On the properties of invariants of forms

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Abstract. This paper provides characterizations of certain invariants in the theory of forms, and discusses their properties.

M.S.C. 2000: 34C14, 11F37, 19GXX.
Key words: Invariant; homogeneous polynomial; weight.

1 Introduction

At first, we point out briefly a geometrical interpretation of the theory of forms. Clearly, the theory of binary forms is identical with the geometry of a line (or a bundle of lines and planes). Analogously, one realizes that a ternary form is identical to the geometry of the plane, namely of algebraic curves. For instance, the elliptic curve $y^2 = x^3 + ax + b$ which plays a key role in analytic function theory [1], can be identified as the algebraic variety corresponding to a certain homogeneous ternary cubic. See [2, 3, 9] for the remarkable connection between elliptic curves, modular forms, and Fermat's Last theorem.

Finally, the theory of quaternary forms is an essential tool in studying the geometry of space, especially of algebraic surfaces. The forms with more than three variables do not admit such a geometric interpretation [4]. Theory of invariants plays a vital role in some areas (see [6] and [5]), specially in the theory of forms. Its origins can be traced back to Cayley (1845), who used the term "hyperdeterminants" for functions possessing the invariant property. So, we are led to essentially new and deep properties of forms through the application of this theory to the theory of forms.

2 Properties of invariants

In the following we shall initially limit ourselves to the theory of binary forms and use them to clarify the general concepts. The generalization of binary forms to forms with arbitrarily many variables poses no difficulties in most cases. We always write the general binary form of order n as:

$$f^{(n)}(x, y) = a_0 x^n + \binom{n}{1} a_1 x^{n-1} y + \cdots + a_n y^n.$$

Applied Sciences, Vol.12, 2010, pp. 109-114.

Example 2.1. Consider the homogeneous quadratic polynomial

$$Q(x,y) = ax^2 + 2bxy + cy^2 \tag{2.1}$$

in two variables x, y. Clearly we can recover the inhomogeneous quadratic polynomial from the associated quadratic form and vice versa.

A glance observation shows that any invertible linear transformation

$$\overline{x} = \alpha x + \beta y, \qquad \overline{y} = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma \neq 0 \tag{2.2}$$

will map a homogeneous polynomial in x and y to a homogeneous polynomial in $\overline{x}$ and $\overline{y}$ according to

$$\overline{Q}(\overline{x},\overline{y}) = \overline{Q}(\alpha x + \beta y, \gamma x + \delta y) = Q(x,y). \tag{2.3}$$

Remarkably, the discriminant of the transformed polynomial is directly related to that of the original quadratic form. A straightforward computation shows that they agree up to the square of the determinant of the coefficient matrix for the linear transformation (2.2):

$$\triangle = ac - b^2 = (\alpha\delta - \beta\gamma)^2(\overline{ac} - \overline{b}^2) = (\alpha\delta - \beta\gamma)^2\overline{\triangle}. \tag{2.4}$$

Therefore, by induction the defining equation of the invariants is

$$I(a_0', a_1', \cdots, a_n') = \delta^p I(a_0, a_1, \cdots, a_n).$$

Definition 2.2. The expressions $g = \nu_0 + \nu_1 + \cdots + \nu_n$ and $\nu_1 + 2\nu_2 + 3\nu_3 + \cdots + n\nu_n$ are called *the degree and the weight* of a function

$$I(a_0, a_1, \ldots, a_n) = \sum Z_{\nu_0\nu_1\cdots\nu_n} a_0^{\nu_0} a_1^{\nu_1} \cdots a_n^{\nu_n},$$

respectively, where $Z_{\nu_0\nu_1\cdots\nu_n}$ are numerical coefficients. See [4] for more details.

Note. In this paper, we shall assume that all terms of a function have the same weight.

Lemma 2.3. *Every linear transformation of binary forms can be composed of the following three types of linear transformations:*

$$x = \alpha x', \qquad y = \delta y', \tag{2.5}$$

$$x = x' + \beta y, \qquad y = y', \tag{2.6}$$

$$x = x', \qquad y = \gamma x' + y'. \tag{2.7}$$

Theorem 2.4. *Every invariant of a form must be homogeneous in its coefficients and every term must have degree $g = \frac{2p}{n}$, where p is the exponent of the transformation determinant by which the invariant changes under substitution of the transformed coefficients. Furthermore, all terms must have the same weight, which is also equal to p.*

Proof. Since a general linear transformation is composed of these three types of transformations, the application of a general transformation will not lead to any new results. We apply the linear transformation

$$x = \alpha x', \qquad y = \delta y'$$

to the binary form

$$f(x,y) = a_0 x^n + \binom{n}{1} a_1 x^{n-1} y + \cdots + \binom{n}{i} a_i x^{n-i} y^i + \cdots + a_n y^n.$$

The given form then becomes:

$$\begin{aligned} f &= f(\alpha x', \delta y') = a_0 \alpha^n x'^n + \tbinom{n}{1} a_1 \alpha^{n-1} x'^{n-1} \delta y' + \cdots \\ &+ \cdots + \tbinom{n}{i} a_i \alpha^{n-i} x'^{n-i} \delta^i y'^i + \cdots a_n \delta^n y'^n \\ &= a_0' x'^n + \tbinom{n}{1} a_1' x'^{n-1} y' + \cdots + \tbinom{n}{i} a_i' x'^{n-i} y'^i + \cdots + a_n' y'^n, \end{aligned}$$

where

$$a_0' = a_0 \alpha^n, a_1' = a_1 \alpha^{n-1} \delta, \ldots, a_i' = a_i \alpha^{n-i} \delta^i, \ldots, a_n' = a_n \delta^n.$$

Now, let

$$I(a_0, a_1, \cdots, a_n) = \sum Z_{\nu_0 \nu_1 \cdots \nu_n} {a_0}^{\nu_0} {a_1}^{\nu_1} \cdots {a_n}^{\nu_n}$$

be an invariant of the form f, where the Z are numerical coefficients. Then we have the identity

$$\begin{aligned} I(a_0', a_1', \cdots, a_n') &= I(a_0 \alpha^n, a_1 \alpha^{n-1} \delta, \cdots, a_n \delta^n) \\ &= \sum \{ Z_{\nu_0 \nu_1 \cdots \nu_n} {a_0}^{\nu_0} \alpha^{n\nu_0} {a_1}^{\nu_1} \alpha^{(n-1)\nu_1} \delta^{\nu_1} \\ &\quad \cdots {a_i}^{\nu_i} \alpha^{(n-i)\nu_i} \delta^{i\nu_i} \cdots {a_n}^{\nu_n} \delta^{n\nu_n} \} \\ &= \sum \{ Z\nu_0\nu_1 \cdots \nu_n {a_0}^{\nu_0} {a_1}^{\nu_1} \cdots {a_i}^{\nu_i} \cdots {a_n}^{\nu_n} \\ &\quad . \alpha^{n\nu_0 + (n-1)\nu_1 + \cdots + (n-i)\nu_i + \cdots + \nu_{n-1}} \delta^{\nu_1 + \cdots + i\nu_i + n\nu_n} \} \\ &= \alpha^p \delta^p \sum Z_{\nu_0 \nu_1 \cdots \nu_n} {a_0}^{\nu_0} {a_1}^{\nu_1} \cdots {a_n}^{\nu_n}, \end{aligned}$$

from which we obtain the identities

$$\begin{aligned} n\nu_0 + (n-1)\nu_1 + \cdots + (n-i)\nu_i + \cdots + \nu_{n-1} &= p, \\ \nu_1 + \cdots + i\nu_i + \cdots + (n-1)\nu_{n-1} + n\nu_n &= p. \end{aligned}$$

Additionally to these formulas we infer: $n(\nu_0 + \nu_1 + \cdots + \nu_n) = 2p$. □

So, the two characteristic equations we have found are the following

$$\nu_1 + 2\nu_2 + 3\nu_3 + \cdots + n\nu_n = p, \qquad ng = 2p.$$

On the other hand, the theorem just proven has evidently a converse.

Theorem 2.5. *Every homogeneous function of the coefficients a, which satisfies the equation* $ng - 2p = 0$, *where* g *is the degree,* p *the weight of this function, and* n *the order of the base form, is an invariant with respect to transformation (2.5).*

The proof is evident; if one substitutes the a' instead of the a into I, then a factor $\alpha^p \delta^p$ appears because of the assumed identities. So, we avoid repeating it.

3 The operation symbols $\mathcal{D}$ and Δ

We introduce two special operational symbols

$$\begin{aligned}\mathcal{D} &= a_0\frac{\partial}{\partial a_1}+2a_1\frac{\partial}{\partial a_2}+3a_2\frac{\partial}{\partial a_3}+\cdots+na_{n-1}\frac{\partial}{\partial a_n},\\ \Delta &= na_1\frac{\partial}{\partial a_0}+(n-1)a_2\frac{\partial}{\partial a_1}+(n-2)a_3\frac{\partial}{\partial a_2}+\cdots+a_n\frac{\partial}{\partial a_{n-1}}.\end{aligned}$$

that are two infinitesimal generators of the unimodular group.

Lemma 3.1. *The operations $\mathcal{D}$ and Δ applied to a homogeneous function, leave the degree g in the a_i unchanged; $\mathcal{D}$ lowers the weight p by 1, that is, to $p-1$; Δ raises it by 1, to $p+1$. Also, homogeneity is preserved by these operations.*

Theorem 3.2. *If $\mathcal{A}$ is a homogeneous function in the a_i, of degree g and weight p, then*

$$(\mathcal{D}\Delta-\Delta\mathcal{D})\mathcal{A}=(ng-2p)\mathcal{A}. \tag{3.1}$$

Proof. We prove this theorem in three steps:
1. If formula (3.1) is valid for two expressions $\mathcal{A}_1$ and $\mathcal{A}_2$ which have the same degree g and the same weight p, then it is also valid for the sum $\mathcal{A}_1+\mathcal{A}_2$. From

$$\begin{aligned}(\mathcal{D}\Delta-\Delta\mathcal{D})\mathcal{A}_1 &= (ng-2p)\mathcal{A}_1,\\ (\mathcal{D}\Delta-\Delta\mathcal{D})\mathcal{A}_2 &= (ng-2p)\mathcal{A}_2,\end{aligned}$$

it follows immediately by addition that

$$(\mathcal{D}\Delta-\Delta\mathcal{D})(\mathcal{A}_1+\mathcal{A}_2)=(ng-2p)(\mathcal{A}_1+\mathcal{A}_2).$$

2. If the formula is valid for $\mathcal{A}_1$(degree= g_1, weight= p_1) and $\mathcal{A}_2$(degree= g_2, weight= p_2), then it is also valid for the product $\mathcal{A}_1\mathcal{A}_2$. The product is easily seen to be a homogeneous function of degree $g=g_1+g_2$ and weight $p=p_1+p_2$. According to our assumptions, we have now

$$\begin{aligned}(\mathcal{D}\Delta-\Delta\mathcal{D})\mathcal{A}_1=(ng_1-2p_1)\mathcal{A}_1,\\ (\mathcal{D}\Delta-\Delta\mathcal{D})\mathcal{A}_2=(ng_2-2p_2)\mathcal{A}_2.\end{aligned}$$

Using the rules, one finds for the product:

$$\begin{aligned}(\mathcal{D}\Delta-\Delta\mathcal{D})(\mathcal{A}_1\mathcal{A}_2) &= \mathcal{D}\Delta(\mathcal{A}_1\mathcal{A}_2)-\Delta\mathcal{D}(\mathcal{A}_1\mathcal{A}_2)\\ &= \mathcal{D}\{\mathcal{A}_1\Delta\mathcal{A}_2+\mathcal{A}_2\Delta\mathcal{A}_1\}-\Delta\{\mathcal{A}_1\mathcal{D}\mathcal{A}_2+\mathcal{A}_2\mathcal{D}\mathcal{A}_1\}\\ &= \mathcal{A}_1\mathcal{D}\Delta\mathcal{A}_2+\{\Delta\mathcal{A}_2\}\{\mathcal{D}\mathcal{A}_1\}+\mathcal{A}_2\mathcal{D}\Delta\mathcal{A}_1+\{\Delta\mathcal{A}_1\}\{\mathcal{D}\mathcal{A}_2\}\\ &\quad -\mathcal{A}_1\Delta\mathcal{D}\mathcal{A}_2-\{\mathcal{D}\mathcal{A}_2\}\{\Delta\mathcal{A}_1\}-\mathcal{A}_2\Delta\mathcal{D}\mathcal{A}_1-\{\mathcal{D}\mathcal{A}_1\}\{\Delta\mathcal{A}_2\}\\ &= \mathcal{A}_1\{\mathcal{D}\Delta\mathcal{A}_2-\Delta\mathcal{D}\mathcal{A}_2\}+\mathcal{A}_2\{\mathcal{D}\Delta\mathcal{A}_1-\Delta\mathcal{D}\mathcal{A}_1\}\\ &= (ng_2-2p_2)\mathcal{A}_1\mathcal{A}_2+(ng_1-2p_1)\mathcal{A}_2\mathcal{A}_1\\ &= (n(g_1+g_2)-2(p_1+p_2))\mathcal{A}_1\mathcal{A}_2;\end{aligned}$$

Therefore, indeed

$$(\mathcal{D}\Delta - \Delta\mathcal{D})(\mathcal{A}_1\mathcal{A}_2) = (ng - 2p)\mathcal{A}_1\mathcal{A}_2$$

3. The formula is valid for $\mathcal{A}$ =Const, since then it only says $0 = 0$, and for $\mathcal{A} = a_i$, we have

$$\begin{aligned} \mathcal{D}\Delta a_i &= \mathcal{D}(n-i)a_{i+1} = (n-i)(i+1)a_i, \\ \Delta\mathcal{D}a_i &= \Delta i a_{i-1} = i(n-i+1)a_i. \end{aligned}$$

Consequently,

$$(\mathcal{D}\Delta - \Delta\mathcal{D})a_i = (ni - i^2 + n - i - in + i^2 - i)a_i = (n - 2i)a_i.$$

Finally the general theorem follows directly from these three facts. □

We shall need to derive two general formulas from formula (3.1). Let $\mathcal{A}$ be a homogeneous function of degree g and weight p. Then $\mathcal{D}\mathcal{A}$ is a homogeneous function of degree g and weight $p-1$, and generally, if we let

$$\mathcal{D}^2 = \mathcal{D}\mathcal{D}, \qquad \mathcal{D}^3 = \mathcal{D}\mathcal{D}^2, \qquad \cdots$$

then $\mathcal{D}^k\mathcal{A}$ is a homogeneous function of degree g and weight $p-k$. Thus, with straightforward computations we have the following formulas:

$$\begin{aligned} \mathcal{D}^2\Delta - \Delta\mathcal{D}^2 &= 2(ng - 2p + 1)\mathcal{D}, \\ \mathcal{D}\Delta^2 - \Delta^2\mathcal{D} &= 2(ng - 2p - 1)\Delta. \end{aligned}$$

Theorem 3.3. *The following two important formulas are valid in general:*

$$\mathcal{D}^k\Delta - \Delta\mathcal{D}^k = k(ng - 2p + k - 1)\mathcal{D}^{k-1}, \tag{3.2}$$

$$\mathcal{D}\Delta^k - \Delta^k\mathcal{D} = k(ng - 2p - k + 1)\Delta^{k-1}. \tag{3.3}$$

Proof. This is easily proven by induction from k to $k+1$. Namely, suppose the formulas (3.1), (3.2) are valid for k. We first apply the operation $\mathcal{D}$ to (3.2), then formula (3.1) to $\mathcal{D}^k$, likewise, we first apply the operation Δ to (3.3), then formula (3.1) to Δ^k. Then we obtain the following identities:

$$\begin{aligned} \mathcal{D}^{k+1}\Delta - \mathcal{D}\Delta\mathcal{D}^k &= k(ng - 2p + k - 1)\mathcal{D}^k, \\ \mathcal{D}\Delta\mathcal{D}^k - \Delta\mathcal{D}^{k+1} &= (ng - 2(p-k))\mathcal{D}^k, \\ \Delta\mathcal{D}\Delta^k - \Delta^{k+1}\mathcal{D} &= k(ng - 2p - k + 1)\Delta^k, \\ \mathcal{D}\Delta^{k+1} - \Delta\mathcal{D}\Delta^k &= (ng - 2(p+k))\Delta^k. \end{aligned}$$

And if one adds the first two formulas on the one hand and the second two on the other hand, then one obtains formulas (3.2) and (3.3) for $k+1$. But since they are valid for $k = 2$, they are valid in general. For $k = 1$, they both turn into formula (3.1), provided one defines $\mathcal{D}^0\mathcal{A} = \Delta^0\mathcal{A} = \mathcal{A}$. □

We shall assume that the formulas we derived are always being applied to a homogeneous function.

4 The smallest system of conditions for the determination of the invariants

Using the relations derived in the previous section, we can substantially simplify the necessary and sufficient conditions for invariants.

Theorem 4.1. *Every homogeneous function I of the coefficients $a_0, a_1, \cdots, a_n$ of degree g and weight p, where $ng = 2p$, is an invariant if it satisfies the differential equation $\mathcal{D}I = 0$.*

Proof. Let $\mathcal{A}$ be a homogeneous function of degree g and weight p, where $ng = 2p$. Suppose further that $\mathcal{A}$ satisfies the differential equation $\mathcal{D}\mathcal{A} = 0$. Then we want to show that $\mathcal{A}$ also satisfies the differential equation $\Delta\mathcal{A} = 0$, and hence it is an invariant. Since $\Delta^k\mathcal{A}$ has degree g for all k, the weight can not take on arbitrary values; therefore, from some Δ^i on, all have to be identically zero. Let this be $\Delta^k\mathcal{A} = 0$, while $\Delta^{k-1}\mathcal{A} \neq 0$. But, according to formula (3.3) from the preceding section, we have

$$\mathcal{D}\Delta^k\mathcal{A} - \Delta^k\mathcal{D}\mathcal{A} = k(-k+1)\Delta^{k-1}\mathcal{A}.$$

Therefore, we must have $k(k-1)\Delta^{k-1}\mathcal{A} = 0$. Here it can neither happen that $\Delta^{k-1}\mathcal{A} = 0$ (because of our assumption), nor that $k = 0$, because then $\mathcal{A}$ would be identically zero, and so it follows that $k = 1$, that is $\Delta\mathcal{A} = 0$, and our assertion is proven. □

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