Finiteness of von Neumann algebras relative to a group of automorphisms

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Abstract. The classification of von Neumann algebras relative to a group of automorphisms is introduced. It is shown that for certain classes of groups of automorphisms of von Neumann algebras there exist mappings similar to those of a centre valued G-trace. The G-dimension is introduced and the equivalence of G-finiteness and the existence of G-dimension function is studied. We introduce the theory with some articular cases, in more general set up. Instead of considering group of automorphisms, we shall consider groups of automorphisms under an action α.

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1 Introduction

Kovacs and Szucs ([8]) introduced the notion of G-finiteness and provided a characterization of G-finite von Neumann algebras. In Kadison and Ringrose ([7]), the result "a von Neumann algebra is finite if and only if it has a dimension function" is proved. To define the finiteness and for the classification of von Neumann algebras, they were mainly using the group of unitary elements. There are studies in which arbitrary groups have been made use to define finiteness. Recently, in 2008, Balasubramani and Ravindran ([2], [3]) had introduced a G-dimension function and proved some results similar to the classical case.

2 Main results

Definition 2.1. Let M be a von Neumann algebra, G a group of automorphisms in M, then the fixed point algebra M^G is defined as the set of all $a \in M$ such that $t(a) = a$ for $t \in G$. Denote P as the set of all projections in M.

Theorem 2.2. *Let M be a von Neumann algebra. If G is either finite group or compact abelian group of automorphisms of M, M^G is the fixed point algebra of M under G then there exists a mapping $T : M \to M^G$ having the following properties.*

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i. T is linear
ii. T is invariant under G
iii. T fixes elements of G
iv. T is σ-weakly continuous and
v. T is positive

Proof. Suppose G is finite, say $G = \{g_1, g_2, \ldots, g_n\}$. α_g denotes the automorphism corresponding to the element $g \in G$. Define T on M as

$$T(a) = \frac{1}{n}\sum_{i=1}^{n} \alpha_{g_i}(a) \text{ for } a \in M.$$

Then T has the properties stated in the theorem.

Now let G be a compact abelian group of automorphisms of M under an action α. Being an action, g $\rightarrow \langle\, \alpha_g(a), \varphi\rangle$ is a bounded (bounded by $\|\alpha\|\,\|\phi\|$) continuous function, where $\langle\, \alpha_g(a), \varphi\rangle$ stands for $\varphi\,(\alpha_g)\,(a)$. So

$$\varphi \rightarrow \int_G \langle \alpha_g\ (a)\,,\, \varphi\rangle\, d\mu\,(g)$$

is a bounded linear functional on M_*, the pre-dual of M, of norm at most $\|\mu\|\,\|a\|$, where μ is a normalized Haar measure on G and $\|\mu\| = \text{Sup}\ \{\sum_{i=1}^{n} |\mu\ Ei|$ such that $\{Ei\}_{i=1}^{n}$ is a Borel partition of G. Since $(M_*)^* = M$, $\exists$ a unique element in M_*, which we shall denote by $T(a)$, such that $\|T\,(a)\| = \|\mu\|\ \|a\|$ and

$$\langle T\,(a)\ ,\, \varphi\rangle = \int_G \langle \alpha_g\ (a)\,,\, \varphi\rangle\, d\mu\,(g)$$

for $a \in M$ and φ in M_*. Using the translation invariance and the normality of Haar measure and linearity of automorphisms, it can be shown that the mapping $a \rightarrow T(a)$ is from M to M^G, T is linear, T is invariant under G and T fixes elements of M^G. Since the automorphisms are positive mappings, T will become positive also. To prove T is σ-weakly continuous, it is enough to prove that $\phi_0 T$ is in M_*. Again this can be done by proving that $\phi_0 T$ is in$(M_*)^*$ whenever φ is in$(M_*)^*$. For φ in$(M_*)^*$, $\phi_0 T$ is positive linear functional on M. Let $\{a_i\}$ be a monotone increasing net in M_* increasing to a in M_*. Then the net $\{\langle\alpha_g\,(ai)\,,\varphi\rangle\}$ of continuous functions on G increases point wise to the continuous function $\{\langle\alpha_g\,(a)\,,\varphi\rangle\}$ on G. By Dini's theorem, the convergence is uniform, since G is compact. So $\{\langle T\,(ai)\ ,\, \varphi\rangle\}$ converges to $\langle T\,(a)\ ,\, \varphi\rangle$. Thus $\phi_0 T$ is normal and belongs to$(M_*)^*$. Hence T is σ-weakly continuous and is the required mapping. □

Definition 2.3. A mapping $T : M \rightarrow M^G$ having the properties in the statement of the above theorem is called a *G–centre valued trace* or a *G–trace.*

Definition 2.4. A von Neumann algebra M is said to be *G–finite* if there is G-trace defined on it.

Now we extend the idea to projections also. Let M be a von Neumann algebra and G a group of automorphisms of M. Let e be a projection in M^G. For each g in G define ${\alpha_g}^e$ on M_e, the reduced algebra of M for the projection e as $\alpha_g^e(eae) = e\alpha_g(a)e$ for a in M. α_g^e is an automorphism on M_e for each $g \in G$ and $\alpha^e : G \rightarrow Aut(M_e)$, the group of all automorphisms on M_e, turns to be an action.

Definition 2.5. The action α^e is called the *reduced action* of α corresponding to the projection e in M^G

Definition 2.6. Let M be a von Neumann algebra and G group of automorphisms of M. A projection e in M^G is said to be *G-finite* if the corresponding reduced algebra is M_e is G_e-finite. (G_e denotes the group G of automorphisms is defined on M_e by the reduced action α^e).

With this definition, in natural way, a von Neumann algebra is G-finite if and only if the identity of M is a G-finite projection.

Definition 2.7. The projections e and f in M are equivalent relative to G (or G-equivalent), denoted by $e \sim_G f$, iff $\exists t \in G$ such that $e = t(f)$. If P denotes the set of all projections in M, then the relation $\sim_G$ is an equivalence relation in P. The central carrier or G-central support of a projection $a \in M$ is the smallest projection in M^G majorizing e and is denoted by C_e^G.

Definition 2.8. Let M be a von Neumann algebra, a *G-dimension function d* on M is a mapping from $P \to M^G$ such that

1. $d(e) > 0$ if $e \neq 0$ in P;
2. $d(e + f) = d(e) + d(f)$ for e and f in P;
3. $d(e) = d(f)$ if $e \sim_G f$, e and f in P
4. $d(q) = q$ if q is a projection in M^G.

Theorem 2.9. *Let s be a bounded linear map from M into a Banach space X. Then the following are equivalent.*

(A) s is invariant under G.

(B) The action of s onto G-equivalent projection in M are equivalent.

Proof. if $e \sim_G f$, then $e = t(f)$ for some $t \in G$. Hence $s(e) = s[t(f)]$. Thus (A) $\Rightarrow$ (B) . For the converse, we have $t(e) = t(e), t \in G$. Hence $e \sim_G t(e)$. Then $s(e) = s[t(e)]$ for all $t \in G, e \in P$. M is the norm closed linear hull of its projection implies that $s(a) = s[t(a)]$ for any $a \in M$. □

Theorem 2.10. *If a von Neumann algebra M is G-finite, then a G-dimension function d exists*

Proof. We have the following result [8]. M is G-finite iff $\exists$ a mapping $T : M \to M^G$ having the following properties

a) T is linear
b) T is invariant under G;
c) T fixes elements of M^G;
d) T is σ-weakly continuous and
e) T is strictly positive.

Now take $d = T_{|P}$. Then d satisfies all the properties. □

Definition 2.11. A von Neumann algebra M is said to be G_p - *finite* if $e \sim_G f \leq e \Rightarrow e = f$ for projections e and f in P.

Theorem 2.12. *If a G-dimension function $T : M \to M^G$ exists, then M is G_p-finite.*

Let e be a projection in P such that $e \sim_G f \leq e$. We have $d(f) = d(e) = d(f + e - f) = d(e - f) + d(f)d(e - f) = 0 \Rightarrow e = f$. Thus M is G_p-finite.

A projection e in M is called a G- minimal projection if e is G-finite and e does not have any non-zero proper G-finite subprojections.

Remark 2.13. Let e in M be a G-minimal projection. Then e is a minimal projection of M^G (in the ordinary sense).

Theorem 2.14. *Let M be a von Neumann algebra and G the group of automorphisms of M. Let $\{e_\alpha\}$ be an orthogonal family of projections in M^G with a sum e in M^G. Then each element in M_e has a direct sum decomposition into elements of $M_\alpha \doteq M_{e_\alpha}$ for each α.*

Proof. Since e in M^G is the direct sum of the family $\{e_\alpha\}$, the range of e the direct sum of the orthogonal family, $\{$ range of $\{e_\alpha\}$ $\}$, of Hilbert spaces. If H is the the Hilbert space on which M is defined, then $e(H) = \sum_\alpha \oplus e_\alpha(H)$. □

Consider an element eae of M_e. Then eae is a bounded linear operator on $e(H)$. So an element $\{\xi_\alpha\}$ of $\sum_\alpha \oplus e_\alpha(H)$ is mapped into an element $\{\eta_\alpha\}$ of $\sum_\alpha \oplus e_\alpha(H)$. If we define a_α to be the mapping $\xi_\alpha \to \eta\alpha$, then $a_\alpha : e_\alpha(H) \to e_\alpha(H)$ and using the properties of eae, and the projection e_α, we get that a_α is a bounded linear operator on $e_\alpha(H)$ for each α and a_α in $M\alpha$. The way in which a_α's are defined directly leads us to conclude that the direct sum direct sum of the operators a_α's is nothing but eae.

From the above result it follows that if M is a von Neumann algebra and G the group of automorphisms of M and $\{e_\alpha\}$ an orthogonal family of projections in M^G, then each element in M^G as a direct sum decomposition into elements of $\mathrm{M}_\alpha^G = \mathrm{M}_{e_\alpha}^G$ for each α.

Theorem 2.15. *Let M be a von Neumann algebra and G group of automorphisms of M. Let $\{e_\alpha\}$ an orthogonal family of G- minimal projections in M^G with a sum e in M^G. Then e is a G-finite projection.*

Proof. For a general element eae in Me, there exists a family $\{a_\alpha\}$ such that a_α is in Mα for each α. Since e_α is a G- minimal projection for each α, M_α^G consists of complex multiples of e_α and so M_α^G is abelian. Since e_α is G-finite, $\exists\ T_\alpha$, a G-trace on M for each α and $T_\alpha\ (a_\alpha)$ is a scalar multiple of e_α. T_α is bounded also with $\|T_\alpha\|$ = 1 and $\|T_\alpha\,(a_\alpha)\| = \|eae\|$ for each α. Hence we can meaningfully define the direct sum of the family of operators $\{T_\alpha\ (a_\alpha)\ \}$. This direct sum is defined on the range of e and belongs to M_e^G. Now define T on Me as $T(eae) = \sum_\alpha \oplus T_\alpha\,(a_\alpha)$. Then it can be shown that T is a G-trace which means that e is a G-finite projection. □

The properties except σ- weak continuity of T to become a G-trace are lifted from the corresponding the properties of members of the family $\{T_\alpha\}$ of G-traces. The previous theorem says that T is from Me to M_e^G. Now, to prove that T is a G-trace, it remains to prove the σ-weak continuity of T. Let $\{ea_ie\}$ be a net in Me converging σ-weakly to eae in Me. We should prove that $\{T(ea_ie)\}$ converges σ-weakly to $T(eae)$

in M_e^G. Since T is positive, it is enough to show that for a sequence $\{\xi n\}$ in $e(H)$ with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$,

$$(2.1) \qquad \sum_{i=1}^{\infty} \langle T\,(ea_ie)\,\xi n, \xi_n\rangle \to \sum_{i=1}^{\infty} \langle T\,(eae)\,\xi_n, \xi_n\rangle$$

By the definition of T and the direct sum expression of $e(H)$ as $\sum_\alpha \oplus e_\alpha(H)$ we can write (2.1) as

$$(2.2) \qquad \sum_{n=1}^{\infty}\sum_{\alpha} \langle T_\alpha\,(a_{i\alpha})\,\xi_n^\alpha, \xi_n^\alpha\rangle \quad \to \sum_{n=1}^{\infty}\sum_{\alpha} \langle T_\alpha\,(a_\alpha)\,\xi_n^\alpha, \xi_n^\alpha\rangle$$

For each n, ξn belongs to $e(H)$ has the form $\{\xi_n^\alpha\}$ as an element of $\sum_\alpha \oplus e_\alpha(H)$ with $\sum_n \|\xi_n^\alpha\|^2 < \infty$. So ξ_n^α is zero for all but a countable number of values of α for each n and hence (2.2) can be reduced to the following form

$$(2.3) \qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle T_m(a_{im})\xi_n^m, \xi_n^m\rangle \quad \to \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle T_m\,(a_m)\,\xi_n^m, \xi_n^m\rangle$$

Now, the σ-weak convergence of $ea_ie = \sum \oplus a_{i\alpha}$ to $eae = \sum \oplus a_\alpha$ implies that $\{a_{i\alpha}\}$ converges σ-weakly to a_α and $T_\alpha(a_{i\alpha}) \to T_\alpha(a_\alpha)$ σ-weakly, for each α. For each α, since $\sum_{n=1}^{\infty} \|\xi_n^\alpha\|^2 \leq \quad \sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$, we have

$$\sum_{n=1}^{\infty} \langle T_\alpha\,(a_{i\alpha})\,\xi_n^\alpha, \xi_n^\alpha\rangle \to \sum_{n=1}^{\infty} \langle T_\alpha\,(a_\alpha)\,\xi_n^\alpha, \xi_n^\alpha\rangle .$$

So the same is true for each $m = 1, 2, 3, \ldots$.

$$(2.4) \qquad \sum_{n=1}^{\infty} \langle T_m(a_{im})\xi_n^m, \xi_n^m\rangle \quad \to \quad \sum_{n=1}^{\infty} \langle T_m\,(a_m)\,\xi_n^m, \xi_n^m\rangle .$$

Set $d_{n,m} = \langle T_m(a_{im})\xi_n^m, \xi_n^m\rangle$ for $n, m = 1, 2, \ldots$ and $\sum_{m=1}^{\infty} |d_{n,m}| = b_n$. Then

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle T_m(a_{im})\xi_n^m, \xi_n^m\rangle| \leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \|T_m\| \quad \|a_{im}\|\,\|\xi_n^m\|^2 < \infty.$$

Thus $\sum_{n=1}^{\infty} b_n$ is convergent for each i and by the properties of double series, we can interchange the order of summation in (2.3). Thus (2.3) can be written as

$$(2.5) \qquad \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle T_m(a_{im})\xi_n^m, \xi_n^m\rangle \quad \to \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \langle T_m\,(a_m)\,\xi_n^m, \xi_n^m\rangle .$$

Now set

$$f_{ai}(m) = \textstyle\sum_{n=1}^{\infty} \langle T_m(a_{im})\xi_n^m, \xi_n^m\rangle$$
$$f_a(m) = \textstyle\sum_{n=1}^{\infty} \langle T_m(a_m)\xi_n^m, \xi_n^m\rangle ,$$

for $m = 1, 2, \ldots$. Then (2.4) can be written as $f_{ai}(m) \to f_a(m)$, for $m = 1, 2, \ldots$. Also,

$$f_{ai}(m) = \sum_{n=1}^{\infty} \langle T_m(a_{im})\xi_n^m, \xi_n^m \rangle \leq \text{ a constant.} \tag{2.6}$$

Now consider $\mathbb{N}$, the set of natural numbers and μ, the counting measure on $\mathbb{N}$. Then for a measurable function f on $\mathbb{N}$, we have

$$\int_{\mathbb{N}} f d\mu = \int_{\bigcup_{n=1}^{\infty}(n)} f d\mu = \sum_{n=1}^{\infty} \int_{\{n\}} f d\mu.$$

By the Lebesgue dominated convergence theorem, (2.4) and (2.6) can be combined together to read

$$\int_{\mathbb{N}} f_{a_i} d\mu \quad \to \int_{\mathbb{N}} f_a d\mu \int_{\mathbb{N}} f_{a_i} d\mu \quad \to \int_{\mathbb{N}} f_a d\mu.$$

That is, $\sum_{m=1}^{\infty} f_{ai}(m) \quad \to \sum_{m=1}^{\infty} f_a(m)$. Further,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle T_m(a_{im})\xi_n^m, \xi_n^m \rangle \quad \to \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \langle T_m(a_m)\xi_n^m, \xi_n^m \rangle,$$

which is (2.5) and hence T is σ-weakly continuous.

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