Relativistic elastic tensor

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Abstract. The aim of this work is to determine the law of transformation of the elastic tensor from one inertial frame to another in the contest of special relativity. By the introduction of the entropy density as function of the component T_{00} of the energy-momentum tensor and of the strain tensor we introduce the relativistic temperature in a generic inertial frame of reference and the relative law of transformation. By mean of a new definition of thermodynamic relativistic stress tensor, compared with the classical one, allows to the law of transformation of the strain tensor. Finally these laws allow to the introduction of the relativistic elastic tensor and, as in the particular case of an isotropic medium, of the dynamic moduli.

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1 Introduction

In a previous paper [3] the dependence of the entropy density on energy momentum tensor allows to obtain the law of transformation of the temperature (in agreement with Otts transformation formula) and therefore a new expression of the relativistic stress tensor in a general inertial frame of reference which take into account the relativistic entropy density.

Since are well known the laws of transformation of every component of the stress tensor it has been possible, by utilizing these laws and our new stress tensor form, to obtain the law of transformation of the strain tensor.

The aim of this work is to introduce the relativistic elastic tensor by the law of transformation of the stress and strain. It will be shown that symmetries of the elastic tensor in the proper frame are not preserved in a generic inertial frame of reference.

Moreover, by specialize to the case of an isotropic medium, it will possible to introduce (as a very particular case) the relativistic dynamic modulus. Even in this case it is shown that in a generic inertial frame the isotropic properties are not preserved, obtaining the components of the relativistic dynamic modulus.

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2 Energy momentum tensor

In [3] a continuum deformable medium in motion with respect to an arbitrary inertial frame $\Sigma \equiv (O, x, y, z, t)$ is considered and the following total density of energy flow is introduced:

$$L_i = E_i + \rho c^2 v_i + v^j \phi_{ji}, \tag{2.1}$$

where:

i) E_i is the vector representing density of energy flow of not mechanical nature (as the heat) [6],[7],

ii) $\rho c^2 v_i$ is the density of energy flow due only to the motion of the medium, ρ is the mass density and c is the scalar velocity of light in vacum,

iii) $v^j \phi_{ji}$ is the density of energy energy flow due to the action of the forces of stress flowing in the positive x_i direction and ϕ_{ji} is the relativistic (no symmetric) stress tensor.

From (2.1), the following total momentum density can be deduced:

$$H_i = \frac{L_i}{c^2} = \rho v_i + \frac{E_i}{c^2} + \frac{v^j \phi_{ji}}{c^2} \tag{2.2}$$

Putting $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$, the following energy momentum tensor $T_{\alpha\beta}$ can be introduced ([12, 17, 14]):

$$T_{\alpha\beta} = \begin{cases} T_{ik} = H_i v_k + \phi_{ik} \\ T_{i0} = T_{0i} = cH_i \\ T_{00} = \rho c^2, \end{cases}$$

in which Latin index assumes the values $1, 2, 3$ and Greek index assumes the values $0, 1, 2, 3$. Introducing the four-vector $W_\alpha \equiv (\rho v_i F_i/c\ ,\ \rho F_i)$, where ρF_i is the unitary volume force, the following tensorial equation can be written:

$$\frac{\partial T_{\alpha\beta}}{\partial x^\beta} = W_\alpha,$$

in which the "temporal component (the upper zero index) represents the balance equation for energy density and the spatial components ("1,2,3" index) represent the balance equation for momentum density. Using the four velocity $V^\alpha \equiv (\alpha\ ,\ \alpha v_i/c)$, the first law of thermodynamics has the form:

$$\frac{\partial T_{\alpha\beta}}{\partial x^\beta} V^\alpha = W_\alpha V^\alpha.$$

3 Density of internal energy

It is known that the coordinate transformation relating two inertial frames Σ and Σ' in relative general configuration are the Lorentz transformation ([12, 16]):

$$\begin{cases} x_i = {x_i}' + \dfrac{\alpha v_i}{c} {x_0}' + (\alpha - 1) \dfrac{v_i v_k}{v^2} {x'}_k \\ x_0 = \alpha \Big({x_0}' + \dfrac{v_i {x'}_i}{c} \Big), \end{cases} \tag{3.1}$$

where $\mathbf{v} \equiv (v_1, v_2, v_3)$ is the uniform velocity of Σ with respect to Σ' and $\alpha = 1/\sqrt{1-(v^2/c^2)}$. So, if we consider $\Sigma' \equiv \Sigma_0$ as a proper reference, the following transformation law for $T_{\alpha\beta}$ can be written:

$$T_{\alpha\beta} = \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^\nu}{\partial x^{\beta'}} T^{(0)}_{\mu\nu}, \tag{3.2}$$

where $T^{(0)}_{\mu\nu}$ is the energy momentum tensor in Σ_0.
By using the transformation laws (3.1) from 3.2 we have [12, 18]:

$$T_{00} = \alpha^2 \rho_0 c^2 + 2\frac{\alpha^2 v^i}{c^2} E_i{}^{(0)} + \frac{\alpha^2}{c^2} v^i v^k \phi^{(0)}_{ik} = \rho c^2. \tag{3.3}$$

The formula (3.3) describes the density internal energy in Σ, where ρ_0 and $\phi^{(0)}_{ik}$ [2], [11] are respectively the mass density and the symmetric Cauchy stress tensor in Σ_0.

4 Relativistic strain and stress tensors.

It is well known that in Σ_0 the entropy of an element of medium with volume $d\tau_0$ is [17]

$$d\sigma_0 = \phi_0 \, d\tau_0,$$

where ϕ_0 is the entropy density which depends on the internal energy $T^{(0)}_{00}$ and on the strain tensor $\gamma^{(0)}_{ik}$, i.e.

$$\phi_0 = \phi_0(T^{(0)}_{00}, \gamma^{(0)}_{ik}).$$

Since $d\sigma_0$ is a relativistic invariant the entropy density will transform in Σ as:

$$\phi = \phi_0 \, \alpha.$$

We assume that in a general inertial frame of reference the entropy density depend on the transform of $T^{(0)}_{00}$ and $\gamma^{(0)}_{ik}$) [15], [9], i.e.

$$\phi = \phi(T_{00}, \gamma_{ik}).$$

Defining in Σ_0 the absolute temperature T_0 as

$$\frac{\partial \phi_0}{\partial T^{(0)}_{00}} = \frac{1}{T_0}$$

in Σ, we have:

$$\frac{\partial \phi}{\partial T_{00}} = \alpha \frac{\partial \phi_0}{\partial T^{(0)}_{00}} \frac{\partial T^{(0)}_{00}}{\partial T_{00}}. \tag{4.1}$$

By virtue of (3.3, one has:

$$\frac{\partial T^{(0)}_{00}}{\partial T_{00}} = \frac{1}{\alpha^2},$$

so the equation (4.1) becomes:

$$\frac{\partial \phi}{\partial T_{00}} = \frac{1}{T_0}\frac{1}{\alpha}.$$

Therefore we define the temperature in Σ as follows

$$\frac{1}{T} = \frac{\partial \phi}{\partial T_{00}} = \frac{1}{T_0}\frac{1}{\alpha}$$

or $T = \alpha T_0$, in agreement with Ott's transformation formula [13].
From equation **??** one has:

$$\frac{\partial \phi}{\partial \gamma_{rs}} = \frac{\partial \phi_0}{\partial \gamma_{ik}^{(0)}}\frac{\partial \gamma_{ik}^{(0)}}{\gamma_{rs}}\alpha, \tag{4.2}$$

and by virtue of the relations:

$$\frac{\partial \phi_0}{\partial \gamma_{ik}^{0}} = \frac{1}{T_0}\,\phi_{ik}^{(0)},$$

the equation (4.2) becomes:

$$\frac{\partial \phi}{\partial \gamma_{rs}} = \frac{1}{T_0}\phi_{ik}^{(0)}\frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}}\alpha = \alpha^2\frac{1}{T}\phi_{ik}^{(0)}\frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}},$$

and so we have

$$T\frac{\partial \phi}{\partial \gamma_{rs}} = \alpha^2\phi_{ik}^{(0)}\frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}}.$$

In the same way, by defining the stress tensor in Σ as follows

$$\phi_{rs} = T\frac{\partial \phi}{\partial \gamma_{rs}},$$

one has

$$\phi_{rs} = \alpha^2\phi_{ik}^{(0)}\frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}}. \tag{4.3}$$

Let us assume that our original system of coordinates Σ is oriented so that the material, at the point of interest in the medium, will be moving with respect to this system with the velocity, $\mathbf{v}$ parallel to the x-axis. Moreover, the system Σ_0 moves with the same velocity $\mathbf{v}$ respect to the system Σ. By virtue of these considerations, the transformation law of the stress tensor are the following [14]:

$$\phi_{11} = \phi_{11}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{11}}, \qquad \phi_{12} = \phi_{12}^{(0)} \alpha = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{12}}, \tag{4.4}$$

$$\phi_{13} = \phi_{13}^{(0)} \alpha = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{13}}, \qquad \phi_{21} = \phi_{21}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{21}}, \tag{4.5}$$

$$\phi_{22} = \phi_{22}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{22}}, \qquad \phi_{23} = \phi_{23}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{23}}, \tag{4.6}$$

$$\phi_{31} = \frac{\phi_{31}^{(0)}}{\alpha} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{31}}, \qquad \phi_{32} = \phi_{32}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{32}}, \tag{4.7}$$

$$\phi_{33} = \phi_{33}^{(0)} = \alpha^2 \phi_{rs}^{(0)} \frac{\partial \gamma_{rs}^{(0)}}{\partial \gamma_{33}}. \tag{4.8}$$

Let us observe that the stress tensor is not symmetric, then by considering the relation (4.3) it results:

$$\alpha^2 \phi_{ik}^{(0)} \frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{rs}} \neq \alpha^2 \phi_{ik}^{(0)} \frac{\partial \gamma_{ik}^{(0)}}{\partial \gamma_{sr}}.$$

Therefore it follows the non symmetry of the strain tensor, i.e., $\gamma_{rs} \neq \gamma_{sr}$. The transformation law of the strain tensor follows from the development of the equations (4.3) and after to observe that the components in Σ of the stress tensor depend only on the corresponding components in Σ_0, we have:

$$\frac{\partial \gamma_{rs}^{(0)}}{\gamma_{ik}} = 0 \quad \text{if} \quad i \neq r,\ k \neq s.$$

This implies that even components in Σ of strain tensor depend only on the corresponding components in Σ_0. Thus the transformation law of the strain tensor is expressed in the following form:

$$\begin{cases} \gamma_{11} = \alpha^2 \gamma_{(11)}^{(0)}, \ \ \gamma_{12} = \alpha \gamma_{(12)}^{(0)}, \ \ \gamma_{13} = \alpha \gamma_{(13)}^{(0)} \\ \gamma_{21} = \alpha^3 \gamma_{(21)}^{(0)}, \ \ \gamma_{22} = \alpha^2 \gamma_{(22)}^{(0)}, \ \ \gamma_{23} = \alpha^2 \gamma_{(23)}^{(0)} \\ \gamma_{31} = \alpha^3 \gamma_{(31)}^{(0)}, \ \ \gamma_{32} = \alpha^2 \gamma_{(32)}^{(0)}, \ \ \gamma_{11} = \alpha^2 \gamma_{(33)}^{(0)}. \end{cases} \tag{4.9}$$

5 Relativistic elastic tensor

We have:

$$\begin{aligned} \phi_{ik}^{(0)} = G_{ikrs}^{(0)} \gamma_{rs}^{(0)} = \quad & G_{ik11}^{(0)} \gamma_{11}^{(0)} + G_{ik12}^{(0)} \gamma_{12}^{(0)} + G_{ik13}^{(0)} \gamma_{13}^{(0)} + \\ & + G_{ik21}^{(0)} \gamma_{21}^{(0)} + G_{ik22}^{(0)} \gamma_{22}^{(0)} + G_{ik23}^{(0)} \gamma_{23}^{(0)} + \\ & + G_{ik31}^{(0)} \gamma_{31}^{(0)} + G_{ik32}^{(0)} \gamma_{32}^{(0)} + G_{ik33}^{(0)} \gamma_{33}^{(0)}, \end{aligned} \tag{5.1}$$

and using (4.9) one has:

$$\phi^{(0)}_{ik} = G^{(0)}_{ik11}\frac{1}{\alpha^2}\gamma_{11} + G^{(0)}_{ik12}\frac{1}{\alpha}\gamma_{12} + G^{(0)}_{ik13}\frac{1}{\alpha}\gamma_{13} + $$
$$+G^{(0)}_{ik21}\frac{1}{\alpha^3}\gamma_{21} + G^{(0)}_{ik22}\frac{1}{\alpha^2}\gamma_{22} + G^{(0)}_{ik23}\frac{1}{\alpha^2}\gamma_{23} +$$
$$+G^{(0)}_{ik31}\frac{1}{\alpha^3}\gamma_{31} + G^{(0)}_{ik32}\frac{1}{\alpha^2}\gamma_{32} + G^{(0)}_{ik33}\frac{1}{\alpha^2}\gamma_{33}.$$

From (4.4)-(4.8), we obtain:

(5.2)
$$G_{11rs} = \begin{pmatrix} \frac{G_{1111}}{\alpha^2} & \frac{G_{1112}}{\alpha} & \frac{G_{1113}}{\alpha} \\ \frac{G_{1121}}{\alpha^3} & \frac{G_{1122}}{\alpha^2} & \frac{G_{1123}}{\alpha^2} \\ \frac{G_{1131}}{\alpha^3} & \frac{G_{1132}}{\alpha^2} & \frac{G_{1133}}{\alpha^2} \end{pmatrix}, \qquad G_{12rs} = \begin{pmatrix} \frac{G_{1211}}{\alpha^2} & G_{1212} & G_{1213} \\ \frac{G_{1221}}{\alpha^2} & \frac{G_{1222}}{\alpha} & \frac{G_{1223}}{\alpha} \\ \frac{G_{1231}}{\alpha^2} & \frac{G_{1232}}{\alpha} & \frac{G_{1233}}{\alpha} \end{pmatrix},$$

(5.3)
$$G_{13rs} = \begin{pmatrix} \frac{G_{1311}}{\alpha} & G_{1312} & G_{1313} \\ \frac{G_{1321}}{\alpha^2} & \frac{G_{1322}}{\alpha} & \frac{G_{1323}}{\alpha} \\ \frac{G_{1331}}{\alpha^2} & \frac{G_{1332}}{\alpha} & \frac{G_{1333}}{\alpha} \end{pmatrix}, \qquad G_{21rs} = \begin{pmatrix} \frac{G_{2111}}{\alpha^3} & \frac{G_{2112}}{\alpha^2} & \frac{G_{2113}}{\alpha^2} \\ \frac{G_{2121}}{\alpha^4} & \frac{G_{2122}}{\alpha^3} & \frac{G_{2123}}{\alpha^3} \\ \frac{G_{2131}}{\alpha^4} & \frac{G_{2132}}{\alpha^3} & \frac{G_{2133}}{\alpha^3} \end{pmatrix},$$

(5.4)
$$G_{22rs} = \begin{pmatrix} \frac{G_{2211}}{\alpha^2} & \frac{G_{2212}}{\alpha} & \frac{G_{2213}}{\alpha} \\ \frac{G_{2221}}{\alpha^3} & \frac{G_{2222}}{\alpha^2} & \frac{G_{2223}}{\alpha^2} \\ \frac{G_{2231}}{\alpha^3} & \frac{G_{2232}}{\alpha^2} & \frac{G_{2233}}{\alpha^2} \end{pmatrix}, \qquad G_{23rs} = \begin{pmatrix} \frac{G_{2311}}{\alpha^2} & \frac{G_{2312}}{\alpha} & \frac{G_{2313}}{\alpha} \\ \frac{G_{2321}}{\alpha^3} & \frac{G_{2322}}{\alpha^2} & \frac{G_{2323}}{\alpha^2} \\ \frac{G_{2331}}{\alpha^3} & \frac{G_{2332}}{\alpha^2} & \frac{G_{2333}}{\alpha^2} \end{pmatrix},$$

(5.5)
$$G_{31rs} = \begin{pmatrix} \frac{G_{3111}}{\alpha^3} & \frac{G_{3112}}{\alpha^2} & \frac{G_{3113}}{\alpha^2} \\ \frac{G_{3121}}{\alpha^4} & \frac{G_{3122}}{\alpha^3} & \frac{G_{3123}}{\alpha^3} \\ \frac{G_{3131}}{\alpha^4} & \frac{G_{3132}}{\alpha^3} & \frac{G_{3133}}{\alpha^3} \end{pmatrix}, \qquad G_{32rs} = \begin{pmatrix} \frac{G_{3211}}{\alpha^2} & \frac{G_{3212}}{\alpha} & \frac{G_{3213}}{\alpha} \\ \frac{G_{3221}}{\alpha^3} & \frac{G_{3222}}{\alpha^2} & \frac{G_{3223}}{\alpha^2} \\ \frac{G_{3231}}{\alpha^3} & \frac{G_{3232}}{\alpha^2} & \frac{G_{3233}}{\alpha^2} \end{pmatrix},$$

(5.6)
$$G_{33rs} = \begin{pmatrix} \frac{G_{3311}}{\alpha^2} & \frac{G_{3312}}{\alpha} & \frac{G_{3313}}{\alpha} \\ \frac{G_{3321}}{\alpha^3} & \frac{G_{3322}}{\alpha^2} & \frac{G_{3323}}{\alpha^2} \\ \frac{G_{3331}}{\alpha^3} & \frac{G_{3332}}{\alpha^2} & \frac{G_{3333}}{\alpha^2} \end{pmatrix},$$

(5.7)
$$G^{(0)}_{ikrs} = G^{(0)}_{iksr} = G^{(0)}_{kirs} = G^{(0)}_{rsik}$$

we have only

(5.8)
$$G_{ikrs} = G_{rsik}\,.$$

For isotropic media we have:

(5.9)
$$G^{(0)}_{ikrs} = -\overset{(0)}{\lambda}\,\delta_{ik}\delta_{rs} - \overset{(0)}{\mu}\,(\delta_{ir}\delta_{ks} + \delta_{is}\delta_{kr})$$

and so the relations (5.2)-(5.6) become:

(5.10)
$$G_{11rs} = \begin{pmatrix} \frac{-\overset{(0)}{\lambda} - \overset{(0)}{\mu}}{\alpha^2} & 0 & 0 \\ 0 & \frac{-\overset{(0)}{\lambda}}{\alpha^2} & 0 \\ 0 & 0 & \frac{-\overset{(0)}{\lambda}}{\alpha^2} \end{pmatrix}, \qquad G_{12rs} = \begin{pmatrix} 0 & -\overset{(0)}{\mu} & 0 \\ \frac{-\overset{(0)}{\mu}}{\alpha^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(5.11)\qquad G_{13rs} = \begin{pmatrix} 0 & 0 & -\overset{(0)}{\mu} \\ 0 & 0 & 0 \\ \frac{-\overset{(0)}{\mu}}{\alpha^2} & 0 & 0 \end{pmatrix}, \qquad G_{21rs} = \begin{pmatrix} 0 & \frac{-\overset{(0)}{\mu}}{\alpha^2} & 0 \\ \frac{-\overset{(0)}{\mu}}{\alpha^4} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(5.12)\qquad G_{22rs} = \begin{pmatrix} \frac{-\overset{(0)}{\lambda}}{\alpha^2} & 0 & 0 \\ 0 & \frac{-\overset{(0)}{\lambda} - \overset{(0)}{\mu}}{\alpha^2} & 0 \\ 0 & 0 & \frac{-\overset{(0)}{\lambda}}{\alpha^2} \end{pmatrix}, \qquad G_{23rs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-\overset{(0)}{\mu}}{\alpha^2} \\ 0 & \frac{-\overset{(0)}{\mu}}{\alpha^2} & 0 \end{pmatrix},$$

$$(5.13)\qquad G_{31rs} = \begin{pmatrix} 0 & 0 & \frac{-\overset{(0)}{\mu}}{\alpha^2} \\ 0 & 0 & 0 \\ \frac{-\overset{(0)}{\mu}}{\alpha^4} & 0 & 0 \end{pmatrix}, \qquad G_{32rs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-\overset{(0)}{\mu}}{\alpha^2} \\ 0 & \frac{-\overset{(0)}{\mu}}{\alpha^2} & 0 \end{pmatrix},$$

$$(5.14)\qquad G_{33rs} = \begin{pmatrix} \frac{-\overset{(0)}{\lambda}}{\alpha^2} & 0 & 0 \\ 0 & \frac{-\overset{(0)}{\lambda} - \overset{(0)}{\mu}}{\alpha^2} & 0 \\ 0 & 0 & \frac{-\overset{(0)}{\lambda}}{\alpha^2} \end{pmatrix},$$

so we have the following symmetries:

$$G_{11rs} = G_{11sr};\ \ G_{22rs} = G_{22sr};\ \ G_{32rs} = G_{32sr};\ \ G_{33rs} = G_{33sr},$$

otherwise

$$(5.15)\qquad G_{12rs} \neq G_{12sr};\ \ G_{13rs} \neq G_{13sr};\ \ G_{21rs} \neq G_{21sr}.$$

6 Dynamic relativistic moduli

From(5.1) and (5.9) it follows

$$(6.1)\qquad \phi_{ik}^{(0)} = \overset{(0)}{k}\,\overset{(0)}{\gamma}\,\delta_{ik} - 2\,\overset{(0)}{\mu}\,d_{rs}^{(0)},$$

in which $d_{rs}^{(0)}$ is the deviator of the strain tensor $\gamma_{rs}^{(0)}$, $\overset{(0)}{\gamma} = \gamma_{rs}^{(0)}\,\delta_{rs}$, $\overset{(0)}{\mu}$ is the shear modulus and

$$\overset{(0)}{k} = -\frac{3\,\overset{(0)}{\lambda} + 2\,\overset{(0)}{\mu}}{3}$$

is the bulk modulus. If only shear strain is considered, following the considerations discussed in [4], we can observe that in the proper frame the dynamic complex modulus can be written:

$$(6.2)\qquad \overset{(0)}{G} = \overset{(0)}{G}_1 + i\,\overset{(0)}{G}_2 = -2\,\overset{(0)}{\mu}\,.$$

and the equation (6.1) assume the form:

$$\phi^{(0)}_{ik} = -2 \overset{(0)}{\mu} d^{(0)}_{rs} = \overset{(0)}{G} d^{(0)}_{rs} . \tag{6.3}$$

In this case the equations (5.10)-(5.15) become:

$$\begin{aligned}
&G_{11rs} = \frac{1}{\alpha^2}\begin{pmatrix} \overset{(0)}{G} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{12rs} = \begin{pmatrix} 0 & \overset{(0)}{G} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{13rs} = \begin{pmatrix} 0 & 0 & \overset{(0)}{G} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&G_{21rs} = \frac{1}{\alpha^4}\begin{pmatrix} 0 & 0 & 0 \\ \overset{(0)}{G} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{22rs} = \frac{1}{\alpha^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \overset{(0)}{G} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{23rs} = \frac{1}{\alpha^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \overset{(0)}{G} \\ 0 & -\overset{(0)}{\mu} & 0 \end{pmatrix}, \\
&G_{31rs} = \frac{1}{\alpha^4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \overset{(0)}{G} & 0 & 0 \end{pmatrix}, \quad G_{32rs} = \frac{1}{\alpha^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \overset{(0)}{G} & 0 \end{pmatrix}, \quad G_{33rs} = \frac{1}{\alpha^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overset{(0)}{G} \end{pmatrix}.
\end{aligned} \tag{6.4}$$

If no shear phenomena occur, i.e. $d^{(0)}_{rs} = 0$, we have (see [4]):

$$\overset{(0)}{G} = \overset{(0)}{G}_1 + i \overset{(0)}{G}_2 = \overset{(0)}{k} .$$

and the equation (6.1) assumes the form:

$$\phi^{(0)}_{ik} = \overset{(0)}{k} \overset{(0)}{\gamma} \delta_{ik} = \overset{(0)}{G} \overset{(0)}{\gamma} \delta_{ik} ,$$

and (5.10)-(5.15) become:

$$\begin{aligned}
&G_{11rs} = \frac{1}{\alpha^2}\begin{pmatrix} \overset{(0)}{G} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{12rs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{13rs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&G_{21rs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{22rs} = \frac{1}{\alpha^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \overset{(0)}{G} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{23rs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&G_{31rs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{32rs} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{33rs} = \frac{1}{\alpha^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overset{(0)}{G} \end{pmatrix}.
\end{aligned}$$

It is interesting to observe that in this case only three diagonal components of the elastic tensor are different from zero and so the transformed. This is in agreement with the principle of relativity.

We conclude by observing that even if we consider only shear or bulk strain the isotropy is not preserved in a general inertial reference frame. The knowledge of the laws of transformation of dynamic modulus and is important for the experimental evaluation of the mechanical properties of materials.

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