On projective regular representations of discrete groups and their infinite tensor products

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Abstract. The goal of this paper is to study the regular projective representations of a discrete group G on a Hilbert space H ([9]). We describe the commuting algebras of the right, respectively left regular projective representations and present the existence theorem for infinite tensor products of projective representations proved by E. Bédos and R. Conti in [2].

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1 Introduction and preliminaries

In the first part of this paper we remind the basic notions of a multiplier on a discrete group, a projective representation of a discrete group and a left, respectively right regular projective representation. In Section 2 we present the description of the commuting algebra of the regular projective representation with the multiplier ω given by Kleppner in [9]. Bédos and Conti initiated the study of infinite tensor products of projective representations of a discrete group in [2]. In Section 3 we prove Bédos and Conti's existence theorem for infinite tensor product of projective representations of a discrete group.

Definition 1.1. ([9]) Let G be a discrete group and let $\mathbb{T}$ be the group of complex numbers of modulus one. A *multiplier* ω on G is a function $\omega\colon G \times G \to \mathbb{T}$ with the properties:

i) $\omega(x, e) = \omega(e, x) = 1$ for all $x \in G$;

ii) $\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z)$ for all $x, y, z \in G$.

If $\omega(x, x^{-1}) = 1$ for all $x \in G$, then the multiplier ω is called *normalized.*

Remark 1.1. ([9]) If ω is normalized, then $\omega(x, y)^{-1} = \omega(y^{-1}, x^{-1})$ for all $x, y \in G$.

Definition 1.2. ([9]) A *projective representation* of G on a Hilbert space H with the associated multiplier ω (more precisely, a unitary ω-representation) is a map $\pi\colon G \to \mathcal{U}(H)$, where $\mathcal{U}(H)$ is the group of unitary operators on H, such that

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i) $\pi(x)\pi(y) = \omega(x, y)\pi(xy)$ for all $x, y \in G$;

ii) $\pi(e) = I_H$, where I_H is the identity operator on H.

Definition 1.3. ([9]) Let ω be a multiplier on G. The map $R^\omega : G \to \mathcal{U}(l^2(G))$ defined by $R^\omega_s f(x) = \omega(x, s)f(xs)$ for all $x \in G$ and $f \in l^2(G)$ is an ω-representation of G called *the right regular ω-representation of G*. The map $L^\omega : G \to \mathcal{U}(l^2(G))$ defined by $L^\omega_s f(x) = \omega(x^{-1}, s)f(s^{-1}x)$ for all $x \in G$ and $f \in l^2(G)$ is an ω-representation of G called *the left regular ω-representation of G*.

2 The commuting algebra of regular projective representations

In this section we consider that ω is a normalized multiplier on G and θ_x is the characteristic function of the point x, for each $x \in G$. In what follows we shall be interested in the right regular ω-representation and the left regular ω^{-1}-representation, which we shall denote by R and L respectively. We have:

$$R_s f(x) = \omega(x, s)f(xs)$$

$$L_s f(x) = \omega(x^{-1}, s)^{-1}f(s^{-1}x) = \omega(s^{-1}, x)f(s^{-1}x)$$

$$R_s\theta_x = \omega(xs^{-1}, s)\theta_{xs^{-1}} = \omega(x, s^{-1})^{-1}\theta_{xs^{-1}} = \omega(s, x^{-1})\theta_{xs^{-1}} \tag{2.1}$$

$$L_s\theta_x = \omega(x^{-1}s^{-1}, s)\theta_{sx} = \omega(x^{-1}, s^{-1})\theta_{sx} = \omega(s, x)^{-1}\theta_{sx}. \tag{2.2}$$

Note that R and L commute for

$$\begin{aligned} L_t R_s f(x) &= \omega(t^{-1}, x)R_s f(t^{-1}x) = \omega(t^{-1}, x)\omega(t^{-1}x, s)f(t^{-1}xs) = \\ &= \omega(x, s)\omega(t^{-1}, xs)f(t^{-1}xs) = \omega(x, s)L_t f(xs) = R_s L_t f(x). \end{aligned}$$

In general, the left and right regular ω-representations do not commute.

Definition 2.1. ([9]) Let T be a linear operator $l^2(G)$ and let f_T be the function $x \longrightarrow (T\theta_e, \theta_x) = T_{x,e}$, where $(T_{x,y})$ is the matrix of the operator T with the entries $T_{x,y} = (T\theta_y, \theta_x)$ such that $T\theta_y = \sum_{x\in G} T_{x,y}\theta_x$.

Lemma 2.1. ([9]) *Let T be a bounded linear operator on $l^2(G)$. Then:*
a) $T \in \mathcal{R}(R, R)$ (the commmuting algebra of the right regular ω-representations) if and only if

$$T_{x,y} = \omega(x, y^{-1})f_T(xy^{-1}),\ \forall x, y \in G. \tag{2.3}$$

b) $T \in \mathcal{R}(L, L)$ (the commmuting algebra of the left regular ω^{-1}-representations) if and only if

$$T_{x,y} = \omega(y^{-1}, x)f_T(y^{-1}x),\ \forall x, y \in G. \tag{2.4}$$

Proof. Suppose that $T \in \mathcal{R}(R, R)$. By Definition 1.1, relation (2.1) and Remark 1.1, we obtain:

$$\begin{aligned} T_{x,y} &= (T\theta_y, \theta_x) = (T\omega(e, y)^{-1}\theta_{ey,x}) = (TR_{y^{-1}}\theta_e, \theta_x) = (R_{y^{-1}}T\theta_e, \theta_x) \\ &= (T\theta_e, R_y\theta_x) = (T\theta_e, \omega(y, x^{-1})\theta_{xy^{-1}}) = (T\theta_e, \omega(x, y^{-1})^{-1}\theta_{xy^{-1}}) \\ &= \omega(x, y^{-1})(T\theta_e, \theta_{xy^{-1}}) = \omega(x, y^{-1})f_T(xy^{-1}). \end{aligned}$$

Conversely, if a) is satisfied, then, by relation (2.1), for each $s \in G$, we have:

$$
\begin{aligned}
(TR_s)_{x,y} = (TR_s\theta_y, \theta_x) &= (T\omega(s, y^{-1})\theta_{ys^{-1}}, \theta_x) = \omega(s, y^{-1})(T\theta_{ys^{-1}}, \theta_x) \\
&= \omega(s, y^{-1})T_{x,ys^{-1}} = \omega(s, y^{-1})\omega(x, sy^{-1})f_T(xsy^{-1}).
\end{aligned} \tag{2.5}
$$

By relation (2.1) and Definition 1.1, we obtain:

$$
\begin{aligned}
(R_sT)_{x,y} &= (R_sT\theta_y, \theta_x) = (T\theta_y, R_{s^{-1}}\theta_x) = (T\theta_y, \omega(x, s)^{-1}\theta_{xs}) \\
&= \omega(x, s)(T\theta_y, \theta_{xs}) = \omega(x, s)T_{xs,y} = \omega(x, s)\omega(xs, y^{-1})f_T(xsy^{-1}) \\
&= \omega(x, sy^{-1})\omega(s, y^{-1})f_T(xsy^{-1}).
\end{aligned} \tag{2.6}
$$

From (2.5) and (2.6), it results that $(TR_s)_{x,y} = (R_sT)_{x,y}$, $\forall x, y \in G$, so $T \in \mathcal{R}(R, R)$. Suppose now tat $T \in \mathcal{R}(L, L)$. Using (2.2) and Definition 1.1, we obtain:

$$
\begin{aligned}
T_{x,y} &= (T\theta_y, \theta_x) = (T\omega(e, y)^{-1}\theta_{ye}, \theta_x) = (TL_y\theta_e, \theta_x) \\
&= (L_yT\theta_e, \theta_x) = (T\theta_e, L_{y^{-1}}\theta_x) = (T\theta_e, \omega(y^{-1}, x)^{-1}\theta_{y^{-1}x}) \\
&= \omega(y^{-1}, x)(T\theta_e, \theta_{y^{-1}x}) = \omega(y^{-1}, x)f_T(y^{-1}x),
\end{aligned}
$$

so we proved (2.4). Conversely, assume that (2.4) is satisfied. For each $s \in G$, using (2.2), we infer:

$$
\begin{aligned}
(TL_s)_{x,y} &= (TL_s\theta_y, \theta_x) = \omega(s, y)^{-1}(T\theta_{sy}, \theta_x) \\
&= \omega(s, y)^{-1}T_{x,sy} = \omega(s, y)^{-1}\omega((sy)^{-1}, x)f_T((sy)^{-1}x) \\
&= \omega(s, y)^{-1}\omega(y^{-1}s^{-1}, x)f_T(y^{-1}s^{-1}x).
\end{aligned} \tag{2.7}
$$

By Definition 1.1 and (2.2), we obtain:

$$
\begin{aligned}
(L_sT)_{x,y} &= (L_sT\theta_y, \theta_x) = (T\theta_y, L_{s^{-1}}\theta_x) = (T\theta_y, \omega(s^{-1}, x)^{-1}\theta_{s^{-1}x}) \\
&= \omega(s^{-1}, x)(T\theta_y, \theta_{s^{-1}x}) = \omega(s^{-1}, x)T_{s^{-1}x,y} \\
&= \omega(s^{-1}, x)\omega(y^{-1}, s^{-1}x)f_T(y^{-1}s^{-1}x) \\
&= \omega(y^{-1}, s^{-1})\omega(y^{-1}s^{-1}, x)f_T(y^{-1}s^{-1}x) \\
&= \omega(s, y)^{-1}\omega(y^{-1}s^{-1}, x)f_T(y^{-1}s^{-1}x).
\end{aligned} \tag{2.8}
$$

From (2.7) and (2.8), it results that $(L_sT)_{x,y} = (TL_s)_{x,y}$, so $T \in \mathcal{R}(L, L)$. □

Theorem 2.1. ([9]) *Let ω be a normalized multiplier on the discrete group G. The commuting algebras of the right regular ω-representation and the left regular ω^{-1}-representation are the commutants of each other.*

Proof. Because R and L commute, $\mathcal{R}(R, R)' \subset \mathcal{R}(L, L)$. Suppose $T \in \mathcal{R}(R, R)$ and $S \in \mathcal{R}(L, L)$. Then, by Lemma 2.1 and Definition 2.1, we have

$$
(TS)_{x,y} = \sum_{z \in G} T_{x,z}S_{z,y} = \sum_{z \in G} \omega(x, z^{-1})\omega(y^{-1}, z)f_T(xz^{-1})f_S(y^{-1}z)
$$

and

$$(2.9) \qquad (ST)_{x,y} = \sum_{z\in G} S_{x,z}T_{z,y} = \sum_{z\in G} \omega(z^{-1},x)\omega(z,y^{-1})f_S(z^{-1}x)f_T(zy^{-1}).$$

In (2.9) we take $z = xz^{-1}y$ and we obtain:

$$(2.10) \qquad \begin{aligned} (ST)_{x,y} &= \sum_{z\in G} \omega(y^{-1}zx^{-1},x)\omega(xz^{-1}y,y^{-1})f_S(z^{-1}x)f_T(zy^{-1}) \\ &= \sum_{z\in G} \omega(y^{-1}zx^{-1},x)\omega(xz^{-1}y,y^{-1})f_S(y^{-1}zx^{-1}x)f_T(xz^{-1}yy^{-1}) \\ &= \sum_{z\in G} \omega(y^{-1}zx^{-1},x)\omega(xz^{-1}y,y^{-1})f_S(y^{-1}z)f_T(xz^{-1}). \end{aligned}$$

On the other hand, by Definition 1.1 and Remark 1.1, we have
(2.11)

$$\begin{aligned} &\omega(y^{-1}zx^{-1},x)\omega(xz^{-1}y,y^{-1}) = \omega(x^{-1},xz^{-1}y)^{-1}\omega(y,y^{-1}zx^{-1})^{-1} \\ &= \omega(x^{-1},x)^{-1}\omega(x^{-1}x,z^{-1}y)^{-1}\omega(x,z^{-1}y)\omega(y,y^{-1})^{-1} \\ &= \omega(x^{-1},x)^{-1}\omega(e,z^{-1}y)^{-1}\omega(x,z^{-1}y)\omega(y,y^{-1})^{-1}\omega(e,zx^{-1})^{-1}\omega(y^{-1},zx^{-1}) \\ &= \omega(x^{-1},x)^{-1}\omega(x,z^{-1}y)\omega(y,y^{-1})^{-1}\omega(y^{-1},zx^{-1}) \\ &= \omega(x,z^{-1}y)\omega(y^{-1},zx^{-1})\omega(x^{-1},x)\omega(y,y^{-1}) \\ &= \omega(x,z^{-1}y)\omega(y^{-1},zx^{-1}) = \omega(x,z^{-1}y)\omega(y^{-1},zx^{-1})\omega(z^{-1},y)\omega(z^{-1},y)^{-1} \\ &= \omega(x,z^{-1}y)\omega(y^{-1},zx^{-1})\omega(z^{-1},y)\omega(y^{-1},z) \\ &= \omega(x,z^{-1}y)\omega(z^{-1},y)\omega(y^{-1},zx^{-1})\omega(y^{-1},z) \\ &= \omega(x,z^{-1}\omega(xz^{-1},y)\omega(y^{-1},zx^{-1})\omega(y^{-1},z) \\ &= \omega(x,z^{-1})\omega(y^{-1},zx^{-1})^{-1}\omega(y^{-1},zx^{-1})\omega(y^{-1},z) = \omega(x,z^{-1})\omega(y^{-1},z). \end{aligned}$$

Using (2.11) in (2.10), we obtain:

$$(ST)_{x,y} = \sum_{z\in G} \omega(x,z^{-1})\omega(y^{-1},z)f_T(xz^{-1})f_S(y^{-1}z) = (TS)_{x,y}.$$

Hence T and S commute and we have $\mathcal{R}(L,L) \subset \mathcal{R}(R,R)'$. □

3 Infinite tensor products of projective representations

For $i = 1,2$, let π_i be an ω_i- representation of G on a Hilbert space H_i. The tensor product representation $\pi_1 \otimes \pi_2$ is an $\omega_1\omega_2$-representation of G on the Hilbert space $H_1 \otimes H_2$.

In the case of ordinary unitary representations of a group, it is known a classical result of Fell that the left regular representation acts in an absorbing way with respect to tensoring (up to multiplicity and equivalence). In the projective case we have the following analogue:

Proposition 3.1. *([2]) Let ω_1, ω_2 two multipliers on the discrete group G and let π be an ω_2- representation of G on a Hilbert space H. Then the tensor product representation $L^{\omega_1} \otimes \pi$ is unitarily equivalent to $L^{\omega_1\omega_2} \otimes I_H$.*

Proof. In the same way as in the nonprojective case the unitary operator that gives the unitarily equivalence is

$$U\colon l^2(G) \otimes H(\simeq l^2(G,H)) \to l^2(G,H)$$

defined by $\quad U(f \otimes h)(x) = f(x)\pi(x^{-1})h.$ □

Let $H = (H_i)$ denote a sequence of Hilbert spaces and let $\xi = (\xi_i)$ be a sequence of unit vectors where $\xi_i \in H_i$ for each $i \geq 1$. We denote by $H^\xi = \bigotimes_i^\xi H_i$ the associated infinite tensor product Hilbert space of the H_i's along the sequence ξ.

Theorem 3.1. *([2]) Let (π_i) be a sequence of ω_i-representations of G acting respectively on the Hilbert spaces H_i and let $\xi = (\xi_i)$ be a sequence of unit vectors where each $\xi_i \in H_i$. Assume that $\otimes_i \pi_i(x)$ exists on H^ξ, for each $x \in G$. Then:*

i) the map $x \longmapsto \pi^\xi(x) := \otimes_i \pi_i(x)$ is an $\omega = \prod_i \omega_i$- representation of G on H^ξ;

ii) if there is one k such that π_k is unitarily equivalent to L^{ω_k}, then π^ξ is unitarily equivalent to $L^\omega \otimes I_H$, for some Hilbert space H;

iii) $L^1 \otimes \pi^\xi$ is unitarily equivalent to $L^\omega \otimes I_{H^\xi}$, where L^1 is the left regular representation of G.

Proof. i) We have

$$\begin{aligned}\pi^\xi(x)\pi^\xi(y) &= (\otimes_i \pi_i(x))(\otimes_i \pi_i(y)) = \otimes_i \pi_i(x)\pi_i(y) = \otimes_i \omega_i(x,y)\pi_i(xy) \\ &= \prod_i \omega_i(x,y) \otimes_i \pi_i(xy) = \omega(x,y)\pi^\xi(xy), \quad \forall\, x, y \in G\end{aligned}$$

The assertions ii) and iii) follow from Proposition 3.1. □

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