On α-Kenmotsu manifolds satisfying certain conditions

Hakan Öztürk, Nesip Aktan and Cengizhan Murathan

Abstract. The object of this paper is to study α-Kenmotsu manifolds which can be derived from almost contact Riemannian manifolds satisfying some certain conditions. We first examine the generalized recurrent α-Kenmotsu manifolds, and next we give some relations about Ricci semi-symmetric and D-conformal curvature tensors. We show that Ricci semi-symmetric α-Kenmotsu manifolds are also Einstein manifolds. Furthermore, we prove that the scalar curvature of α-Kenmotsu manifolds with η-parallel Ricci tensors is constant. We conclude the paper with an example on α-Kenmotsu manifolds.

M.S.C. 2000: 53D10, 53C25, 53C21.
Key words: α-Kenmotsu manifold; Generalized recurrent manifold; Ricci semi symmetric; D-conformal curvature tensor; η-parallel Ricci tensor.

1 Introduction

A $(2n+1)$-dimensional differentiable manifold M of class C^∞ is said to have an almost contact structure if the structural group of its tangent bundle reduces to $U(n) \times 1$, (see [2], [19]) equivalently an almost contact structure is given by a triple (ϕ, ξ, η) satisfying certain conditions (see Section 2). Many different types of almost contact structures are defined in the literature (cosymplectic, almost cosymplectic, Sasakian, Quassi Sasakian, α-Kenmotsu, almost α-Kenmotsu,..., [12], [21]).

The main purpose of this paper is to investigate the class of almost contact metric manifolds which are called α-Kenmotsu manifolds. These manifolds appear for the first time in (see [9]), where they have been locally classified. The author characterized the warped product space by tensor equations.

In Section 1, we give some basic definitions on Riemannian manifolds. According to these definitions, we introduce almost contact structure and α-Kenmotsu manifolds for n-dimension in Section 2. In Section 3, we study generalized recurrent α-Kenmotsu manifolds and we prove some theorems about the scalar curvature of the manifolds. In Section 4, we consider the Ricci semi-symmetric condition and show that Ricci semi-symmetric α-Kenmotsu manifolds are Einstein manifolds too. In Section 5, we deal with α-Kenmotsu manifolds whose D-conformal curvature tensor is irrotational.

Applied Sciences, Vol.12, 2010, pp. 115-126.

In Section 6, we examine α-Kenmotsu manifold with η-parallel Ricci tensor. We prove that α-Kenmotsu manifolds with η-parallel Ricci tensors have constant scalar curvatures. Finally, in Section 7 we construct an illustrating example on α-Kenmotsu manifolds.

2 Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold. We denote by ∇ the covariant differentiation with respect to the Riemannian metric g. Then we have

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor and the Ricci tensor of M are defined by

$$R(X,Y,Z,W) = g\left(R(X,Y)Z, W\right),$$

and

$$S(X,Y) = trace\left\{Z \to R(X,Z)Y\right\},$$

respectively. Locally, the Ricci tensor is also given by

$$S(X,Y) = \sum_{i=1}^{n} R(X, E_i, Y, E_i),$$

where $\{E_1, E_2, ..., E_n\}$ is a local orthonormal frame and X, Y, Z, W are vector fields on M.

The Ricci operator Q is a tensor field of type $(1,1)$ on M defined by

$$g(QX, Y) = S(X, Y),$$

for all vector fields on M.

A Riemannian manifold (M, g) is called a generalized recurrent Riemannian manifold (see [3]) if the curvature tensor R satisfies the following condition

$$(\nabla_X R)(Y,Z)W = \psi(X) R(Y,Z)W + \beta(X)\left[g(Z,W)Y - g(Y,W)Z\right], \tag{2.1}$$

where ψ and β are two 1-forms, β is non-zero and these are defined by

$$\psi(X) = g(X, A), \quad \beta(X) = g(X, B), \tag{2.2}$$

where A, B are vector fields associated with 1-forms α and β, respectively and ∇ is the Riemannian connection of g.

A Riemannian manifold (M, g) is called generalized Ricci recurrent (see [3]) if its Ricci tensor S satisfies the following condition

$$(\nabla_X S)(Y,Z) = \psi(X) S(Y,Z) + (n-1)\,\beta(X) g(Y,Z), \tag{2.3}$$

where α and β are defined as in (2.2).

A Riemannian manifold (M, g) is called generalized concircular recurrent if its concircular curvature tensor $\overline{C}$ (see [13])

$$\overline{C}(X,Y)Z = R(X,Y)Z - \frac{\tau}{n(n-1)}(g(Y,Z)X - g(X,Z)Y), \tag{2.4}$$

satisfies the following condition

$$(\nabla_X \overline{C})(Y,Z)W = \psi(X)\overline{C}(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{2.5}$$

(see [13]), where ψ and β are defined as in (2.2) and τ is the scalar curvature of (M, g).

3 α-Kenmotsu manifolds

Let M be a real n-dimensional C^∞ manifold and $\chi(M)$ the Lie algebra of C^∞ vector fields on M. An almost contact structure on M is defined by a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η on M such that for any point $p \in M$ we have

$$\eta_p(\xi_p) = 1, \quad \phi_p^2 = -I + \eta_p \otimes \xi_p, \tag{3.1}$$

which implies

$$\phi_p(\xi_p) = 0, \quad \eta_p \circ \phi_p = 0 \quad rank(\phi_p) = n-1, \tag{3.2}$$

where I denotes the identity transformation of the tangent space T_pM at the point of p. Manifolds equipped with an almost contact structure are called almost contact manifolds. A Riemannian manifold M with metric tensor g and with a triple (ϕ, ξ, η) such that

$$g(\phi X, \phi Y) = g(X,Y) - \eta\left(X\right)\eta\left(Y\right), \tag{3.3}$$

and

$$g(\xi, X) = \eta(X), \tag{3.4}$$

where $X, Y \in \chi(M)$, is an almost contact metric manifold. Then M is said to have a (ϕ, ξ, η, g)-structure.

An almost contact metric manifold M is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant where the 2-form Φ is defined as

$$\Phi(X,Y) = g(\phi X, Y).$$

We have known that an almost contact metric manifold (M, ϕ, ξ, η, g) is said to be normal if the Nijenhuis torsion

$$N_\phi(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X,Y] + 2d\eta(X,Y)\xi,$$

vanishes for any $X, Y \in \chi(M)$. Remarking that a normal almost α-Kenmotsu manifold is said to be α-Kenmotsu manifold ($\alpha \neq 0$), (see [12]).

Moreover, if the manifold M satisfies the following relations

$$\left(\triangledown_X \phi\right)Y = -\alpha\left[g(X, \phi Y)\xi + \eta\left(Y\right)\phi X\right], \tag{3.5}$$

and

$$\nabla_X \xi = -\alpha\phi^2 X, \tag{3.6}$$

then $(M^n, \phi, \xi, \eta, g)$ is called an α-Kenmotsu manifold (see [2], [12]), where $\bigtriangledown$ denotes the Riemannian connection of g.

On an α-Kenmotsu manifold M, the following relations are held

$$S(X,\xi) = -\alpha^2(n-1)\eta(X), \tag{3.7}$$

$$R(\xi,X)Y = \alpha^2\left[-g(X,Y)\xi + \eta(Y)X\right] \tag{3.8}$$

$$R(X,Y)\xi = \alpha^2\left[\eta(X)Y - \eta(Y)X\right], \tag{3.9}$$

$$g(R(\xi,X)Y,\xi) = \alpha^2\left[-g(X,Y) + \eta(X)\eta(Y)\right], \tag{3.10}$$

$$R(\xi,X)\xi = \alpha^2\left[X - \eta(X)\xi\right] = -\alpha^2\phi^2 X, \tag{3.11}$$

for any $X, Y \in \chi(M)$.

Since $g(QX,Y) = S(X,Y)$, we have

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y),$$

where Q is the Ricci operator.

Using the properties $g(X,\phi Y) = -g(\phi X, Y)$, $Q\phi = \phi Q$, (3.1) and (3.7), we obtain

$$S(\phi X, \phi Y) = S(X,Y) + \alpha^2(n-1)\eta(X)\eta(Y). \tag{3.12}$$

Also, we have

$$(\nabla_X \eta)(Y) = \alpha\left[g(X,Y) - \eta(X)\eta(Y)\right]. \tag{3.13}$$

4 Generalized recurrent α-Kenmotsu manifolds

In this section, we give some theorems about generalized recurrent α-Kenmotsu manifolds. At first, we give the following theorems

Theorem 4.1. *If M is a generalized recurrent α-Kenmotsu manifold, then $\alpha^2\psi - \beta$ is everywhere zero.*

Proof. Assume that M is a generalized recurrent α-Kenmotsu manifold. Then the curvature tensor R of M satisfies the condition (2.1) for any $X, Y, Z, W \in \chi(M)$. Taking $Y = W = \xi$ in (2.1), we have

$$(\nabla_X R)(\xi, Z)\xi = \psi(X)R(\xi,Z)\xi + \beta(X)[g(Z,\xi)\xi - g(\xi,\xi)Z]. \tag{4.1}$$

On the other hand, it is well-known that

$$\begin{aligned}(\nabla_X R)(\xi, Z)\xi &= \nabla_X R(\xi,Z)\xi - R(\nabla_X\xi, Z)\xi \\ &\quad -R(\xi, \nabla_X Z)\xi - R(\xi,Z)\nabla_X\xi.\end{aligned} \tag{4.2}$$

So using (3.8), (3.9) and (3.11), the equation (4.2) can be written as

$$(\nabla_X R)(\xi, Z)\xi = 0.$$

From the left hand side of the equation (4.1) and using (3.11), (3.4), we get

$$\left[\alpha^2\psi(X) - \beta(X)\right]\left[\eta(Z)\xi - Z\right] = 0.$$

Remarking that the equality $\eta(Z)\xi - Z = 0$ does not hold for an α-Kenmotsu manifold. So we obtain

$$\alpha^2\psi(X) - \beta(X) = 0,$$

for all $X \in \chi(M)$. Since the equation $\alpha^2\psi(X) - \beta(X) = 0$ is independent from the choice of the vector field X, we have $\alpha^2\psi - \beta = 0$ on M. $\square$

Theorem 4.2. *If M is a generalized Ricci-recurrent α-Kenmotsu manifold, then $\alpha^2\psi = \beta$.*

Proof. Assume that M is a generalized Ricci-recurrent α-Kenmotsu manifold. Then the Ricci tensor S of M satisfies the condition (2.3) for any $X, Y, Z \in \chi(M)$. Now, putting $Y = Z = \xi$ in (2.3), we obtain

$$(\nabla_X S)(\xi, \xi) = -\alpha^2(n-1)\psi(X) + (n-1)\beta(X). \tag{4.3}$$

On the other hand, it is known that

$$(\nabla_X S)(\xi, \xi) = 0. \tag{4.4}$$

By using (4.4) in (4.3), we get

$$-\alpha^2(n-1)\psi(X) + (n-1)\beta(X) = 0. \tag{4.5}$$

Since the equation $(n-1)\left[-\alpha^2\psi(X) + \beta(X)\right] = 0$ is independent from the choice of the vector field X, we obtain $-\alpha^2\psi + \beta = 0$ on M. This completes the proof of the theorem. $\square$

In order to find the scalar curvature of the manifold M, we have the following theorems

Theorem 4.3. *Let M be a generalized recurrent α-Kenmotsu manifold. Then the scalar curvature τ of M satisfies the below condition*

$$\tau\eta(A) = (1-n)\left[n\eta(B) + 2\alpha^2\eta(A)\right].$$

Proof. Suppose that M is a generalized recurrent α-Kenmotsu manifold. So by the using of second Bianchi identity, we get

$$\begin{aligned} &\psi(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z] \\ &+\psi\{Y)R\{Z,X)W + \beta(Y)[g(X,W)Z - g(Z,W)X] \\ &+\psi(Z)R(X,Y)W + \beta(Z)[g(Y,W)X - g(X,W)Y] \quad = 0. \end{aligned} \tag{4.6}$$

By a contraction of (4.6) with respect to Y, we have

$$\begin{aligned} &\psi(X)S(Z,W) + n\beta(X)g(Z,W) + R(Z,X,W,A) \\ &\beta(Z)g(X,W) - \beta(X)g(Z,W) \\ &-\psi(Z)S(X,W) - n\beta(Z)g(X,W) = 0. \end{aligned} \tag{4.7}$$

Contracting (4.7) over Z and W, we get

$$\tau\psi(X) + n(n-1)\beta(X) - 2S(X,A) = 0. \tag{4.8}$$

Putting $X = \xi$ in (4.8) and using (2.2) and (3.7), we obtain

$$\tau\eta(A) + n(n-1)\eta(B) + 2\alpha^2\,(n-1)\,\eta(A) = 0. \tag{4.9}$$

From the equation (4.9), we have the desired result. $\square$

Theorem 4.4. *Let M be a generalized concircular recurrent α-Kenmotsu manifold. Then the following relation*

$$\left[\left(\alpha^2+\frac{\tau}{n(n-1)}\right)\psi(X)-\beta(X)-\frac{X\left[\tau\right]}{n(n-1)}\right]=0,$$

holds for every vector field X on M, where $X[\tau]$ denote the derivative of τ with respect to the vector field X.

Proof. Suppose that M is a generalized concircular recurrent α-Kenmotsu manifold. Then the concircular curvature tensor $\overline{C}$ of M satisfies the condition (2.5) for all $X, Y, Z, W \in \chi(M)$. Taking $Y = W = \xi$ in (2.5), we get

$$(\nabla_X\overline{C})(\xi,Z)\xi = a(X)\overline{C}(\xi,Z)\xi+\beta(X)[g(Z,\xi)\xi-g(\xi,\xi)Z]. \tag{4.10}$$

On the other hand, from the definition of covariant derivative we have

$$\begin{aligned}(\nabla_X\overline{C})(\xi,Z)\xi &= \nabla_X\overline{C}(\xi,Z)\xi-\overline{C}(\nabla_X\xi,Z)\xi\\ &\quad -\overline{C}\left(\xi,\nabla_XZ\right)\xi-\overline{C}\left(\xi,Z\right)\nabla_X\xi.\end{aligned} \tag{4.11}$$

So using (2.4) and (3.8), the equation (4.11) can be written as follows

$$\begin{aligned}(\nabla_X\overline{C})(\xi,Z)\xi &= \nabla_X\left[\left(\alpha^2+\frac{\tau}{n(n-1)}\right)(Z-\eta(Z)\xi)\right]\\ &\quad -\left[\alpha\left(\alpha^2+\frac{\tau}{n(n-1)}\right)\right](\eta(X)\eta(Z)\xi-\eta\left(Z\right)X)\\ &\quad -\left[\left(\alpha^2+\frac{\tau}{n(n-1)}\right)\right](\nabla_XZ-\eta(\nabla_XZ)\xi)\\ &\quad -\left[\alpha\left(\alpha^2+\frac{\tau}{n(n-1)}\right)\right](\eta(Z)\eta(X)\xi-g\left(X,Z\right)\xi).\end{aligned}$$

Hence, we obtain

$$(\nabla_X\overline{C})(\xi,Z)\xi=\frac{1}{n(n-1)}X\left[\tau\right](Z-\eta(Z)\xi). \tag{4.12}$$

On the other hand, by using (4.12), the equation (4.10) can be written as

$$\begin{aligned}&\psi(X)\overline{C}(\xi,Z)\xi+\beta(X)\left(\eta(Z)\xi-Z\right)\\ =\ &(Z-\eta(Z)\xi)\left(\left[\left(\alpha^2+\frac{\tau}{n(n-1)}\right)\psi(X)-\beta(X)\right]\right).\end{aligned} \tag{4.13}$$

So from the equation (4.12) and (4.13), we get

$$\left[\left(\alpha^2+\frac{\tau}{n(n-1)}\right)\psi(X)-\beta(X)-\frac{X\left[\tau\right]}{n(n-1)}\right](Z-\eta(Z)\xi)=0. \tag{4.14}$$

Since the equality $Z-\eta(Z)\xi=0$ does not hold, the equation (4.14) clearly completes the proof. □

5 Ricci semi-symmetric α-Kenmotsu manifolds

In this section, we suppose that n-dimensional α-Kenmotsu manifolds which satisfy the following condition

$$(R(X,Y)\cdot S)(Z,U)=0. \tag{5.1}$$

So we have

$$(R(X,Y)\cdot S)(Z,U)=R(X,Y)S(Z,U)-S(R(X,Y)Z,U)-S(Z,R(X,Y)U)=0,$$

for any X,Y,Z and $U\in\chi(M)$. Taking $X=\xi$ and $Z=\xi$ in (5.1), we get

$$S(Y,U)=-\alpha^2(n-1)g(Y,U).$$

by using (3.8) and (3.11). Hence, we can give the following theorem

Theorem 5.1. *A Ricci semi-symmetric α-Kenmotsu manifold is an Einstein manifold.*

In a similar, if $(R(X,Y)\cdot R)(Z,U)=0$, we obtain

$$S(Y,U)=(-\alpha^2 n+1)g(Y,U),$$

for any X,Y,Z and $U\in\chi(M)$. It is important to note that $R\cdot R=0\subset R\cdot S=0$. Since $R\cdot R=0$ implies $R\cdot S=0$, we can state the following corollary

Corollary 5.1. *A semi-symmetric α-Kenmotsu manifold is an Einstein manifold.*

6 D-conformal curvature tensor on α-Kenmotsu manifolds

In this section, we give definitions related to the D-conformal curvature tensor

Definition 6.1. The D-conformal curvature tensor B on a Riemannian manifold M $(n>4)$ is defined as

$$\begin{aligned} B(X,Y)Z \;=\;& R(X,Y)Z+\frac{1}{n-3}[S(X,Z)Y-S(Y,Z)X+g(X,Z)QY\\ &-g(Y,Z)QX-S(X,Z)\eta(Y)\xi+S(Y,Z)\eta(X)\xi\\ &-\eta(X)\eta(Z)QY+\eta(Y)\eta(Z)QX]\\ &-\frac{(k-2)}{(n-3)}[g(X,Z)Y-g(Y,Z)X]\\ &+\frac{k}{(n-3)}[g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(Z)\xi X,\\ &+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)] \end{aligned} \tag{6.1}$$

where $k=\dfrac{\tau+2(n-1)}{(n-2)}$ and τ is the scalar curvature of the manifold M.

Definition 6.2. The rotation (curl) of the D-conformal curvature tensor B on a Riemannian manifold is given by

$$\begin{aligned} RotB &= (\nabla_V B)(X,Y,Z) + (\nabla_X B)(V,Y,Z) \\ &\quad +(\nabla_Y B)(V,Y,Z) - (\nabla_Z B)(X,Y,V). \end{aligned} \tag{6.2}$$

By virtue of second Bianchi identity

$$(\nabla_V B)(X,Y)Z) + (\nabla_X B)(Y,V)Z) + (\nabla_Y B)(V,X)Z) = 0, \tag{6.3}$$

(6.2) reduces to

$$\operatorname{curl} B = -(\nabla_Z B)(X,Y)V. \tag{6.4}$$

If the D-conformal curvature tensor is irrotational then $\operatorname{curl} B = 0$ and we have

$$(\nabla_Z B)(X,Y)V = 0,$$

which implies

$$\nabla_Z \{B(X,Y)V\} = B(\nabla_Z X,Y)V + B(X,\nabla_Z Y)V + B(X,Y)\nabla_Z V. \tag{6.5}$$

Putting $V = \xi$ in (6.5), we get

$$\nabla_Z \{B(X,Y)\xi\} = B(\nabla_Z X,Y)\xi + B(X,\nabla_Z Y)\xi + B(X,Y)\nabla_Z \xi. \tag{6.6}$$

So we can give the following lemmas

Lemma 6.1. *The D-conformal curvature tensor B on α-Kenmotsu manifolds satisfies the following relation*

$$B(X,Y)\xi = \lambda\left[\eta(X)Y - \eta(Y)X\right], \tag{6.7}$$

where $\lambda = -2\dfrac{(\alpha^2-1)}{(n-3)}$.

Proof. Using (3.7) and (3.9) in (6.1), we obtain (6.7). □

Lemma 6.2. *If the D-conformal curvature tensor B on α-Kenmotsu manifolds is irrotational, then the D-conformal curvature tensor B is given by*

$$B(X,Y)Z = \lambda\left[g(X,Z)Y - g(Y,Z)X\right]. \tag{6.8}$$

Proof. Using (6.7) and (3.6) in (6.6), we have

$$\begin{aligned} \nabla_Z\left(\lambda\left[\eta(X)Y - \eta(Y)X\right]\right) &= \lambda\left[\eta(X)\nabla_Z Y - \eta(\nabla_Z Y)X\right] \\ &\quad +\lambda\left[\eta(\nabla_Z X)Y - \eta(Y)\nabla_Z X\right] \\ &\quad +\alpha\left[B(X,Y)(Z - \eta(Z)\xi)\right]. \end{aligned}$$

We simplify the above equation, we get (6.8). □

Now, we can state the following theorem in order to find the scalar curvature of the manifold M

Theorem 6.1. *If the D-conformal curvature tensor B on α-Kenmotsu manifolds is irrotational, then the scalar curvature of M is given by*

$$\tau = -\left[\alpha(n-1)\right]^2 - n^2 + 1, \tag{6.9}$$

where τ is the scalar curvature of the manifold M.

Proof. By using the definition of the D-conformal curvature tensor B and (6.8), we get

(6.10)
$$\begin{aligned} R(X,Y)Z = \; & \lambda\left[g(X,Z)Y - g(Y,Z)X\right] - \tfrac{1}{n-3}[S(X,Z)Y - S(Y,Z)X \\ & + g(X,Z)QY - g(Y,Z)QX - S(X,Z)\eta(Y)\xi \\ & + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX] \\ & + \tfrac{(k-2)}{(n-3)}\left[g(X,Z)Y - g(Y,Z)X\right] \\ & - \tfrac{k}{(n-3)}\left[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(Z)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\right]. \end{aligned}$$

Let $\{E_1, E_2, ..., E_{n-1}, \xi\}$ be an orthonormal basis of the tangent space at any point. First, we apply inner product with any vector field W both two sides. Then the sum for $1 \le i \le n$ of the relation (6.10) with $X = W = E_i$, yields

$$\begin{aligned} & \left(\lambda(1-n) + \frac{\tau}{n-3} + \frac{(k-2)(1-n)}{n-3} + \frac{k}{n-3}\right) g(Y,Z) \\ & - \left(\frac{2\alpha^2(n-1) + \tau + k(2-n)}{n-3}\right)\eta(Y)\eta(Z) = 0. \end{aligned} \tag{6.11}$$

By a contraction of (6.11) with respect to Y and Z, we have

$$\tau = \lambda n(n-3) + (k-2)n + k\left(\frac{n}{1-n}\right) - k\left(\frac{2-n}{1-n}\right) + 2\alpha^2.$$

Simplifying the above equation, the proof of the theorem is clear. □

7 α-Kenmotsu manifolds with η-parallel Ricci tensor

In this section, we examine the notion of Ricci η-parallelity for an α-Kenmotsu manifold. At first, we give the definition of η-parallel Ricci tensor

Definition 7.1. The Ricci tensor S of an α-Kenmotsu manifold M is called η-parallel, if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \text{ for all } X, Y, Z \in \chi(M). \tag{7.1}$$

Let us consider an n-dimensional α-Kenmotsu manifold M with η-parallel Ricci tensor. Then we have

$$(\nabla_X S)(\phi Y, \phi Z) = \nabla_X S(\phi Y, \phi Z) - S(\nabla_X \phi Y, \phi Z) - S(\phi Y, \nabla_X \phi Z). \tag{7.2}$$

Using (3.5), (3.7), (3.12) and (3.2) in (7.2), we obtain

$$\begin{aligned} (\nabla_X S)(\phi Y, \phi Z) = \; & \nabla_X S(Y,Z) + \alpha^2(n-1)\left[\eta(Z)\nabla_X\eta(Y) + \eta(Y)\nabla_X\eta(Z)\right] \\ & + \alpha\eta(Y)\left[S(X,Z) + \alpha^2(n-1)\eta(X)\eta(Z)\right] \\ & + \alpha\eta(Z)\left[S(Y,X) + \alpha^2(n-1)\eta(X)\eta(Y)\right] - S(\nabla_X Y, Z) \\ & - S(Y, \nabla_X Z) - \alpha^2(n-1)\left[\eta(Z)\eta(\nabla_X Y) + \eta(Y)\eta(\nabla_X Z)\right]. \end{aligned} \tag{7.3}$$

Also, we have

$$(\nabla_X \eta)Y = \nabla_X \eta(Y) - \eta(\nabla_X Y), \tag{7.4}$$

and

$$\nabla_X S(Y,Z) = (\nabla_X S)(Y,Z) + S(\nabla_X Y, \phi Z) + S(Y, \nabla_X Z). \tag{7.5}$$

Using (7.4), (7.5) and (3.13), the relation (7.3) reduces to

$$\begin{aligned}(\nabla_X S)(\phi Y, \phi Z) &= (\nabla_X S)(Y,Z) + \alpha^3(n-1)\left[\eta(Z)g(X,Y) + \eta(Y)g(X,Z)\right] \\ &\quad +\alpha\left[\eta(Z)S(Y,X) + \eta(Y)S(X,Z)\right].\end{aligned} \tag{7.6}$$

Then using (7.1) in (7.6), we obtain

$$\begin{aligned}(\nabla_X S)(Y,Z) &= -\alpha^3(n-1)\left[\eta(Z)g(X,Y) + \eta(Y)g(X,Z)\right] \\ &\quad -\alpha\left[\eta(Z)S(Y,X) + \eta(Y)S(X,Z)\right].\end{aligned} \tag{7.7}$$

Hence, we can give the following proposition

Proposition 7.1. *An α-Kenmotsu manifold M has η-parallel Ricci tensor if and only if the above equation (7.7) holds.*

Now, let $\{E_1, E_2, ..., E_{n-1}, \xi\}$ be an orthonormal basis of the tangent space at any point for $i = 1, 2, ..., n$. Taking $Y = Z = E_i$ in (7.7) and then taking summation over the index i, we get

$$\begin{aligned}\sum_{i=1}^{n}(\nabla_X S)(E_i, E_i) &= -\alpha^3(n-1)\sum_{i=1}^{n}\eta(E_i)g(X,E_i) + \eta(E_i)g(X,E_i) \\ &\quad -\alpha\sum_{i=1}^{n}\eta(E_i)S(E_i,X) + \eta(E_i)S(X,E_i) \\ &= -2\alpha^3(n-1)\eta(X) + 2\alpha^3(n-1)\eta(X)\end{aligned}$$

So we have $d\tau(X) = 0$, which implies τ is constant, where τ is the scalar curvature of the manifold M. Thus we can give the following theorem

Theorem 7.1. *If an α-Kenmotsu manifold M has η-parallel Ricci tensor, then the scalar curvature is constant.*

8 Example

We consider the 3-dimensional manifold $M^3 = \left\{(x,y,z) \in \mathbb{R}^3\right\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields are

$$e_1 = f_1(z)\frac{\partial}{\partial x} + f_2(z)\frac{\partial}{\partial y}, \;\; e_2 = -f_2(z)\frac{\partial}{\partial x} + f_1(z)\frac{\partial}{\partial y}, \;\; e_3 = \frac{\partial}{\partial z},$$

where f_1, f_2 are given by $f_1(z) = c_2 e^{-\alpha z}$, $f_2(z) = c_1 e^{-\alpha z}$, with $c_1^2 + c_2^2 \neq 0$, $\alpha \neq 0$ for constants c_1, c_2 and α. It is obvious that $\{e_1, e_2, e_3\}$ are linearly independent at each point of M^3. Let g be the Riemannian metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1, \qquad g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0$$

and given by the tensor product $g = (f_1^2 + f_2^2)^{-1}(dx \otimes dx + dy \otimes dy) + dz \otimes dz$. Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X on M^3 and ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then using linearity of g and ϕ, we have

$$\phi^2 X = -X + \eta(X)e_3, \qquad \eta(e_3) = 1, \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields on M^3.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we get

$$[e_1, e_3] = \alpha e_1, \qquad [e_2, e_3] = \alpha e_2, \qquad [e_1, e_2] = 0.$$

Using Koszul's formula, the Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) - g(Z,[X,Y]). \end{aligned}$$

Koszul's formula yields

$$\begin{array}{llllll} \nabla_{e_1} e_1 = -\alpha e_3, & \nabla_{e_1} e_2 = -e_3, & \nabla_{e_1} e_3 = \alpha e_1, \\ \nabla_{e_2} e_1 = -e_3, & \nabla_{e_2} e_2 = -\alpha e_3, & \nabla_{e_2} e_3 = \alpha e_2, \\ \nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = 0. \end{array}$$

Thus it can be easily seen that $(M^3, \phi, \xi, \eta, g)$ is an α-Kenmotsu manifold. Hence, one can easily obtain by simple calculation that the curvature tensor components are as follows

$$\begin{array}{ll} R(e_1,e_2)e_1 = \alpha(\alpha e_2 - e_1), & R(e_1,e_2)e_2 = \alpha(e_2 - \alpha e_1), \\ R(e_1,e_2)e_3 = 0, & R(e_1,e_3)e_1 = \alpha^2 e_3, \\ R(e_1,e_3)e_2 = \alpha e_3, & R(e_1,e_3)e_3 = -\alpha^2 e_1, \\ R(e_2,e_3)e_1 = \alpha e_3, & R(e_2,e_3)e_2 = \alpha^2 e_3, \\ R(e_2,e_3)e_3 = -\alpha^2 e_2. & \end{array}$$

References

[1] K. Arslan, C. Murathan, C. Özgür, *On ϕ-conformally flat contact metric manifolds*, Balkan J. Geom. Appl. 5, 2 (2000), 1-7.

[2] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics 203, Birkhâuser 2002.

[3] U. C. De and N. Guha, *On generalized recurrent manifolds*, Proc. Math. Soc 7 (1991), 7-11.

[4] U.C. De, N. Guha and D. Kamilya, *On generalized Ricci recurrent manifolds*, Tensor(N.S), 56, 3 (1995), 312-317.

[5] R.S. Dubey, *Generalized recurrent spaces*, Indian J. Pure and Appl. Math., 10, 12 (1979), 1508-1512.

[6] N. Guha, *On generalised Ricci-recurrent Sasakian manifolds*, Bull. Calcutta Math. Soc. 92, 5 (2000), 361-364.

[7] S. Hong, C. Özgür and M.M. Tripathi, *On some special classes of Kenmotsu manifolds*, Kuwait J. Sci. Eng. (to appear).

[8] J.B. Jun, U.C. De and G. Pathak, *On Kenmotsu Manifolds*, J. Korean Math. Soc. 42, 3 (2005), 435-445.

[9] K. Kenmotsu, *A class of contact Riemannian manifold*, Tohoku Math. Jour. 24 (1972), 93-103.

[10] Q. Khan, *On generalized recurrent Sasakian manifolds*, Kyungpook Math. J. 44, 2 (2004), 167-172.

[11] J.S. Kim, R. Prasad and M. M. Tripathi, *On generalized Ricci-recurrent trans-Sasakian manifolds*, J. Korean Math. Soc. 39, 6 (2002), 953-961.

[12] T.W. Kim, H.K. Pak, *Canonical foliations of certain classes of almost contact metric structures*, Acta Math. Sinica, Eng. Ser. Aug., 21, 4 (2005), 841-846.

[13] Y. B. Maralabhavi and M. Rathnamma, *Generalized recurrent and con-circular recurrent manifolds*, Indian J. Pure and Appl. Math., 30, 11 (1999), 1167-1171.

[14] C. Özgür, *Contact metric manifolds with cyclic-parallel Ricci tensor*, Differ. Geom. Dyn. Syst. 4 (2002), 21-25.

[15] C. Özgür, *On weak symmetries of Lorentzian para Sasakian manifolds*, Radovi Matamatički 11 (2002), 263-270.

[16] C. Özgür, *On weakly symmetric Kenmotsu manifolds*, Differ. Geom. Dyn. Syst. 8 (2006), 204-209.

[17] C. Özgür, *On Generalized Recurrent Kenmotsu Manifolds*, World Applied Sciences Journal, 2, 1 (2007), 29-33.

[18] E.M. Patterson, *Some theorems on Ricci-recurrent spaces*, J. London Math. Soc. 27 (1952), 287-295

[19] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, 3. World Scientific 1984.

[20] A. Yildiz, U.C. De, *On a type of Kenmotsu manifolds*, Differ. Geom. Dyn. Syst. (DGDS) 12 (2010), 289-298.

[21] Z. Olszak, *On almost cosymplectic manifolds*, Kodai Math, 4, 2 (1981), 239-250.

Authors' addresses:

Hakan Öztürk and Nesip Aktan
Afyon Kocatepe University, Faculty of Art and Sciences,
Department of Mathematics, Afyonkarahisar, Turkey.
E-mail: hozturk@aku.edu.tr, naktan@aku.edu.tr

Cengizhan Murathan
Uludağ University, Faculty of Art and Sciences,
Department of Mathematics, Bursa, Turkey.
E-mail: cengiz@uludag.edu.tr