is an automorphism of the groupoid $\langle A^k, \circ\rangle$ and of the groupoid $\langle A^k, *\rangle$, considered with the operation "$*$" component-wise defined on A^k.

Consequently one can prove:

Theorem 2.6. *a)* **([11])** *Let A be a semigroup, $l \geqslant 2$, $k \geqslant 2$. Then, within the semigroup A^k, the following identity holds*[2]

$$[\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_{l-1}\mathbf{x}_l]_{l,k} = \mathbf{x}_1\mathbf{x}_2^{f_1}\ldots\mathbf{x}_{l-1}^{f_1^{l-2}}\mathbf{x}_l^{f_1^{l-1}}.$$

b) **([12])** *Let A be a groupoid with unity, which contains an element distinct from the unity; let $k \geq l \geq 2$ and $s \geqslant 0$. Then the $(sk+l)$-ary operation $[\ \ ]_{sk+l,k}$, defined on A^k is not semi-associative, and hence, is not associative.*

3 The ℓ-ary operation $[\ \ ]_{l,\sigma,k}$

Formulas (2.3) and (2.4) lead to considering new multiple operations which involve the composition $[\ \ ]_{n,n-1}$.

Let A be a semigroup, $k \geqslant 2$, $l \geqslant 2$; let σ be a permutation of S_k. We define on A^k the l-ary operation

$$
\begin{aligned}
[\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_l]_{l,\sigma,k} = \ & (x_{11}x_{2\sigma(1)}\ldots x_{(l-1)\sigma^{l-2}(1)}x_{l\sigma^{l-1}(1)}, \ldots \\
& \ldots, x_{1k}x_{2\sigma(k)}\ldots x_{(l-1)\sigma^{l-2}(k)}x_{l\sigma^{l-1}(k)}).
\end{aligned} \tag{3.1}
$$

[2]The article [18] contains detailed information on the l-ary groupoid $\langle A^k, [\ \ ]_{l,k}\rangle$.

Example 3.1. The operations $[\ \ ]_{3,\sigma_1,3}$, $[\ \ ]_{3,\sigma_2,3}$ and $[\ \ ]_{4,\sigma,3}$, where $\sigma_1 = (12) \in S_3$, $\sigma_2 = (13) \in S_3$, $\sigma = (132) \in S_3$, have the form

$$\begin{aligned} [\mathbf{xyz}]_{3,\sigma_1,3} &= (x_1y_2z_1, x_2y_1z_2, x_3y_3z_3), \\ [\mathbf{xyz}]_{3,\sigma_2,3} &= (x_1y_3z_1, x_2y_2z_2, x_3y_1z_3), \\ [\mathbf{xyzu}]_{4,\sigma,3} &= (x_1y_3z_2u_1, x_2y_1z_3u_2, x_3y_2z_1u_3). \end{aligned}$$

Regarding the associativity of the operation $[\ \ ]_{l,\sigma,k}$, we have

Theorem 3.2. [11, 14, 13]. *Let A be a semigroup, $k \geqslant 2$, $l \geqslant 2$, σ a permutation from S_k, which satisfies the condition $\sigma^l = \sigma$. Then the l-ary operation $[\ \ ]_{l,\sigma,k}$ is associative.*

We show that if the permutation σ from the definition of the operation $[\ \ ]_{l,\sigma,k}$ does not satisfy the condition $\sigma^l = \sigma$, then the l-ary operation $[\ \ ]_{l,\sigma,k}$ might be non-associative.

Example 3.3. Replacing $l = k = 3$ and $\sigma = (132)$ in (3.1), we define on $\mathbb{R}^3$ the ternary operation $[\ \ ] = [\ \ ]_{3,(132),3}$, as follows

$$[\mathbf{xyz}] = (x_1y_{\sigma(1)}z_{\sigma^2(1)}, x_2y_{\sigma(2)}z_{\sigma^2(2)}, x_3y_{\sigma(3)}z_{\sigma^2(3)}) = (x_1y_3z_2, x_2y_1z_3, x_3y_2z_1).$$

Since σ^3 is the identity permutation, then the condition $\sigma^3 = \sigma$ does not hold true; as well, since

$$\begin{aligned} [[\mathbf{xyz}]\mathbf{uv}] &= [(x_1y_3z_2, x_2y_1z_3, x_3y_2z_1)\ \mathbf{uv}] = (x_1y_3z_2u_3v_2, x_2y_1z_3u_1v_3, x_3y_2z_1u_2v_1), \\ [\mathbf{x}[\mathbf{yzu}]\mathbf{v}] &= [\mathbf{x}(y_1z_3u_2, y_2z_1u_3, y_3z_2u_1)\mathbf{v}] = (x_1y_3z_2u_1v_2, x_2y_1z_3u_2v_3, x_3y_2z_1u_3v_1), \\ [\mathbf{xy}[\mathbf{zuv}]] &= [\mathbf{xy}(z_1u_3v_2, z_2u_1v_3, z_3u_2v_1)] = (x_1y_3z_2u_1v_3, x_2y_1z_3u_2v_1, x_3y_2z_1u_3v_2), \end{aligned}$$

the elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ can be chosen in such way, that the following relations hold true:

$$\begin{aligned} &[[\mathbf{xyz}]\mathbf{uv}] \neq [\mathbf{x}[\mathbf{yzu}]\mathbf{v}], && x_1 = y_3 = z_2 = v_2 = 1, && u_3 \neq u_1; \\ &[[\mathbf{xyz}]\mathbf{uv}] \neq [\mathbf{xy}[\mathbf{zuv}]], && x_1 = y_3 = z_2 = u_3 = u_1 = 1, && v_2 \neq v_3; \\ &[\mathbf{x}[\mathbf{yzu}]\mathbf{v}] \neq [\mathbf{xy}[\mathbf{zuv}]], && x_1 = y_3 = z_2 = u_1 = 1, && v_2 \neq v_3. \end{aligned}$$

From any of these relations follows the non-associativity of the operation $[\ \ ]$. From the second relation, it results that the operation $[\ \ ]$ is not semi-associative, i.e., the following equality does not hold in $\mathbb{R}^3$:

$$[[\mathbf{xyz}]\mathbf{uv}] = [\mathbf{xy}[\mathbf{zuv}]].$$

We further show that a ternary operation defined on $\mathbb{R}^3$ which is not associative, can still be semi-associative.

Example 3.4. We define on $\mathbb{R}^3$ the ternary operation

$$[\mathbf{xyz}] = (x_1y_3z_1, x_2y_1z_2, x_3y_2z_3).$$

Due to the relations

$$[[\mathbf{xyz}]\mathbf{uv}] = [(x_1y_3z_1, x_2y_1z_2, x_3y_2z_3)\mathbf{uv}] = (x_1y_3z_1u_3v_1, x_2y_1z_2u_1v_2, x_3y_2z_3u_2v_3),$$

$$[\mathbf{xy}[\mathbf{zuv}]] = [\mathbf{xy}(z_1u_3v_1, z_2u_1v_2, z_3u_2v_3)] = (x_1y_3z_1u_3v_1, x_2y_1z_2u_1v_2, x_3y_2z_3u_2v_3),$$

$$[\mathbf{x}[\mathbf{yzu}]\mathbf{v}] = [\mathbf{x}(y_1z_3u_1, y_2z_1u_2, y_3z_2u_3)\mathbf{v}] = (x_1y_3z_2u_3v_1, x_2y_1z_3u_1v_2, x_3y_2z_1u_2v_3),$$

we deduce that $[[\mathbf{xyz}]\mathbf{uv}] = [\mathbf{xy}[\mathbf{zuv}]], \forall \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, but there exist $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$ such that $[\mathbf{xy}[\mathbf{zuv}]] \neq [\mathbf{x}[\mathbf{yzu}]\mathbf{v}]$. Consequently, the operation $[\ \ ]$ is semi-associative, but not associative. We remark that the operation $[\ ]$ from this example is not of the form $(\ )_{l,\sigma,k}$.

Example 3.5. In Theorem 3.2 we replace: $A = \mathbb{R}$, i.e., the semigroup with the usual operation of multiplication of reals and $k = 4$. We shall describe all the associative operations on $\mathbb{R}^4$ which are of the form $[\ \ ]_{l,\sigma,4}$, where $\sigma \in S_4$, and $l-1$ is the order of the permutation σ. Each of the six transpositions

$$(12), (13), (14), (23), (24), (34) \in S_4,$$

considered as elements of order 2, define on $\mathbb{R}^4$ a ternary associative operation. We describe, as an example, the explicit form of the operation $[\ \ ]_{3,(24),4}$:

$$[x_1x_2x_3]_{3,(24),4} = (x_{11}x_{21}x_{31}, x_{12}x_{24}x_{32}, x_{13}x_{23}x_{33}, x_{14}x_{22}x_{34}).$$

These associative ternary operations define as well three elements of order two:

$$(12)(34), (13)(24), (14)(23) \in S_4.$$

As an example, the element of order two $[\ \ ]_{3,(14)(23),4}$ has the explicit form:

$$[x_1x_2x_3]_{3,(14)(23),4} = (x_{11}x_{24}x_{31}, x_{12}x_{23}x_{32}, x_{13}x_{22}x_{33}, x_{14}x_{21}x_{34}).$$

Each of the eight cycles $(123), (124), (132), (134), (142), (143), (234), (243) \in S_4$, as an element of order three, defines on $\mathbb{R}^4$ a 4-ary associative operation. E.g., the explicit form of the operation $[\ \ ]_{4,(143),4}$ is:

$$[x_1x_2x_3x_4]_{4,(143),4} = (x_{11}x_{24}x_{33}x_{41}, x_{12}x_{22}x_{32}x_{42}, x_{13}x_{21}x_{34}x_{43}, x_{14}x_{23}x_{31}x_{44}).$$

Each of the six cycles $(1234), (1243), (1324), (1342), (1423), (1432) \in S_4$, as element of order 4, defines on $\mathbb{R}^4$ a 5-ary associative operation. The explicit form of all these operations follows:

$$\begin{aligned}
[x_1x_2x_3x_4x_5]_{5,(1234),4} = \ & (x_{11}x_{22}x_{33}x_{44}x_{51}, x_{12}x_{23}x_{34}x_{41}x_{52}, \\
& \ x_{13}x_{24}x_{31}x_{42}x_{53}, x_{14}x_{21}x_{32}x_{43}x_{54}); \\
[x_1x_2x_3x_4x_5]_{5,(1243),4} = \ & (x_{11}x_{22}x_{34}x_{43}x_{51}, x_{12}x_{24}x_{33}x_{41}x_{52}, \\
& \ x_{13}x_{21}x_{32}x_{44}x_{53}, x_{14}x_{23}x_{31}x_{42}x_{54}); \\
[x_1x_2x_3x_4x_5]_{5,(1324),4} = \ & (x_{11}x_{23}x_{32}x_{44}x_{51}, x_{12}x_{24}x_{31}x_{43}x_{52}, \\
& \ x_{13}x_{22}x_{34}x_{41}x_{53}, x_{15}x_{21}x_{33}x_{42}x_{54}); \\
[x_1x_2x_3x_4x_5]_{5,(1342),4} = \ & (x_{11}x_{23}x_{34}x_{42}x_{51}, x_{12}x_{21}x_{33}x_{44}x_{52}, \\
& \ x_{13}x_{24}x_{32}x_{41}x_{53}, x_{15}x_{22}x_{31}x_{43}x_{54});
\end{aligned}$$

$$[x_1x_2x_3x_4x_5]_{5,(1423),4} = (x_{11}x_{24}x_{32}x_{43}x_{51}, x_{12}x_{23}x_{31}x_{44}x_{52}, x_{13}x_{21}x_{34}x_{42}x_{53}, x_{14}x_{22}x_{33}x_{41}x_{54});$$

$$[x_1x_2x_3x_4x_5]_{5,(1432),4} = (x_{31}x_{24}x_{33}x_{42}x_{51}, x_{12}x_{21}x_{34}x_{43}x_{52}, x_{13}x_{22}x_{31}x_{44}x_{53}, x_{15}x_{23}x_{32}x_{41}x_{54}).$$

We note that the constructed on $\mathbb{R}^4$ binary operation

$$x_1x_2 = (x_{11}x_{21}, x_{12}x_{22}, x_{13}x_{23}, x_{14}x_{24})$$

is defined by the identity permutation $\varepsilon \in S_4$, i.e., it coincides with $[\ \ ]_{2,\varepsilon,4}$.

Besides the before mentioned 24 associative operations there also exist on $\mathbb{R}^4$ other associative polyadic operations. Since for any permutation $\sigma \in S_4$ of order r and for any integer $t \geqslant 1$ the permutation σ^{rt} is the identity permutation, then by Theorem 3.2, $[\ \ ]_{rt+1,\sigma,4}$ is an associative $(rt+1)$-ary operation, defined on $\mathbb{R}^4$. As an example, $[\ \ ]_{7,(13)(24),4}$ and $[\ \ ]_{7,(134),4}$ are 7-ary associative operations. For the first mentioned operation, we have $r = 2, t = 3$, and for the second, $r = 3, t = 2$.

Moreover, the permutation σ satisfies the condition $\sigma^l = \sigma$, where $l \geqslant 2$, then for the inverse permutation σ^{-1}, it holds true the equality $(\sigma^{-1})^l = \sigma^{-1}$. Hence, from Theorem 3.2, it follows

Corollary 3.6. [11, 13]. *Let A be a semigroup, and $k \geqslant 2$, $l \geqslant 2$; let γ be a permutation from S_k, which satisfies the condition $\gamma^l = \gamma$, $\sigma = \gamma^{-1}$. Then the l-ary operation $[\ \ ]_{l,\sigma,k}$ is associative.*

E.g., in Example 3.5, the associativity of the operation $[\ \ ]_{5,(1324),4}$ can be regarded as consequence of the associativity of the operation $[\ \ ]_{5,(1423),4}$, since the permutations (1324) and (1423) are inverse to each other.

Theorem 3.7. [12, 14]. *Let A be a semigroup with unity, $k \geqslant 2$, $l \geqslant 2$, and let σ be a permutation from S_k, which satisfies the condition $\sigma^l \neq \sigma$. Then the l-ary operation $[\ \ ]_{l,\sigma,k}$ is not semi-associative, and in particular, are not associative.*

Proposition 3.8. *If $\langle A, +, \times\rangle$ is an associative algebra over the field P, then $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ is a $(2,l)$-algebra over P. Moreover, if $\sigma^l = \sigma$, then $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ is an associative $(2,l)$-algebra over P.*

Proposition 3.9. *Let the semigroup A contain the unity and an element distinct from unity. If the permutation $\sigma \in S_k$ is not the identical permutation, then the l-ary groupoid $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$ is not abelian.*

Proposition 3.10. *If the permutation $\sigma \in S_k$ satisfies the condition $\sigma^l = \sigma$, then from the commutativity of the semigroup A follows the semi-commutativity of the l-ary semigroup $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$. If the semigroup A contains the unity, then the converse is true, i.e., from the semi-commutativity of the l-ary semigroup $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$, it follows the commutativity of the semigroup A.*

Proposition 3.11. *If A is a group, then $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$ is an l-ary quasigroup. If, moreover, the condition $\sigma^l = \sigma$ holds true, then $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$ is an l-ary group.*

Proposition 3.12. *If $A = \{0\}$ is the null semigroup, then $\underbrace{(0, \ldots, 0)}_{k}$ is the zero element of the l-ary groupoid $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$. If, moreover, $l \geqslant 3$, then in the l-ary groupoid $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$ all the elements are zero-divisors.*

Proposition 3.13. *If the semigroup A contains more than one element, and σ is not the identity permutation from S_k, then $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$ contains no unity.*

If $\langle A, +, \times\rangle$ is an algebra over the field P, $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in A^k$, then we shall denote

$$\overline{\mathbf{a}} = (a_1, -a_2, \ldots, -a_k) \in A^k.$$

Proposition 3.14. *If A is an associative algebra over the field P, then*

$$\overline{[\mathbf{x_1 x_2} \ldots \mathbf{x}_l]}_{l,\sigma,k} = [\overline{\mathbf{x}}_\mathbf{1} \mathbf{x_2} \ldots \mathbf{x}_l]_{l,\sigma,k}.$$

If σ is a cycle of length k in S_k, which satisfies the condition $\sigma^l = \sigma$, then

$$\overline{[\mathbf{x_1 x_2} \ldots \mathbf{x}_l]}_{l,\sigma,k} = \begin{cases} [\overline{\mathbf{x}}_\mathbf{1} \overline{\mathbf{x}}_\mathbf{2} \ldots \overline{\mathbf{x}}_l]_{l,\sigma,k}, & \text{for even } (l-1)(k-1)/k, \\ -[\overline{\mathbf{x}}_\mathbf{1} \overline{\mathbf{x}}_\mathbf{2} \ldots \overline{\mathbf{x}}_l]_{l,\sigma,k}, & \text{for odd } (l-1)(k-1)/k. \end{cases}$$

Corollary 3.15. *If A is an associative algebra over the field P, σ is a cycle of length k from S_k, then*

$$\overline{[\mathbf{x_1 x_2} \ldots \mathbf{x_{k+1}}]}_{k+1,\sigma,k} = \begin{cases} [\overline{\mathbf{x}}_\mathbf{1} \overline{\mathbf{x}}_\mathbf{2} \ldots \overline{\mathbf{x}}_\mathbf{k+1}]_{k+1,\sigma,k}, & \text{for odd } k, \\ -[\overline{\mathbf{x}}_\mathbf{1} \overline{\mathbf{x}}_\mathbf{2} \ldots \overline{\mathbf{x}}_\mathbf{k+1}]_{k+1,\sigma,k}, & \text{for even } k. \end{cases}$$

Corollary 3.16. *If A is an associative algebra over the field P and σ is a cycle of length k from S_k, then*

$$\overline{[\mathbf{x_1 x_2} \ldots \mathbf{x_{2k-1}}]}_{2k-1,\sigma,k} = [\overline{\mathbf{x}}_\mathbf{1} \overline{\mathbf{x}}_\mathbf{2} \ldots \overline{\mathbf{x}}_\mathbf{2k-1}]_{2k-1,\sigma,k}.$$

Proposition 3.17. *If A is a group, 1 is its unity, $\sigma = \sigma_1 \ldots \sigma_p$ is the decomposition into product of independent cycles (excepting cycles of length 1) of a permutation $\sigma \in S_k$ which satisfies the condition $\sigma^l = \sigma$, then the element $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ is idempotent in $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$ if and only if the components ε_m whose index m remains fixed under the permutation σ satisfy the condition*

$$\varepsilon_m^{l-1} = 1,$$

while the components, whose indices appear in the expression of the cycle σ_r ($r = 1, \ldots, p$), satisfy the condition

$$\varepsilon_{i_r} \varepsilon_{\sigma(i_r)} \varepsilon_{\sigma^2(i_r)} \ldots \varepsilon_{\sigma^{l-2}(i_r)} = 1,$$

where i_r is an arbitrary symbol which appears in the expression of the cycle σ_r.

Corollary 3.18. *If A is a group, 1 is its unity, and the cycle $\sigma \in S_k$ of length k satisfies the condition $\sigma^l = \sigma$, then*

$$I(A^k, [\ \ ]_{l,\sigma,k}) = \{(\varepsilon_1, \ldots, \varepsilon_k) \in A^k | (\varepsilon_1 \varepsilon_{\sigma(1)} \varepsilon_{\sigma^2(1)} \ldots \varepsilon_{\sigma^{k-1}(1)})^{\frac{l-1}{k}} = 1\}.$$

In particular,

$$I(A^k, [\ \ ]_{k+1,\sigma,k}) = \{(\varepsilon_1, \ldots, \varepsilon_k) \in A^k | \varepsilon_1 \varepsilon_{\sigma(1)} \varepsilon_{\sigma^2(1)} \ldots \varepsilon_{\sigma^{k-1}(1)} = 1\}.$$

Corollary 3.19. *If A is an abelian (commutative) group, 1 is its unity, and the cycle $\sigma \in S_k$ satisfies the condition $\sigma^l = \sigma$, then*

$$I(A^k, [\ \ ]_{l,\sigma,k}) = \{(\varepsilon_1, \ldots, \varepsilon_k) \in A^k | (\varepsilon_1\varepsilon_2 \ldots \varepsilon_k)^{\frac{l-1}{k}} = 1\}.$$

In particular,

$$I(A^k, [\ \ ]_{k+1,\sigma,k}) = \{(\varepsilon_1, \ldots, \varepsilon_k) \in A^k | \varepsilon_1\varepsilon_2 \ldots \varepsilon_k = 1\}.$$

Remark. In Proposition 3.17, the condition

$$\varepsilon_{i_r}\varepsilon_{\sigma(i_r)}\varepsilon_{\sigma^2(i_r)} \ldots \varepsilon_{\sigma^{l-2}(i_r)} = 1$$

can be replaced with

$$\varepsilon_{\sigma(i_r)}\varepsilon_{\sigma^2(i_r)} \ldots \varepsilon_{\sigma^{l-2}(i_r)}\varepsilon_{i_r} = 1.$$

Similar replacements hold valid in Corollaries 3.18 and 3.19.

Theorem 3.20. [13, 14]. *Let $\langle A, +, \times\rangle$ be an associative algebra over the field P, 0 - its zero element, $k \geqslant 2$, $l \geq 3$, such that k divides $l-1$, and σ a cycle of length k from S_k. Then:*

1) $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ is an associative $(2,l)$-algebra over P, whose all its elements are zero-divisors of its zero element $\underbrace{(0, \ldots, 0)}_{k}$;

2) if $\langle A, +, \times\rangle$ is commutative, then $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ is semi-abelian (semi-commutative);

3) if $\langle A\backslash\{0\}, \times\rangle$ is a group, then $\langle (A\backslash\{0\})^k, [\ \ ]_{l,\sigma,k}\rangle$ is an l-ary group;

4) for any $j \in \{1, \ldots, k\}$ and any $\mathbf{a} = (a_1, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_k) \in A^k$, we have

$$[\underbrace{\mathbf{a}, \ldots, \mathbf{a}}_{l}]_{l,\sigma,k} = (\underbrace{0, \ldots, 0}_{k});$$

5) if the algebra $\langle A, +, \times\rangle$ contains more than one element and has a unity, then $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ is non-abelian;

6) if A contains more than one element, then $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ does not contain a unity;

7) if $\langle A\backslash\{0\}, \times\rangle$ is a group and 1 is its unity, then

$$I(A^k, [\ \ ]_{l,\sigma,k}) = \{(\varepsilon_1, \ldots, \varepsilon_k) \in A^k | (\varepsilon_1\varepsilon_{\sigma(1)} \ldots \varepsilon_{\sigma^{l-2}(1)})^{\frac{l-1}{k}} = 1\} \cup \{(\underbrace{0, \ldots, 0}_{k})\};$$

8) we have the following:

$$\overline{[\mathbf{x_1}\mathbf{x_2} \ldots \mathbf{x}_l]}_{l,\sigma,k} = \begin{cases} [\overline{\mathbf{x}}_\mathbf{1}\overline{\mathbf{x}}_\mathbf{2} \ldots \overline{\mathbf{x}}_l]_{l,\sigma,k}, & \text{for even } (l-1)(k-1)/k, \\ -[\overline{\mathbf{x}}_\mathbf{1}\overline{\mathbf{x}}_\mathbf{2} \ldots \overline{\mathbf{x}}_l]_{l,\sigma,k}, & \text{for odd } (l-1)(k-1)/k. \end{cases}$$

Proof. 1) follows from Propositions 3.8 and 3.12; 2) follows from Proposition 3.10; 3) follows from Proposition 3.11; 4) is straightforward; 5) follows from Proposition 3.9; 6) follows from Proposition 3.13; 7) follows from Corollary 3.18 and item 4); 8) follows from Proposition 3.14. □

Replacing $l = k + 1$ in Theorem 3.20, we infer

Corollary 3.21. *Let $\langle A, +, \times\rangle$ be an associative algebra over the field P, 0 its zero element, $k \geqslant 2$, and σ a cycle of length k from S_k. Then:*

1) $\langle A^k, +, [\ \ ]_{k+1,\sigma,k}\rangle$ is an associative $(2, k+1)$-algebra over P, in which all its elements are divisors of its zero element $(\underbrace{0,\ldots,0}_{k})$;

2) if $\langle A, +, \times\rangle$ is commutative, then $\langle A^k, +, [\ \ ]_{k+1,\sigma,k}\rangle$ is semi-abelian;

3) if $\langle A\backslash\{0\}, \times\rangle$ is a group, then $\langle (A\backslash\{0\})^k, [\ \ ]_{k+1,\sigma,k}\rangle$ is a $(k+1)$-ary group;

4) for any $j \in \{1,\ldots,k\}$ and any $a = (a_1,\ldots,a_{j-1},0,a_{j+1},\ldots,a_k) \in A^k$ we have

$$[\underbrace{a,\ldots,a}_{k+1}]_{k+1,\sigma,k} = (\underbrace{0,\ldots,0}_{k});$$

5) if the algebra $\langle A, +, \times\rangle$ contains more than one element and has a unity, then $\langle A^k, +, [\ \ ]_{k+1,\sigma,k}\rangle$ is non-abelian;

6) if A contains more than one element, then $\langle A^k, +, [\ \ ]_{k+1,\sigma,k}\rangle$ has no unity;

7) if $\langle A\backslash\{0\}, \times\rangle$ is a group, 1 is its unity, then

$$I(A^k, +, [\ \ ]_{k+1,\sigma,k}) = \{(\varepsilon_1,\ldots,\varepsilon_k) \in A^k | \varepsilon_1\varepsilon_{\sigma(1)}\ldots\varepsilon_{\sigma^{l-2}(1)} = 1\} \cup \{(\underbrace{0,\ldots,0}_{k})\};$$

8) we have the following:

$$\overline{[\mathbf{x_1x_2\ldots x_{k+1}}]}_{k+1,\sigma,k} = \begin{cases} [\mathbf{\overline{x}_1\overline{x}_2\ldots\overline{x}_{k+1}}]_{k+1,\sigma,k}, & \text{for odd } k,\\ -[\mathbf{\overline{x}_1\overline{x}_2\ldots\overline{x}_{k+1}}]_{k+1,\sigma,k}, & \text{for even } k.\end{cases}$$

We further prove that the operations $[\ \ ]_{l,k}$ and $[\ \ ]_{l,(12\ldots k),k}$ coincide.

Proposition 3.22. *Let A be a semigroup, $l \geqslant 2$, $k \geqslant 2$, $\alpha = (12\ldots k) \in S_k$. Then the operations $[\ \ ]_{l,k}$ and $[\ \ ]_{l,\alpha,k}$ coincide: $[\ \ ]_{l,k} = [\ \ ]_{l,\alpha,k}$.*

Proof. Let $l = sk + r$, $s \geqslant 0$, $1 \leqslant r \leqslant k$. Using Theorem 2.6, items 1) and 2) from Lemma 2.5, the definition of the transformation f_i and the equalities

$$\alpha(j) = j+1,\ \alpha^2(j) = j+2,\ \ldots,\ \alpha^{k-j}(j) = k,$$
$$\alpha^{k-j+1}(j) = 1,\ \ldots,\ \alpha^{k-1}(j) = j-1,\ \alpha^k(j) = j,$$

we get

$$\begin{aligned}
&[\mathbf{x_1x_2\ldots x_l}]_{l,k} = [\mathbf{x_1x_2\ldots x_kx_{k+1}x_{k+2}\ldots x_{2k}x_{2k+1}x_{2k+2}\ldots}\\
&\quad\mathbf{\ldots x_{(s-1)k}x_{(s-1)k+1}x_{(s-1)k+2}\ldots x_{sk}x_{sk+1}x_{sk+2}\ldots x_{sk+r}}]_{l,k} =\\
&= \mathbf{x_1x_2^{f_1}\ldots x_k^{f_{k-1}}x_{k+1}^{f_k}x_{k+2}^{f_{k+1}}\ldots x_{2k}^{f_{2k-1}}x_{2k+1}^{f_{2k}}x_{2k+2}^{f_{2k+1}}\ldots}\\
&\quad\mathbf{\ldots x_{(s-1)k}^{f_{(s-1)k-1}}x_{(s-1)k+1}^{f_{(s-1)k}}x_{(s-1)k+2}^{f_{(s-1)k+1}}\ldots x_{sk}^{f_{sk-1}}x_{sk+1}^{f_{sk}}x_{sk+2}^{f_{sk+1}}\ldots x_{sk+r}^{f_{sk+r-1}}} =\\
&= \mathbf{x_1x_2^{f_1}\ldots x_k^{f_{k-1}}x_{k+1}x_{k+2}^{f_1}\ldots x_{2k}^{f_{k-1}}x_{2k+1}x_{2k+2}^{f_1}\ldots}\\
&\quad\mathbf{\ldots x_{(s-1)k}^{f_{k-1}}x_{(s-1)k+1}x_{(s-1)k+2}^{f_1}\ldots x_{sk}^{f_{k-1}}x_{sk+1}x_{sk+2}^{f_1}\ldots x_{sk+r}^{f_{r-1}}} =\\
&= (x_{11},\ldots,x_{1k})(x_{22},\ldots,x_{2k},x_{21})\ldots\\
&\quad\ldots(x_{kk},x_{k1},\ldots,x_{k(k-1)})(x_{(k+1)1},\ldots,x_{(k+1)k})\\
&\qquad(x_{(k+2)2},\ldots,x_{(k+2)k},x_{(k+2)1})\ldots(x_{(2k)k},x_{(2k)1},\ldots,x_{(2k)(k-1)})\\
&\qquad(x_{(2k+1)1},\ldots,x_{(2k+1)k})(x_{(2k+2)2},\ldots,x_{(2k+2)k},x_{(2k+2)1})\ldots
\end{aligned}$$

$$\ldots(x_{((s-1)k)k}, x_{((s-1)k)1}, \ldots, x_{((s-1)k)(k-1)})(x_{((s-1)k+1)1}, \ldots, x_{((s-1)k+1)k})$$
$$(x_{((s-1)k+2)2}, \ldots, x_{((s-1)k+2)k}, x_{((s-1)k+2)1})(x_{(sk)k}, x_{(sk)1}, \ldots, x_{(sk)(k-1)})$$
$$(x_{(sk+1)1}, \ldots, x_{(sk+1)k})(x_{(sk+2)2}, \ldots, x_{(sk+2)k}, x_{(sk+2)1})\ldots$$
$$\ldots(x_{(sk+r)r}, \ldots, x_{(sk+r)k}, x_{(sk+r)1}, \ldots, x_{(sk+r)(r-1)}) =$$
$$= (x_{11}, \ldots, x_{1k})(x_{2\alpha(1)}, \ldots, x_{2\alpha(k-1)}, x_{2\alpha(k)})\ldots$$
$$\ldots(x_{k\alpha^{k-1}(1)} x_{k\alpha^{k-1}(2)}, \ldots, x_{k\alpha^{k-1}(k)})(x_{(k+1)1}, \ldots, x_{(k+1)k})$$
$$(x_{(k+2)\alpha(1)}, \ldots, x_{(k+2)\alpha(k)})\ldots(x_{(2k)\alpha^{k-1}(1)}, \ldots, x_{(2k)\alpha^{k-1}(k)})$$
$$(x_{(2k+1)1}, \ldots, x_{(2k+1)k})(x_{(2k+2)\alpha(1)}, \ldots, x_{(2k+2)\alpha(k)})\ldots$$
$$\ldots(x_{((s-1)k)\alpha^{k-1}(1)}, \ldots, x_{((s-1)k)\alpha^{k-1}(k)})\ldots(x_{((s-1)k+1)1}, \ldots, x_{((s-1)k+1)k})$$
$$(x_{((s-1)k+2)\alpha(1)}, \ldots, x_{((s-1)k+2)\alpha(k)})\ldots(x_{(sk)\alpha^{k-1}(1)}, \ldots, x_{(sk)\alpha^{k-1}(k)})$$
$$(x_{(sk+1)1}, \ldots, x_{(sk+1)k})(x_{(sk+2)\alpha(1)}, \ldots, x_{(sk+2)\alpha(k)})\ldots$$
$$\ldots(x_{(sk+r)\alpha^{r-1}(1)}, \ldots, x_{(sk+r)\alpha^{r-1}(k)}) =$$
$$= (x_{11} x_{2\alpha(1)} \ldots x_{k\alpha^{k-1}(1)} x_{(k+1)1} x_{(k+2)\alpha(1)} \ldots x_{(2k)\alpha^{k-1}(1)} \cdots$$
$$\cdots x_{((s-1)k+1)1} x_{((s-1)k+2)\alpha(1)} \cdots$$
$$\cdots x_{(sk)\alpha^{k-1}(1)} x_{(sk+1)1} x_{(sk+2)\alpha(1)} \cdots x_{(sk+r)\alpha^{r-1}(1)}, \ldots x_{1k} x_{2\alpha(k)} \cdots,$$
$$\cdots x_{k\alpha^{k-1}(k)} x_{(k+1)k} x_{(k+2)\alpha(k)} \cdots x_{(2k)\alpha^{k-1}(k)} \cdots$$
$$\cdots x_{((s-1)k+1)k} x_{((s-1)k+2)\alpha(k)} \cdots x_{(sk)\alpha^{k-1}(k)} \cdots$$
$$\cdots x_{(sk+1)k} x_{(sk+2)\alpha(k)} x_{(sk+r)\alpha^{r-1}(k)}) =$$
$$= (x_{11} x_{2\alpha(1)} \cdots x_{k\alpha^{k-1}(1)} x_{(k+1)\alpha^{k}(1)} x_{(k+2)\alpha^{k+1}(1)} \cdots x_{(2k)\alpha^{2k-1}(1)} \cdots$$
$$\cdots x_{((s-1)k+1)\alpha^{(s-1)k}(1)} x_{((s-1)k+2)\alpha^{(s-1)k+1}(1)} \cdots$$
$$\cdots x_{(sk)\alpha^{sk-1}(1)} x_{(sk+1)\alpha^{sk}(1)} x_{(sk+2)\alpha^{sk+1}(1)} \cdots$$
$$\cdots x_{(sk+r)\alpha^{sk+r-1}(1)}, \ldots, x_{1k} x_{2\alpha(k)} \cdots x_{k\alpha^{k-1}(k)} x_{(k+1)\alpha^{k}(k)} x_{(k+2)\alpha^{k+1}(k)} \cdots$$
$$\cdots x_{(2k)\alpha^{2k-1}(k)} \cdots x_{((s-1)k+1)\alpha^{(s-1)k}(k)} x_{((s-1)k+2)\alpha^{(s-1)k+1}(k)} \cdots$$
$$\cdots x_{(sk)\alpha^{sk-1}(k)} x_{(sk+1)\alpha^{sk}(k)} x_{(sk+2)\alpha^{sk+1}(k)} \cdots x_{(sk+r)\alpha^{sk+r-1}(k)}) =$$
$$= (x_{11} x_{2\alpha(1)} \cdots x_{(sk+r)\alpha^{sk+r-1}(1)}, \ldots, x_{1k} x_{2\alpha(k)} \cdots x_{(sk+r)\alpha^{sk+r-1}(k)}) =$$
$$= (x_{11} x_{2\alpha(1)} \cdots x_{l\alpha^{l-1}(1)}, \ldots, x_{1k} x_{2\alpha(k)} \cdots x_{l\alpha^{l-1}(k)}) = [x_1 x_2 \ldots x_l]_{l,\alpha,k}.$$

Hence $[x_1 x_2 \ldots x_l]_{l,k} = [x_1 x_2 \ldots x_l]_{l,\alpha,k}$, and the Theorem is proved. □

This result shows that Theorem 6.1 and Corollary 6.2 from [18] are particular cases of Theorem 3.20 and respectively Corollary 3.21 - since the last ones follow by the replacements $l = s(n-1)$, $k = n-1$ ($n \geqslant 3$) and $\sigma = (12 \ldots n-1)$. We remark that the corresponding results from [18] are particular cases of the assertions (3.8) – (3.19) as well.

4 The n-ary operation $[\ \ ]_{n,m,m(n-1)}$

Let B be a set, $m \geqslant 1$, $n \geqslant 3$, $A = B^m$ an m-ary Cartesian power of the set B, $\langle A, \times \rangle$ a semigroup, whose operation shall be sometimes omitted, for brevity.

We note that if B is a semigroup, then for the product $\times$ we may consider the operation which is componentwise defined on $A = B^m$. We define on $A^{n-1} = B^{m(n-1)}$ the n-ary operation $[\ ]_{n,m,m(n-1)}$ as follows. If, for $i = 1, \dots, n$, we denote

$$\boldsymbol{\alpha_i} = (\alpha_{i1}^{(1)}, \dots, \alpha_{im}^{(1)}, \alpha_{i1}^{(2)}, \dots, \alpha_{im}^{(2)}, \dots \alpha_{i1}^{(n-1)}, \dots, \alpha_{im}^{(n-1)}) \in B^{m(n-1)},$$

then

$$\begin{aligned}[\boldsymbol{\alpha_1\alpha_2 \dots \alpha_n}]_{n,m,m(n-1)} &= (y_{11}, \dots, y_{1m}, y_{21}, \dots, y_{2m}, \dots \\ &\dots, y_{(n-1)1}, \dots, y_{(n-1)m}) \in B^{m(n-1)},\end{aligned} \tag{4.1}$$

where, for $j = 1, \dots, n-1$, the components y_{ij} are defined by

$$\begin{aligned}(y_{j1}, \dots, y_{jm}) &= (\alpha_{11}^{(j)}, \dots, \alpha_{1m}^{(j)}) \times (\alpha_{21}^{(j+1)}, \dots, \alpha_{2m}^{(j+1)}) \times \dots \\ &\dots (\alpha_{(n-j)1}^{(n-1)}, \dots, \alpha_{(n-j)m}^{(n-1)}) \times (\alpha_{(n-j+1)1}^{(1)}, \dots, \alpha_{(n-j+1)m}^{(1)}) \times \dots \\ &\dots (\alpha_{(n-1)1}^{(j-1)}, \dots, \alpha_{(n-1)m}^{(j-1)}) \times (\alpha_{n1}^{(j)}, \dots, \alpha_{nm}^{(j)}) \in B^m.\end{aligned} \tag{4.2}$$

If one makes the replacements

$$\boldsymbol{\alpha_{ij}} = (\alpha_{i1}^{(j)}, \dots, \alpha_{im}^{(j)}) \in B^m, \quad \mathbf{y_j} = (y_{j1}, \dots, y_{jm}), \ \ j \in \{1, \dots, n-1\},$$

then (4.2) becomes

$$\mathbf{y_j} = \boldsymbol{\alpha_{ij}\alpha_{2(j+1)} \dots \alpha_{(n-j)(n-1)}\alpha_{(n-j+1)1} \dots \alpha_{(n-1)(j-1)}\alpha_{nj}} \in B^m.$$

It is obvious, that using the relation (2.2) for $m = 1$, the n-ary operation $[\ ]_{n,m,m(n-1)}$ coincides with the n-ary operation $[\ ]_{n,n-1}$: $[\ ]_{n,n-1} = [\ ]_{n,1,n-1}$.

Theorem 4.1. [17, 18, 13]. *The n-ary operation $[\ ]_{n,m,m(n-1)}$ is associative.*

If in this Theorem we replace $m = 2$, then we get

Corollary 4.2. *The n-ary operation $[\ ]_{n,2,2(n-1)}$ is associative.*

If in Corollary 4.2 we replace $n = 3$, $B = \mathbb{R}$, and if for the operation "$\times$" we take the multiplication of complex (or dual/double) numbers, then we get three distinct associative ternary operations, defined on $\mathbb{R}^4$. The explicit forms of these operations were determined in [17] and [18].

Generally speaking, if we take for the operation $\times$, the multiplication of complex (or dual/double) numbers, then according to Corollary 4.2, for any $n \geqslant 3$, on the Cartesian power $\mathbb{R}^{2(n-1)}$ one can build three distinct associative n-ary operations. We shall further describe the form of these operations, for the case of multiplication of complex numbers, for $n = 4$.

Corollary 4.3. [17, 18, 13]. *The 4-ary operation defined on $\mathbb{R}^6$:*

$$\begin{aligned}&[(x_1, x_2, x_3, x_4, x_5, x_6)(y_1, y_2, y_3, y_4, y_5, y_6) \\ &\quad (z_1, z_2, z_3, z_4, z_5, z_6)(u_1, u_2, u_3, u_4, u_5, u_6)]_{4,2,6} = (r_1, r_2, r_3, r_4, r_5, r_6),\end{aligned}$$

where

$$\begin{cases} r_1 = & x_1y_3z_5u_1 - x_2y_4z_5u_1 - x_1y_4z_6u_1 - x_2y_3z_6u_1 - \\ & -x_1y_3z_6u_2 + x_2y_4z_6u_2 - x_1y_4z_5u_2 - x_2y_3z_5u_2, \\ r_2 = & x_1y_3z_5u_2 - x_2y_4z_5u_2 - x_1y_4z_6u_2 - x_2y_3z_6u_2 + x_1y_3z_6u_1 - \\ & -x_2y_4z_6u_1 + x_1y_4z_5u_1 + x_2y_3z_5u_1, \\ r_3 = & x_3y_5z_1u_3 - x_4y_6z_1u_3 - x_3y_6z_2u_3 - x_4y_5z_2u_3 - x_3y_5z_2u_4 + \\ & +x_4y_6z_2u_4 - x_3y_6z_1u_4 - x_4y_5z_1u_4, \\ r_4 = & x_3y_5z_1u_4 - x_4y_6z_1u_4 - x_3y_6z_2u_4 - x_4y_5z_2u_4 + x_3y_5z_2u_3 - \\ & -x_4y_6z_2u_3 + x_3y_6z_1u_3 + x_4y_5z_1u_3, \\ r_5 = & x_5y_1z_3u_5 - x_6y_2z_3u_5 - x_5y_2z_4u_5 - x_6y_1z_4u_5 - x_5y_1z_4u_6 + \\ & +x_6y_2z_4u_6 - x_5y_2z_3u_6 - x_6y_1z_3u_6, \\ r_6 = & x_5y_1z_3u_6 - x_6y_2z_3u_6 - x_5y_2z_4u_6 - x_6y_1z_4u_6 + x_5y_1z_4u_5 - \\ & -x_6y_2z_4u_5 + x_5y_2z_3u_5 + x_6y_1z_3u_5, \end{cases}$$

is associative.

If in Theorem 4.1 we replace $m = 4$, $B = \mathbb{R}$, and for the operation "$\times$" we consider the multiplication of quaternions, then for any $n \geqslant 3$, on the Cartesian power $\mathbb{R}^{4(n-1)}$ we can define an associative n-ary operation. In [17, 18] is presented the explicit form of such a operation for $m = 4$, $n = 3$ (i.e., a ternary operation on $\mathbb{R}^8$).

5 The ℓ-ary operation $[\ \ ]_{l,\sigma,m,mk}$

As shown before, for $m = 1$ the n-ary operation $[\ \ ]_{n,m,m(n-1)}$ coincides with the n-ary operation $[\ \ ]_{n,n-1}$, which is a particular case of the operation $[\ \ ]_{l,k}$ for $l = n$, $k = n-1$. The last one, in its turn, is a particular case of the operation $[\ \ ]_{l,\sigma,k}$ for $\sigma = (12\dots k)$. Then the following task appears: to generalize the operation $[\ \ ]_{n,m,m(n-1)}$ in such a way, that for $m = 1$ it coincides with the operation $[\ \ ]_{l,\sigma,k}$.

Let B be a set, $m \geqslant 1$, $l \geqslant 2$, $k \geqslant 2$, $\sigma \in S_k$, $A = B^m$ the m-ary Cartesian power of the set B and $\langle A, \times\rangle$ a semigroup. Like before, in several places we shall omit the multiplication sign $\times$, for brevity.

We shall define on B^{mk} an ℓ-ary operation $[\ \ ]_{l,\sigma,m,mk}$ as follows. If

$$\boldsymbol{\alpha_i} = (\alpha_{i1}^{(1)}, \dots, \alpha_{im}^{(1)}, \alpha_{i1}^{(2)}, \dots, \alpha_{im}^{(2)}, \dots, \alpha_{i1}^{(k)}, \dots, \alpha_{im}^{(k)}) \in B^{mk},\ i \in \{1, \dots, l\},$$

then

$$(5.1)\quad [\boldsymbol{\alpha_1\alpha_2\dots\alpha_l}]_{l,\sigma,m,mk} = (y_{11}, \dots, y_{1m}, y_{21}, \dots, y_{2m}, \dots, y_{k1}, \dots, y_{km}) \in B^{mk},$$

where y_{ij} is defined by

$$(5.2)\quad \begin{aligned} (y_{j1}, \dots, y_{jm}) = \ & (\alpha_{11}^{(j)}, \dots, \alpha_{1m}^{(j)}) \times (\alpha_{21}^{(\sigma(j))}, \dots, \alpha_{2m}^{(\sigma(j))}) \times \dots \\ & \dots (\alpha_{(l-1)1}^{(\sigma^{l-2}(j))}, \dots, \alpha_{(l-1)m}^{(\sigma^{l-2}(j))}) \times (\alpha_{lm}^{(\sigma^{l-1}(j))}, \dots, \alpha_{lm}^{(\sigma^{l-1}(j))}). \end{aligned}$$

If we replace

$$\boldsymbol{\alpha_{ij}} = (\alpha_{i1}^{(j)}, \ldots, \alpha_{im}^{(j)}) \in B^m,\ \boldsymbol{y_j} = (y_{j1}, \ldots, y_{jm}),\ j \in \{1, \ldots, k\},$$

then (5.2) gets the form

$$\boldsymbol{y_j} = \boldsymbol{\alpha_{1j}\alpha_{2\sigma(j)} \ldots \alpha_{(l-1)\sigma^{l-2}(j)}\alpha_{l\sigma^{l-1}(j)}} \in B^m.$$

It is clear that for $m = 1$, due to (2.4), the ℓ-ary operation $[\ \ ]_{l,\sigma,m,mk}$ coincides with the ℓ-ary operation $(\ )_{l,\sigma,k}$. But if $\ell = n$, $k = n-1$, and $\sigma = (12\ldots n-1)$, then (5.1) and (5.2) get the form (4.1) and respectively (4.2), and the operation $[\ \ ]_{l,\sigma,m,mk}$ coincides with the operation $[\ \ ]_{n,m,m(n-1)}$. In this way, the posed problem of extending the operation $[\ \ ]_{n,m,m(n-1)}$ is solved.

We examine on $A^k = \underbrace{B^m \times \ldots \times B^m}_{k}$ the ℓ-ary operation $(\ )_{l,\sigma,k}$ and we describe its explicit form. To this goal, for any $i \in \{1, \ldots, l\}$, we put

$$\mathbf{x_i} = (\mathbf{x_{i1}}, \ldots, \mathbf{x_{ik}}) \in A^k,\ \mathbf{x_{ij}} = (x_{i1}^{(j)}, \ldots, x_{im}^{(j)}) \in B^m,\ j = 1, \ldots, k,$$

i.e.,

$$\mathbf{x_i} = ((x_{i1}^{(1)}, \ldots, x_{im}^{(1)}), (x_{i1}^{(2)}, \ldots, x_{im}^{(2)}), \ldots, (x_{i1}^{(k)}, \ldots, x_{im}^{(k)})) \in A^k.$$

Using (3.1) in the definition of the operation $(\ )_{l,\sigma,k}$, we infer

$$[\mathbf{x_1 x_2} \ldots \mathbf{x}_l]_{l,\sigma,k} = (\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_k}),$$

where

$$\mathbf{y_j} = \boldsymbol{x_{1j}x_{2\sigma(j)} \ldots x_{(l-1)\sigma^{(l-2)}(j)}x_{l\sigma^{(l-1)}(j)}} \in B^m,$$

or

$$\mathbf{y_j} = (x_{11}^{(j)}, \ldots, x_{1m}^{(j)})(x_{21}^{(\sigma(j))}, \ldots, x_{2m}^{(\sigma(j))}) \ldots (x_{(l-1)1}^{(\sigma^{l-2}(j))}, \ldots, x_{(l-1)m}^{(\sigma^{l-2}(j))}) \cdot$$
$$\cdot (x_{l1}^{(\sigma^{l-1}(j))}, \ldots, x_{lm}^{(\sigma^{l-1}(j))}).$$

Lemma 5.1. *The universal algebras $\langle B^{mk}, [\ \ ]_{l,\sigma,m,mk}\rangle$ and $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$ are isomorphic.*

Proof. It is clear that the mapping ψ, which puts into correspondence the element

$$(\alpha_1^{(1)}, \ldots, \alpha_m^{(1)}, \alpha_1^{(2)}, \ldots, \alpha_m^{(2)}, \ldots, \alpha_1^{(k)}, \ldots, \alpha_m^{(k)}) \in B^{mk}$$

with the element

$$((\alpha_1^{(1)}, \ldots, \alpha_m^{(1)}), (\alpha_1^{(2)}, \ldots, \alpha_m^{(2)}), \ldots, (\alpha_1^{(k)}, \ldots, \alpha_m^{(k)})) \in A^k$$

is a bijection from B^{mk} to A^k. Moreover, we have

$$
\begin{aligned}
&[\boldsymbol{\alpha_1\alpha_2\dots\alpha_l}]^{\psi}_{l,\sigma,m,mk} = (y_{11},\dots,y_{1m},y_{21},\dots,y_{2m},\dots,y_{k1},\dots,y_{km})^{\psi} = \\
&\quad = ((y_{11},\dots,y_{1m}),(y_{21},\dots,y_{2m}),\dots,(y_{k1},\dots,y_{km})) = (\mathbf{y_1},\mathbf{y_2},\dots,\mathbf{y_k}) = \\
&\quad = (\boldsymbol{\alpha_{11}\alpha_{2\sigma(1)}\dots\alpha_{(l-1)\sigma^{l-2}(1)}\alpha_{l\sigma^{l-1}(1)}},\dots,\boldsymbol{\alpha_{1k}\alpha_{2\sigma(k)}}\dots \\
&\qquad \dots\boldsymbol{\alpha_{(l-1)\sigma^{l-2}(k)}\alpha_{l\sigma^{l-1}(k)}}) = \\
&\quad = ((\boldsymbol{\alpha_{11}},\dots,\boldsymbol{\alpha_{1k}})(\boldsymbol{\alpha_{21}},\dots,\boldsymbol{\alpha_{2k}})\dots(\boldsymbol{\alpha_{l1}},\dots,\boldsymbol{\alpha_{lk}}))_{l,\sigma,k} = \\
&\quad = (((\alpha^{(1)}_{11},\dots,\alpha^{(1)}_{1m}),\dots,(\alpha^{(k)}_{11},\dots,\alpha^{(k)}_{1m})) \\
&\qquad\qquad ((\alpha^{(1)}_{21},\dots,\alpha^{(1)}_{2m}),\dots,(\alpha^{(k)}_{21},\dots,\alpha^{(k)}_{2m}))\dots \\
&\qquad \dots((\alpha^{(1)}_{l1},\dots,\alpha^{(1)}_{lm}),\dots,(\alpha^{(k)}_{l1},\dots,\alpha^{(k)}_{lm})))_{l,\sigma,k} = \\
&\quad = ((\alpha^{(1)}_{11},\dots,\alpha^{(1)}_{1m},\dots,\alpha^{(k)}_{11},\dots,\alpha^{(k)}_{1m})^{\psi}(\alpha^{(1)}_{21},\dots,\alpha^{(1)}_{2m},\dots,\alpha^{(k)}_{21},\dots,\alpha^{(k)}_{2m})^{\psi}\dots \\
&\qquad \dots(\alpha^{(1)}_{l1},\dots,\alpha^{(1)}_{lm},\dots,\alpha^{(k)}_{l1},\dots,\alpha^{(k)}_{lm})^{\psi})_{l,\sigma,k} = [\boldsymbol{\alpha_1^{\psi}\alpha_2^{\psi}\dots\alpha_l^{\psi}}]_{l,\sigma,k},
\end{aligned}
$$

i.e.,

$$[\boldsymbol{\alpha_1\alpha_2\dots\alpha_l}]^{\psi}_{l,\sigma,m,mk} = [\boldsymbol{\alpha_1^{\psi}\alpha_2^{\psi}\dots\alpha_l^{\psi}}]_{l,\sigma,k}.$$

Consequently, ψ is the claimed isomorphism, and the Lemma is proved. □

Lemma 5.1 and Theorem 3.2 provide a sufficiency condition for associativity:

Theorem 5.2. [13]. *If the permutation σ satisfies the condition $\sigma^l = \sigma$, then the l-ary operation $[\ \ ]_{l,\sigma,m,mk}$ is associative.*

This result follows from Theorem 5.2 for $l = n$, $k = n-1$ and $\sigma = (12\dots n-1)$.

If in Theorem 5.2 we put $m = 2$, then we get

Corollary 5.3. *If permutation σ satisfies the condition $\sigma^l = \sigma$, then the l-ary operation $[\ \ ]_{l,\sigma,2,2k}$ is associative.*

We remark that the 4-ary operation from Corollary 4.3 coincides with the operation $[\ \ ]_{4,(123),2,6}$, i.e., it is an operation of the form $[\ \ ]_{l,\sigma,m,mk}$ for $m = 2$, $l = 4$, $k = 3$, and $\sigma = (123)$ a permutation of order 3 from S_3. But the permutation $(132) \in S_3$ satisfies as well the condition $\sigma^4 = \sigma$. Hence, for $B = \mathbb{R}$ and $\langle A = \mathbb{R}^2, \times\rangle$ – the semigroup of complex numbers, and, by replacing $\ell = 4$, $k = 3$ and $\sigma = (132)$ in Corollary 5.3, we get

Corollary 5.4. *The 4-ary operation defined on $\mathbb{R}^6$:*

$$
\begin{aligned}
&[(x_1,x_2,x_3,x_4,x_5,x_6)(y_1,y_2,y_3,y_5,y_6) \\
&\quad (z_1,z_2,z_3,z_4,z_5,z_6)(u_1,u_2,u_3,u_4,u_5,u_6)]_{4,(132),2,6} = (r_1,r_2,r_3,r_4,r_5,r_6),
\end{aligned}
$$

where

$$\begin{cases} r_1 = & x_1y_5z_3u_1 - x_2y_6z_3u_1 - x_1y_6z_4u_1 - x_2y_5z_4u_1 - \\ & -x_1y_5z_4u_2 + x_2y_6z_4u_2 - x_1y_6z_3u_2 - x_2y_5z_3u_2, \\ r_2 = & x_1y_5z_3u_2 - x_2y_6z_3u_2 - x_1y_6z_4u_2 - x_2y_5z_4u_2 + \\ & +x_1y_5z_4u_1 - x_2y_6z_4u_1 + x_1y_6z_3u_1 + x_2y_5z_3u_1, \\ r_3 = & x_3y_1z_5u_3 - x_4y_2z_5u_3 - x_3y_2z_6u_3 - x_4y_1z_6u_3 - \\ & -x_3y_1z_6u_4 + x_4y_2z_6u_4 - x_3y_2z_5u_4 - x_4y_1z_5u_4, \\ r_4 = & x_3y_1z_5u_4 - x_4y_2z_5u_4 - x_3y_2z_6u_4 - x_4y_1z_6u_4 + \\ & +x_3y_1z_6u_3 - x_4y_2z_6u_3 + x_3y_2z_5u_3 + x_4y_1z_5u_3, \\ r_5 = & x_5y_3z_1u_5 - x_6y_4z_1u_5 - x_5y_4z_2u_5 - x_6y_3z_2u_5 - \\ & -x_5y_3z_2u_6 + x_6y_4z_2u_6 - x_5y_4z_1u_6 - x_6y_3z_1u_6, \\ r_6 = & x_5y_3z_1u_6 - x_6y_4z_1u_6 - x_5y_4z_2u_6 - x_6y_3z_2u_6 + \\ & +x_5y_3z_2u_5 - x_6y_4z_2u_5 + x_5y_4z_1u_5 + x_6y_3z_1u_5, \end{cases}$$

is associative.

We remark that, corresponding to the definition of the operation $[\ \ ]_{l,\sigma,m,mk}$, the components $r_1, \ldots, r_6$ are implicitly defined by the relations

$$\begin{cases} (r_1, r_2) = (x_1, x_2) \times (y_5, y_6) \times (z_3, z_4) \times (u_1, u_2), \\ (r_3, r_4) = (x_3, x_4) \times (y_1, y_2) \times (z_5, z_6) \times (u_3, u_4), \\ (r_5, r_6) = (x_5, x_6) \times (y_3, y_4) \times (z_1, z_2) \times (u_5, u_6). \end{cases}$$

Lemma 5.1 and Theorem 3.7 allow us to state the following

Theorem 5.5. [13]. *If the semigroup $\langle A, \times\rangle$ from the definition of the operation $[\ \ ]_{l,\sigma,m,mk}$ contains the unity, and if the permutation σ satisfies the condition $\sigma^l \neq \sigma$, then the l-ary operation $[\ \ ]_{l,\sigma,m,mk}$ is not semi-associative and, in particular, it is non-associative.*

If in Theorem 5.5 one replaces $m = 2$, then it follows

Corollary 5.6. *If the semigroup $\langle A, \times\rangle$ from the definition of the operation $[\ \ ]_{l,\sigma,m,mk}$ contains the unity, and if the permutation σ satisfies the condition $\sigma^l \neq \sigma$, then the l-ary operation $[\ \ ]_{l,\sigma,2,2k}$ is not semi-associative and, in particular, it is non-associative.*

We shall provide now examples of multiple non-associative operations of the form $[\ \ ]_{l,\sigma,m,mk}$.

Example 5.7. Let $\langle A, \times\rangle$ be the semigroup of complex (or dual/double) numbers, let $m = 2$ and $k = 3$. If $l = 3$, then according to Corollary 5.6, the ternary operations $[\ \ ]_{3,(123),2,6}$ and $[\ \ ]_{3,(132),2,6}$ are not associative. But if $\ell = 4$, then, due to the same Corollary, the 4-ary operations $[\ \ ]_{4,(12),2,6}$, $[\ \ ]_{4,(13),2,6}$ and $[\ \ ]_{4,(23),2,6}$ are non-associative. All the five provided examples are defined on the Cartesian power $\mathbb{R}^6$.

Consider, as before, a set B, let $m \geqslant 1$, $n \geqslant 3$, and let $A = B^m$ be the m-ary Cartesian power of the set B. Moreover, let $\langle A, +\rangle$ be a groupoid. We shall define on B^{mk} a binary operation $\widetilde{+}$, as follows. If

$$\boldsymbol{\alpha} = (\alpha_{11}, \dots, \alpha_{1m}, \dots, \alpha_{k1}, \dots, \alpha_{km}),$$
$$\boldsymbol{\beta} = (\beta_{11}, \dots, \beta_{1m}, \dots, \beta_{k1}, \dots, \beta_{km}) \in B^{mk},$$

then

$$\boldsymbol{\alpha} \widetilde{+} \boldsymbol{\beta} = (u_{11}, \dots, u_{1m}, \dots, u_{k1}, \dots, u_{km}) \in B^{mk},$$

where, for any $j \in \{1, \dots, k\}$,

$$(u_{j1}, \dots, u_{jm}) = (\alpha_{j1}, \dots, \alpha_{jm}) + (\beta_{j1}, \dots, \beta_{jm}) \in B^{mk}.$$

Remark. If on the set B we define an operation "$+$", this defines on the set $A = B^m$ in componentwise manner, a corresponding operation "$+$", then:

$$\boldsymbol{\alpha} \widetilde{+} \boldsymbol{\beta} = (\alpha_{11} + \beta_{11}, \dots, \alpha_{1m} + \beta_{1m}, \dots, \alpha_{k1} + \beta_{k1}, \dots, \alpha_{km} + \beta_{km}),$$

i.e., in this case, the operation "$\widetilde{+}$" coincides with the operation "$+$", componentwise defined on B^{mk}.

For the same assumptions on m, n, B and A we define a multiplication of the elements of the field P to the elements from $A = B^m$:

$$\lambda \mathbf{a} = \lambda(a_1, \dots, a_m) = (u_1, \dots, u_m).$$

We define the product "$\circ$" between elements $\lambda \in P$ with elements from B^{mk}, as follows. If

$$\boldsymbol{\alpha} = (\alpha_{11}, \dots, \alpha_{1m}, \dots, \alpha_{k1}, \dots, \alpha_{km}) \in B^{mk},$$

then

$$\lambda \circ \boldsymbol{\alpha} = (u_{11}, \dots, u_{1m}, \dots, u_{k1}, \dots, u_{km}),$$

where, for any $j \in \{1, \dots, k\}$,

$$(u_{j1}, \dots, u_{jm}) = \lambda(\alpha_{j1}, \dots, \alpha_{jm}).$$

Remark. If $\lambda \mathbf{a} = \lambda(a_1, \dots, a_m) = (\lambda a_1, \dots, \lambda a_m)$, then

$$\lambda \circ \boldsymbol{\alpha} = (\lambda\alpha_{11}, \dots, \lambda\alpha_{1m}, \dots, \lambda\alpha_{k1}, \dots, \lambda\alpha_{km}).$$

If $\langle A = B^m, +, \times \rangle$ is an algebra, then for any

$$\boldsymbol{\alpha} = (\alpha_{11}, \dots, \alpha_{1m}, \alpha_{21}, \dots, \alpha_{2m}, \dots, \alpha_{k1}, \dots, \alpha_{km}) \in B^{mk}$$

we put

$$\underline{\boldsymbol{\alpha}} = (\alpha_{11}, \dots, \alpha_{1m}, \beta_{21}, \dots, \beta_{2m}, \dots, \beta_{k1}, \dots, \beta_{km}) \in B^{mk},$$

where $(\beta_{i1}, \dots, \beta_{im}) = -(\alpha_{11}, \dots, \alpha_{im}),\ i = 2, \dots, k$. As consequence of the relations

$$\begin{cases} \boldsymbol{\alpha}^{\boldsymbol{\psi}} = ((\alpha_{11}, \dots, \alpha_{1m}), (\alpha_{21}, \dots, \alpha_{2m}), \dots, (\alpha_{k1}, \dots, \alpha_{km})), \\ \overline{\boldsymbol{\alpha}^{\boldsymbol{\psi}}} = ((\alpha_{11}, \dots, \alpha_{1m}), -(\alpha_{21}, \dots, \alpha_{2m}), \dots, -(\alpha_{k1}, \dots, \alpha_{km})) = \\ \quad = ((\alpha_{11}, \dots, \alpha_{1m}), (\beta_{21}, \dots, \beta_{2m}), \dots, (\beta_{k1}, \dots, \beta_{km})), \\ (\overline{\boldsymbol{\alpha}^{\boldsymbol{\psi}}})^{\boldsymbol{\psi}^{-1}} = (\alpha_{11}, \dots, \alpha_{1m}, \beta_{21}, \dots, \beta_{2m}, \dots, \beta_{k1}, \dots, \alpha_{km}), \end{cases}$$

we infer the following results

Lemma 5.8. *For any $\boldsymbol{\alpha} \in B^{mk}$, it holds the equality $\underline{\boldsymbol{\alpha}} = (\overline{\boldsymbol{\alpha}^{\psi}})^{\psi^{-1}}$.*

Theorem 5.9. [14, 13]. *Let B be a set, and $m \geqslant 1$, $k \geqslant 2$, $\ell \geqslant 3$, such that k divides $l-1$. Let σ be a cycle of length k from S_k, $\langle A = B^m, +, \times\rangle$ an associative algebra over the field P and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)$ is its zero element. Then:*

1) $\langle B^{mk}, \widetilde{+}, [\ \ ]_{l,\sigma,m,mk}\rangle$ is an associative $(2,l)$-algebra over the field P, which is isomorphic to the $(2,l)$-algebra $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$;

2) in $\langle B^{mk}, \widetilde{+}, [\ \ ]_{l,\sigma,m,mk}\rangle$, all its elements are divisors of its zero element,

$$(\underbrace{\theta_1, \ldots, \theta_m, \ldots, \theta_1, \ldots, \theta_m}_{k});$$

3) if $\langle A, +, \times\rangle$ is commutative, then $\langle B^{mk}, \widetilde{+}, [\ \ ]_{l,\sigma,m,mk}\rangle$ is semi-abelian;

4) if $\langle A^* = A\backslash\{\theta\}, \times\rangle$ is a group, then $\langle \widetilde{B}, [\ \ ]_{l,\sigma,m,mk}\rangle$ is an l-ary group, where $\widetilde{B}$ is the set of elements

$$(b_{11}, \ldots, b_{1m}, \ldots, b_{k1}, \ldots, b_{km}) \in B^{mk}$$

such that $(b_{j1}, \ldots, b_{jm}) \neq (\theta_1, \ldots, \theta_m)$ for any $j = 1, \ldots, k$;

5) for any elements

$$\mathbf{b} = (b_{11}, \ldots, b_{1m}, \ldots, b_{k1}, \ldots, b_{km}) \in B^{mk}$$

such that $(b_{j1}, \ldots, b_{jm}) = (\theta_1, \ldots, \theta_m)$ for some $j \in \{1, \ldots, k\}$, we have

$$[\underbrace{\mathbf{b} \ldots \mathbf{b}}_{l}]_{l,\sigma,m,mk} = (\underbrace{\theta_1, \ldots, \theta_m, \ldots, \theta_1, \ldots, \theta_m}_{k});$$

6) if the set B contains more than one element and $\langle A, +, \times\rangle$ has a unity, then $\langle B^{mk}, \widetilde{+}, [\ \ ]_{l,\sigma,m,mk}\rangle$ is non-abelian;

7) if the set B contains more than one element, then $\langle B^{mk}, \widetilde{+}, [\ \ ]_{l,\sigma,m,mk}\rangle$ contains no unity;

8) if $\langle A^*, \times\rangle$ is a group and $\mathbf{e}$ is its unity, then

$$I(B^{mk}, \widetilde{+}, [\ \ ]_{l,\sigma,m,mk}) = J \cup \{(\underbrace{\theta_1, \ldots, \theta_m, \ldots, \theta_1, \ldots, \theta_m}_{k})\},$$

where J is the set of all the elements

$$(\varepsilon_{11}, \ldots, \varepsilon_{1m}, \ldots, \varepsilon_{(k-1)1}, \ldots, \varepsilon_{(k-1)m}, \varepsilon_{k1}, \ldots, \varepsilon_{km}) \in B^{mk},$$

such that

$$(\boldsymbol{\varepsilon_1} \boldsymbol{\varepsilon_{\sigma(1)}} \ldots \boldsymbol{\varepsilon_{\sigma^{l-2}(1)}})^{\frac{l-1}{k}} = \mathbf{e},$$

where

$$\boldsymbol{\varepsilon_1} = (\varepsilon_{11}, \ldots, \varepsilon_{1m}), \ldots, \boldsymbol{\varepsilon_k} = (\varepsilon_{k1}, \ldots, \varepsilon_{km}) \in A.$$

9) we have the following:

$$\underline{[\boldsymbol{\alpha_1}\boldsymbol{\alpha_2}\ldots\boldsymbol{\alpha_l}]}_{l,\sigma,m,mk} = \begin{cases} [\underline{\boldsymbol{\alpha_1}}\ \underline{\boldsymbol{\alpha_2}} \ldots \underline{\boldsymbol{\alpha_l}}]_{l,\sigma,m,mk}, & \text{for even } (l-1)(k-1)/k \\ -[\underline{\boldsymbol{\alpha_1}}\ \underline{\boldsymbol{\alpha_2}} \ldots \underline{\boldsymbol{\alpha_l}}]_{l,\sigma,m,mk}, & \text{for odd } (l-1)(k-1)/k. \end{cases}$$

Proof. 1) We note that due to Proposition 3.8, $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ is an associative $(2,l)$-algebra over P. It is clear that the mapping φ, which relates the element

$$((\alpha_1^{(1)},\dots,\alpha_m^{(1)}),(\alpha_1^{(2)},\dots,\alpha_m^{(2)}),\dots,(\alpha_1^{(k)},\dots,\alpha_m^{(k)}))\in A^k$$

to the element

$$(\alpha_1^{(1)},\dots,\alpha_m^{(1)},\alpha_1^{(2)},\dots,\alpha_m^{(2)},\dots,\alpha_1^{(k)},\dots,\alpha_m^{(k)})\in B^{mk}$$

is a bijection between A^k and B^{mk}. Since $\varphi=\psi^{-1}$, where ψ is the mapping from Lemma 5.1, we infer that φ is an the isomorphism from $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ to $\langle B^{mk}, [\ \ ]_{l,\sigma,m,mk}\rangle$. Consequently,

$$[\boldsymbol{\alpha_1}\boldsymbol{\alpha_2}\dots\boldsymbol{\alpha_l}]^{\varphi}_{l,\sigma,k}=[\boldsymbol{\alpha_1^{\varphi}}\boldsymbol{\alpha_2^{\varphi}}\dots\boldsymbol{\alpha_l^{\varphi}}]_{l,\sigma,m,mk}. \tag{5.3}$$

Let

$$\begin{aligned}\boldsymbol{\alpha}&=((\alpha_{11},\dots,\alpha_{1m}),\dots,(\alpha_{k1},\dots,\alpha_{km})),\\ \boldsymbol{\beta}&=((\beta_{11},\dots,\beta_{1m}),\dots,(\beta_{k1},\dots,\beta_{km}))\end{aligned}$$

be arbitrary elements from A^k. Then

$$\begin{aligned}(\boldsymbol{\alpha}+\boldsymbol{\beta})^{\boldsymbol{\varphi}} &= (((\alpha_{11},\dots,\alpha_{1m}),\dots,(\alpha_{k1},\dots,\alpha_{km}))+\\ &\quad +((\beta_{11},\dots,\beta_{1m}),\dots,(\beta_{k1},\dots,\beta_{km})))^{\varphi}=\\ &= ((\alpha_{11},\dots,\alpha_{1m})+(\beta_{11},\dots,\beta_{1m}),\dots,(\alpha_{k1},\dots,\alpha_{km})+\\ &\quad +(\beta_{k1},\dots,\beta_{km}))^{\varphi}=\\ &= ((v_{11},\dots,v_{1m}),\dots,(v_{k1},\dots,v_{km}))^{\varphi}=\\ &= (v_{11},\dots,v_{1m},\dots,v_{k1},\dots,v_{km}),\end{aligned}$$

where for any $j=1,\dots,k$ we put

$$(v_{j1},\dots,v_{jm})=(\alpha_{j1},\dots,\alpha_{jm})+(\beta_{j1},\dots,\beta_{jm}). \tag{5.4}$$

Moreover,

$$\begin{aligned}\boldsymbol{\alpha^{\varphi}}\widetilde{+}\boldsymbol{\beta^{\varphi}} &= ((\alpha_{11},\dots,\alpha_{1m}),\dots,(\alpha_{k1},\dots,\alpha_{km}))^{\varphi}\widetilde{+}\\ &\quad \widetilde{+}((\beta_{11},\dots,\beta_{1m}),\dots,(\beta_{k1},\dots,\beta_{km}))^{\varphi}=\\ &= (\alpha_{11},\dots,\alpha_{1m},\dots,\alpha_{k1},\dots,\alpha_{km})\widetilde{+}\\ &\quad \widetilde{+}(\beta_{11},\dots,\beta_{1m},\dots,\beta_{k1},\dots,\beta_{km})=\\ &= (u_{11},\dots,u_{1m},\dots,u_{k1},\dots,u_{km}),\end{aligned}$$

where, according to the definition of the operation $\widetilde{+}$, for any $j=\{1,\dots,k\}$ we have

$$(u_{j1},\dots,u_{jm})=(\alpha_{j1},\dots,\alpha_{jm})+(\beta_{j1},\dots,\beta_{jm}). \tag{5.5}$$

Since the right sides of (5.4) and (5.5) are equal, it follows that

$$(\boldsymbol{\alpha}+\boldsymbol{\beta})^{\boldsymbol{\varphi}}=\boldsymbol{\alpha^{\varphi}}\widetilde{+}\boldsymbol{\beta^{\varphi}}. \tag{5.6}$$

Let $\boldsymbol{\alpha} = ((\alpha_{11}, \ldots, \alpha_{1m}), \ldots, (\alpha_{k1}, \ldots, \alpha_{km}))$ be an arbitrary element from A^k. Then

$$\begin{aligned}(\lambda\boldsymbol{\alpha})^{\varphi} &= (\lambda((\alpha_{11}, \ldots, \alpha_{1m}), \ldots, (\alpha_{k1}, \ldots, \alpha_{km})))^{\varphi} = \\ &= (\lambda(\alpha_{11}, \ldots, \alpha_{1m}), \ldots, \lambda(\alpha_{k1}, \ldots, \alpha_{km}))^{\varphi} = \\ &= ((v_{11}, \ldots, v_{1m}), \ldots, (v_{k1}, \ldots, v_{km}))^{\varphi} = \\ &= (v_{11}, \ldots, v_{1m}, \ldots, v_{k1}, \ldots, v_{km}),\end{aligned}$$

where we put

$$(v_{j1}, \ldots, v_{jm}) = \lambda(\alpha_{j1}, \ldots, \alpha_{jm}). \tag{5.7}$$

Moreover, we have

$$\begin{aligned}\lambda \circ \boldsymbol{\alpha}^{\boldsymbol{\varphi}} &= \lambda \circ ((\alpha_{11}, \ldots, \alpha_{1m}), \ldots, (\alpha_{k1}, \ldots, \alpha_{km}))^{\varphi} = \\ &= \lambda \circ (\alpha_{11}, \ldots, \alpha_{1m}, \ldots, \alpha_{k1}, \ldots, \alpha_{km}) = \\ &= (u_{11}, \ldots, u_{1m}, \ldots, u_{k1}, \ldots, u_{km}),\end{aligned}$$

where, according to the definition of the product "$\circ$", for any $j = \{1, \ldots, k\}$,

$$(u_{j1}, \ldots, u_{jm}) = \lambda(\alpha_{j1}, \ldots, \alpha_{jm}). \tag{5.8}$$

Since the right sides of (5.7) and (5.8) are equal, then

$$(\lambda\boldsymbol{\alpha})^{\varphi} = \lambda \circ \boldsymbol{\alpha}^{\boldsymbol{\varphi}} \tag{5.9}$$

From (5.3), (5.6) and (5.9) it follows that φ is an isomorphism from $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ to $\langle B^{mk}, \widetilde{+}, [\ \ ]_{l,\sigma,m,mk}\rangle$. But since $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$ is an associative $(2,l)$-algebra over P, then $\langle B^{mk}, \widetilde{+}, [\ \ ]_{l,\sigma,m,mk}\rangle$ is an associative $(2,l)$-algebra over P.

2) According to item 1) in Theorem 3.20, in the $(2,l)$-algebra $\langle A^k, +, [\ \ ]_{l,\sigma,k}\rangle$, all the elements are divisors of its zero element

$$(\underbrace{\boldsymbol{\theta} \ldots \boldsymbol{\theta}}_{k}) = ((\theta_1, \ldots, \theta_m), \ldots, (\theta_1, \ldots, \theta_m)).$$

Further, we apply the isomorphism φ defined in 1). For proving the items 3), 5), 6), 7) and 8), we respectively use the items 2), 4), 5), 6) and 7) of Theorem 3.20 and apply the isomorphism φ. For item 4), we use item 3) from Theorem 3.20 and the equality $((A\backslash\{(\theta_1, \ldots, \theta_m)\}^k)^{\varphi} = \widetilde{B}$. For 9), we use Lemma 5.8, and we get $\underline{\boldsymbol{\alpha}} = (\overline{\boldsymbol{\alpha}^{\boldsymbol{\psi}}})^{\varphi}$ for any $\boldsymbol{\alpha} \in B^{mk}$, where ψ is the isomorphism from Lemma 5.1, and $\varphi = \psi^{-1}$ is the isomorphism from item 1). Then, according to item 8) of Theorem 3.20, for even $(l-1)(k-1)/k$ we have

$$\begin{aligned}&\underline{[\boldsymbol{\alpha_1}\boldsymbol{\alpha_2} \ldots \boldsymbol{\alpha_l}]_{l,\sigma,m,mk}} = (\overline{[\boldsymbol{\alpha_1}\boldsymbol{\alpha_2} \ldots \boldsymbol{\alpha_l}]^{\psi}_{l,\sigma,m,mk}})^{\varphi} = \\ &= (\overline{[\boldsymbol{\alpha_1^{\psi}}\boldsymbol{\alpha_2^{\psi}} \ldots \boldsymbol{\alpha_l^{\psi}}]_{l,\sigma,mk}})^{\varphi} = ([\overline{\boldsymbol{\alpha_1^{\psi}}}\ \overline{\boldsymbol{\alpha_2^{\psi}}} \ldots \overline{\boldsymbol{\alpha_l^{\psi}}}]_{l,\sigma,mk})^{\varphi} = \\ &= [(\overline{\boldsymbol{\alpha_1^{\psi}}})^{\varphi}\ (\overline{\boldsymbol{\alpha_2^{\psi}}})^{\varphi} \ldots (\overline{\boldsymbol{\alpha_l^{\psi}}})^{\varphi}]_{l,\sigma,m,mk} = [\underline{\boldsymbol{\alpha_1}}\ \underline{\boldsymbol{\alpha_2}} \ldots \underline{\boldsymbol{\alpha_l}}]_{l,\sigma,m,mk}.\end{aligned}$$

For odd $(l-1)(k-1)/k$, we apply again item 8) of Theorem 3.20, and we get

$$[\underline{\boldsymbol{\alpha_1}\boldsymbol{\alpha_2}\ldots\boldsymbol{\alpha_l}}]_{l,\sigma,m,mk} = (\overline{[\boldsymbol{\alpha_1}\boldsymbol{\alpha_2}\ldots\boldsymbol{\alpha_l}]^{\psi}_{l,\sigma,m,mk}})^{\varphi} = (\overline{[\boldsymbol{\alpha_1^{\psi}}\boldsymbol{\alpha_2^{\psi}}\ldots\boldsymbol{\alpha_l^{\psi}}]}_{l,\sigma,mk})^{\varphi} =$$
$$= (-[\overline{\boldsymbol{\alpha_1^{\psi}}}\ \overline{\boldsymbol{\alpha_2^{\psi}}}\ldots\overline{\boldsymbol{\alpha_l^{\psi}}}]_{l,\sigma,mk})^{\varphi} = -([\overline{\boldsymbol{\alpha_1^{\psi}}}\ \overline{\boldsymbol{\alpha_2^{\psi}}}\ldots\overline{\boldsymbol{\alpha_l^{\psi}}}]_{l,\sigma,m,mk})^{\varphi} =$$
$$= -[(\overline{\boldsymbol{\alpha_1^{\psi}}})^{\varphi}\ (\overline{\boldsymbol{\alpha_2^{\psi}}})^{\varphi}\ldots(\overline{\boldsymbol{\alpha_l^{\psi}}})^{\varphi}]_{l,\sigma,m,mk} = -[\underline{\boldsymbol{\alpha_1}}\ \underline{\boldsymbol{\alpha_2}}\ldots\underline{\boldsymbol{\alpha_l}}]_{l,\sigma,m,mk}.$$

Hence the Theorem is proved. □

If in Theorem 5.9 we put $m=2$, $l=4$, $k=3$, $\sigma=(132)$, $B=\mathbb{R}$ and $\langle A=\mathbb{R}^2,+,\times\rangle$ is the algebra of complex numbers, then we get

Corollary 5.10. *The following assertions hold true:*

1) $\langle\mathbb{R}^6,+,[\ \ ]_{4,(132),2,6}\rangle$ *is associative, non-abelian, semi-abelian (2, 4)-algebra over* $\mathbb{R}$*, in which all the elements are divisors of its zero* $(0,\ 0,\ 0,\ 0,\ 0,\ 0)$*, and which has no unity;*

2) $\langle\widetilde{\mathbb{R}},[\ \ ]_{4,(132),2,6}\rangle$ *is a 4-ary group, where*

$$\begin{aligned}\widetilde{\mathbb{R}} = \mathbb{R}^6 \setminus (\ & \{(0,\ 0,\ a,\ b,\ c,\ d) \mid a,\ b,\ c,\ d\in\mathbb{R}\} \\ & \bigcup\{(a,\ b,\ 0,\ 0,\ c,\ d) \mid a,\ b,\ c,\ d\in\mathbb{R}\} \\ & \bigcup\{(a,\ b,\ c,\ d,\ 0,\ 0) \mid a,\ b,\ c,\ d\in\mathbb{R}\}\ );\end{aligned}$$

3) the set of all the multiplicative idempotents of $\langle\mathbb{R}^6,+,[\ \ ]_{4,(132),2,6}\rangle$ *has the form*

$$I(\mathbb{R}^6,+,[\ \ ]_{4,2,6}) = \left\{(a,\ b,\ c,\ d,\ \frac{ac-bd}{(a^2+b^2)(c^2+d^2)},\ \frac{-ad-bc}{(a^2+b^2)(c^2+d^2)})\ \middle|\ a,\ b,\ c,\ d\in\mathbb{R},\ a^2+b^2\neq 0,\ \ c^2+d^2\neq 0\right\} \bigcup\ \{(0,\ 0,\ 0,\ 0,\ 0,0)\}.$$

4) for any $\boldsymbol{\alpha_1},\ldots,\boldsymbol{\alpha_6}\in\mathbb{R}^6$ *we have*

$$[\underline{\boldsymbol{\alpha_1}\boldsymbol{\alpha_2}\ldots\boldsymbol{\alpha_6}}]_{4,(132),2,6} = [\underline{\boldsymbol{\alpha_1}}\ \underline{\boldsymbol{\alpha_2}}\ldots\underline{\boldsymbol{\alpha_6}}]_{4,(132),2,6}.$$

Remark. The assertion 3) from above emerges from Corollary 3.19, which states that for the abelian group A and arbitrary permutations σ and $\tau\in S_k$ which satisfy the conditions $\sigma^l=\sigma$, $\tau^l=\tau$, we have $I(A^k,[\ \ ]_{l,\sigma,k}) = I(A^k,[\ \ ]_{l,\tau,k})$.

6 The corresponding group of the ℓ-ary group $\langle A^k,[\ \ ]_{l,\sigma,k}\rangle$

If A is a group, and the condition $\sigma^l=\sigma$ holds true, then according to Proposition 3.11, $\langle A^k,[\ \ ]_{l,\sigma,k}\rangle$ is an ℓ-ary group. But since according to Post [22], any l-ary group has a corresponding group, then appears the question of finding the corresponding Post group $(A^k)_0$ of the ℓ-ary group $\langle A^k,[\ \ ]_{l,\sigma,k}\rangle$.

Proposition 6.1. *If* A *is a group,* $l\geqslant 3$*,* $k\geqslant 2$*,* σ *a permutation from* S_k *which satisfies the condition* $\sigma^l=\sigma$*, then the corresponding Post group* $(A^k)_0$ *of the l-ary group* $\langle A^k,[\ \ ]_{l,\sigma,k}\rangle$ *is isomorphic to the direct product* A^k *of* k *copies of the group* A*,* $(A^k)_0\simeq A^k$*.*

Proof. We put $\boldsymbol{e} = (\underbrace{1,\ldots,1}_{k})$, where 1 is the unity of the group A. According to Proposition 1.6.1 from [15], the group $(A^k)_0$ is isomorphic to the group $\langle A^k, ⓔ\rangle$ whose operation is defined as

$$\mathbf{x}\,ⓔ\,\mathbf{y} = [\mathbf{x}\ \underbrace{\boldsymbol{e}\ldots\boldsymbol{e}}_{l-2}\ \mathbf{y}]_{l,\sigma,k}.$$

Since $\sigma^{l-1}(j) = j$ for any $j \in \{1,2,\ldots,k\}$, then putting $\mathbf{x} = (x_{11},\ldots,x_{1k})$, $\mathbf{y} = (x_{l1},\ldots,x_{lk})$, we get

$$\begin{aligned}
\mathbf{x}\,ⓔ\,\mathbf{y} &= [\mathbf{x}\ \underbrace{\boldsymbol{e}\ldots\boldsymbol{e}}_{l-2}\ \mathbf{y}]_{l,\sigma,k} = [(x_{11},\ldots,x_{1k})(x_{21}=1,\ldots,x_{2k}=1)\ldots \\
&\qquad \ldots(x_{(l-1)1}=1,\ldots,x_{(l-1)k}=1)(x_{l1},\ldots,x_{lk})]_{l,\sigma,k} = \\
&= (x_{11}x_{2\sigma(1)}\ldots x_{(l-1)\sigma^{l-2}(1)}x_{l\sigma^{l-1}(1)},\ldots \\
&\qquad \ldots, x_{1k}x_{2\sigma(k)}\ldots x_{(l-1)\sigma^{l-2}(k)}x_{l\sigma^{l-1}(k)}) = \\
&= (x_{11}\underbrace{1\ldots1}_{l-2}x_{l1},\ldots,x_{1k}\underbrace{1\ldots1}_{l-2}x_{lk}) = (x_{11}x_{l1},\ldots,x_{1k}x_{lk}) = \\
&= (x_{11},\ldots,x_{1k})(x_{l1},\ldots,x_{lk}) = \mathbf{xy}.
\end{aligned}$$

Hence, $\mathbf{x}\,ⓔ\,\mathbf{y} = \mathbf{xy}$. Then the operation ⓔ coincides with the operation of the direct product A^k of k copies of the group A, and the Proposition is proved. □

Corollary 6.2. *If A is a group, $l \geqslant 3$ and $k \geqslant 2$, then for any permutations σ, $\tau \in S_k$ which satisfy $\sigma^l = \sigma$, $\tau^l = \tau$, the corresponding Post groups of the l-ary groups $\langle A^k, [\ ]_{l,\sigma,k}\rangle$ and $\langle A^k, [\ ]_{l,\tau,k}\rangle$ are isomorphic.*

Proposition 6.1 is of notable importance, since using the corresponding results form the theory of polyadic groups, one can obtain new information about the ℓ-ary group $\langle A^k, [\ ]_{l,\sigma,k}\rangle$.

As an example, we prove the following

Proposition 6.3. *If A is a group, $l \geqslant 3$, $k \geqslant 2$ and σ is a permutation from S_k which satisfies the condition $\sigma^l = \sigma$, then the l-ary group $\langle A^k, [\ ]_{l,\sigma,k}\rangle$ is not semicyclic.*

Proof. A polyadic group is called *semicyclic* [15], if its corresponding Post group is cyclic. Since the direct product A^k is not a cyclic group, then according to Proposition 6.1, the corresponding Post group $(A^k)_0$ of the ℓ-ary group $\langle A^k, [\ ]_{l,\sigma,k}\rangle$ is not cyclic as well. Hence the ℓ-ary group $\langle A^k, [\ ]_{l,\sigma,k}\rangle$ is not cyclic, and the Proposition is proved. □

We note that the non-cyclicity of the l-ary group $\langle A^k, [\ ]_{l,\sigma,k}\rangle$ follows from Proposition 3.9, according to which $\langle A^k, [\ ]_{l,\sigma,k}\rangle$ is non-abelian.

Finally, from Propositions 3.10 and 6.3, we get the following

Corollary 6.4. *If A is an abelian group, $l \geqslant 3$, $k \geqslant 2$, σ a permutation from S_k which satisfies the condition $\sigma^l = \sigma$, then the l-ary group $\langle A^k, [\ ]_{l,\sigma,k}\rangle$ is semi-abelian, but is not semi-cyclic.*

Since any semi-cyclic ℓ-ary group is semi-abelian, then from Corollary 6.4 it follows that for any $\ell \geqslant 3$, the class of all semi-abelian ℓ-ary groups is larger than the class of all poly-cyclic ℓ-ary groups. Proposition 6.1 can be used not only for obtaining new results, but also for simplifying the proofs of already known results. As an example, according to the Post criterion, the semi-commutativity of polyadic groups is equivalent to the commutativity of the corresponding Post group. Therefore, if A is an abelian group and $\sigma^l = \sigma$, then from the commutativity of the direct product A^k, according to Proposition 6.1, it follows the semi-commutativity of the ℓ-ary group $\langle A^k, [\ \ ]_{l,\sigma,k}\rangle$.

7 Particular cases. Applications

The described above n-ary operations are tightly related to multilinear forms (covariant tensors) defined on Cartesian powers of the field of real numbers. We shall further present illustrative examples which relate the prior developed theory - by means of multilinear forms, to the Berwald-Moor, Chernov and Bogoslovski geometric structures used in Relativity Theory.

If G is a multiplicative group we define on G^m the induced $n-$ary operation $\mu_{n,m} = [\ \cdot\ , \dots ,\ \cdot\ ]_{n,m} : (G^m)^n \to G^m$, given by

$$(7.1)\qquad \mu_{n,m}(x_1, \dots x_n) \stackrel{\text{not}}{=} [x_1, \dots x_n]_{n,m} \stackrel{\text{def}}{=} (p_1, \dots p_m),$$

for all $x_k = (x_{k1}, \dots , x_{km}) \in G^m$, $k \in \overline{1, n}$, where

$$p_k = \prod_{j=1}^{n} x_{j\tau(j,k)},\ \ \tau(j, k) = mod_m(j + k - 2) + 1,\quad k \in \overline{1, m}.$$

Consider now for the multiplicative group G, the abelian multiplicative group of positive reals $(\mathbb{R}^*_+ = (0, \infty),\ \ \cdot\ )$, and the mapping $\theta : G^m \to G$,

$$(7.2)\qquad \theta(p) = p_1 + \ldots + p_m, \forall p = (p_1, \dots , p_m) \in G.$$

We note that both the mappings $\mu_{n,m}$ and θ are both additive and positive-homogeneous relative to the vectors of G^m. Hence the composition $\theta \circ \mu_{n,m} : (G^m)^n \to G$ is positive $n-$multilinear and defines by extension to $V = \mathbb{R}^m \supset G^m$ a tensor $\mathcal{A} \in \mathcal{T}^0_n(V) = \otimes^n V^*$ whose coefficients are

$$(7.3)\qquad \mathcal{A}_{i_1\dots i_n} = \begin{cases} 1, & \text{if } \exists j \in \overline{1, m}, \text{ s.t. } i_k = \sigma^j(mod_m(k - 1) + 1),\ \forall k = \overline{1, n}, \\ 0, & \text{the rest,} \end{cases}$$

where σ is the cycle $(1 \dots m) \in \sigma_m$ (the roll-left operator).

We shall further provide a series of notable particular cases, which provide the structures for alternative models of Relativity Theory.

Applications.

1. The Bogoslovsky case. The particular case $\mu_{n,n-1}$ provides the rank-n reduced Bogoslovsky tensor $\mathcal{A}_{rB} = \theta \circ \mu_{n,n-1}$ on $\mathbb{R}^{n-1}$, whose nontrivial coefficients are

$$(\mathcal{A}_{rB})_{i_1 \dots i_n} = 1, \text{ for } (i_1 \dots i_{n-1}) \in \{\sigma^j(1 \dots n-1) \mid j = \overline{0, n-2}\} \text{ and } i_n = i_1,$$

where we denoted by σ the cycle $(1 \dots n-1) \in \mathcal{O}_{n-1}$. This tensor has $n-1$ nonzero components, and leads by symmetrization to the full Bogoslovsky tensor, of $C_n^2 \cdot (n-1)!$ nontrivial coefficients

$$(\mathcal{A}_B)_{i_1 \dots i_n} = \frac{1}{C_n^2 \cdot (n-2)!}, \text{ for } \{i_1, \dots, i_n\} = \{1, \dots, n-1\}.$$

Both of them provide the $m-$root Finsler norm

$$\begin{aligned} F_B(y) &= \mathcal{A}_B(y, \dots, y) = \mathcal{A}_{rB}(y, \dots, y) = \\ &= \sqrt[n]{y^1 \cdot \dots \cdot y^{n-1} \sum_{k=1}^{n-1} y^k}, \quad \forall y = (y^1, \dots, y^{n-1}) \in (\mathbb{R}_+^*)^{n-1}. \end{aligned}$$

2. The Berwald-Moor case. The particular case $\mu_{n,n}$ provides the rank-n reduced Berwald-Moor tensor $\mathcal{A}_{rBM} = \theta \circ \mu_{n,n}$ on $\mathbb{R}^n$, whose nontrivial coefficients are

$$(\mathcal{A}_{rBM})_{i_1 \dots i_n} = 1, \text{ if } \exists j \in \overline{1, n}, \ \sigma^j(1 \dots n) = (i_1 \dots i_n),$$

where σ is the cycle $(1 \dots n) \in \mathcal{O}_n$, and which has n nonzero components. Its symmetrization leads to the full Berwald-Moor tensor of $n!$ nontrivial coefficients

$$(\mathcal{A}_{BM})_{i_1 \dots i_n} = \frac{1}{n!}, \text{ for } \{i_1, \dots, i_n\} = \{1, \dots, n\}.$$

Both of them produce the $m-$root Finsler norm

$$\begin{aligned} F_{BM}(y) &= \mathcal{A}_{BM}(y, \dots, y) = \mathcal{A}_{rBM}(y, \dots, y) = \\ &= \sqrt[n]{y^1 \cdot \dots \cdot y^n}, \quad \forall y = (y^1, \dots, y^{n-1}) \in (\mathbb{R}_+^*)^{n-1}. \end{aligned}$$

3. The Chernov case. The particular case $\mu_{n-1,n}$ provides the rank- $n-1$ reduced Chernov tensor $\mathcal{A}_{rC} = \theta \circ \mu_{n-1,n}$ on $\mathbb{R}^n$ of n nontrivial coefficients

$$(\mathcal{A}_{rC})_{i_1 \dots i_n} = 1, \text{ if } i_k = mod_n(i_k) + 1, \ \forall k \in \overline{1, n-1}.$$

Its symmetrization leads to the full Chernov tensor of $n!$ nontrivial coefficients

$$(\mathcal{A}_C)_{i_1 \dots i_n} = \frac{1}{(n-1)!}, \text{ for card} \{i_1, \dots, i_{n-1}\} = n-1, \ \ i_1, \dots, i_{n-1} \in \overline{1, n}.$$

Both of them provide the $m-$root Finsler norm[3]

$$\begin{aligned} F_C(y) &= \mathcal{A}_C(y, \dots, y) = \mathcal{A}_{rC}(y, \dots, y) = \\ &= \sqrt[n-1]{\sum_{k=1}^{n} y^1 \cdot \dots \hat{y}^k \cdot \dots \cdot y^n}, \quad \forall y = (y^1, \dots, y^n) \in (\mathbb{R}_+^*)^n. \end{aligned}$$

[3] The hat denotes absence of the corresponding factor.

We note that the algebraic properties of these tensors have been intensive subject of recent research, especially due to the existing interrelation between the properties of their attached algebras, and the Finsler geometry lying beyond their related physical models ([19, 20, 21, 9, 7, 8, 1, 2, 6, 3, 4, 5]).

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