A note on the Uzawa-Lucas model with unskilled labor

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Abstract. In his work on Uzawa-Lucas model with unskilled labor, Robertson [8] asserts that his results do not substantially change if one relaxes the assumption of a total utility function depending on the size of the population, which means that the total utility depends only on the per capita felicity discounted by the rate of time preference. In this paper, Robertson's statement is proved analytically.

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1 Introduction

The Uzawa-Lucas model has been at the heart of macroeconomic research for many decades, and it is still widely considered as a fundamental benchmark for ongoing research programs in many fields. Elaborating on some ideas of Uzawa [9], Lucas [5] extends the optimal growth model, with its single physical production sector, to include a second sector in which human capital is produced. This two-sector model, with two controls and two state variables, induces a much larger set of optimality conditions compared to the typical Ramsey model [7], and it is an endogenous growth model. In the Uzawa-Lucas model, human capital accumulation is equivalent to unskilled labor augmentation. As noted by Barro and Sala-i-Martin [1], this means that the elasticity of output with respect to human capital is equal to the elasticity of output with respect to labor. As well, it means that the elasticity of the marginal product of physical capital with respect to human capital is equal to the elasticity of the marginal product of physical capital with respect to labor. This symmetry is a useful simplification in some applications, but inappropriate for the study of the effects of initial factor endowments on growth paths, if labor is a feature of the economy. Recently, Robertson [8] has extended the Uzawa-Lucas model, incorporating labor factor inputs, as distinct from human capital. He relaxed the assumption that human capital accumulation is equivalent to labor augmentation by assuming that aggregate human inputs are a concave function of unskilled labor and human capital. The transitional growth path of his model, under condition of unskilled labor, is very similar to the standard neoclassical model. The economy is predicted to exhibit high

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levels of physical capital investment, relatively high income growth and low rates of human capital accumulation. These results reverse the conclusion of Mulligan and Sala-i-Martin [6], and Barro and Sala-i-Martin [1], regarding the transition path of developing economies in the Uzawa-Lucas model. In a footnote of his paper, Robertson asserts that "This utility function weights consumption per person by the size of the population. Sometimes this weight is ignored, though this will not substantially change the results in the paper." The purpose of this paper is to prove analytically this statement.

2 The model

We follow Robertson [8] by considering an economy that produces a single homogeneous final good Y according to the Cobb-Douglas production function $y = k^\alpha (uh)^\beta$, where k is physical capital stock per worker, h is human capital stock per worker, u is the fraction of human capital allocated to final output, and $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$. The accumulation equations for physical and human capital are

$$\dot{k} = k^\alpha (uh)^\beta - c - nk, \tag{2.1}$$

$$\dot{h} = \delta(1-u)h - (n-m)h, \tag{2.2}$$

where c is consumption per worker, $n = \dot{N}/N$ is the exogenous growth rate of unskilled labor, $m \leq n$ is the natural growth rate of human capital, and δ is a positive parameter. If $m = n$, then we have Lucas' [5] human capital function, where labor force growth does not affect the growth rate of human capital per worker. The problem for the representative consumer is to maximize intertemporal utility given by

$$\int_0^\infty \frac{c^{1-1/\sigma}}{1-1/\sigma} e^{-\rho t} dt, \tag{2.3}$$

where $\rho > 0$ is the constant rate of time preference, and σ is the intertemporal consumption elasticity of substitution. Contrary to Robertson [8], the number of family members receiving the given utility level is no taken into account. Since empirical estimates of σ are typically around unity or smaller (see Barro and Sala-i-Martin [1]), we assume $\sigma < 1$. Notice that $\alpha < 1$ implies $\sigma\alpha < 1$. To find an optimal set of paths $\{c, k, h, u\}$ that maximizes (2.3) subject to (2.1),(2.2) and the initial endowments k_0, h_0, we consider the current-value Hamiltonian for the problem

$$H = \frac{c^{1-1/\sigma}}{1-1/\sigma} + \lambda[k^\alpha (uh)^\beta - c - nk] + \mu[\delta(1-u)h - (n-m)h],$$

where λ, μ are co-state variables. According to the Pontryagin maximum principle, the necessary conditions for a maximum are

$$H_c = 0 \;\Rightarrow\; c^{1-1/\sigma} - \lambda = 0, \tag{2.4}$$

$$H_u = 0 \quad \Rightarrow \quad \beta k^\alpha (uh)^\beta h\lambda - \delta h\mu = 0, \tag{2.5}$$

$$\dot{\lambda} = \rho\lambda - H_k \quad \Rightarrow \quad \dot{\lambda} = \lambda\left[\rho - \alpha k^{\alpha-1}(uh)^\beta + n\right], \tag{2.6}$$

$$\dot{\mu} = \rho\mu - H_h \quad \Rightarrow \quad \dot{\mu} = \rho\mu - \lambda[\beta k^\alpha (uh)^{\beta-1} u] - \mu[\delta(1-u) - (n-m)], \tag{2.7}$$

plus equations (2.1),(2.2). Notice that equation (2.5) gives

$$\lambda = \frac{\delta\mu}{\beta k^\alpha (uh)^{\beta-1}}. \tag{2.8}$$

Using this to eliminate λ, we have that (2.7) simplifies to

$$\frac{\dot{\mu}}{\mu} = \rho - \delta + (n-m).$$

This completes the necessary conditions for the model. Sufficient conditions for a maximum require consideration of the balanced growth path, or steady state.

3 Balanced growth path and stability

We define a balanced growth path as a situation in which $\dot{u} = 0$, and $\dot{c}/c = M$, where M is a positive constant.

Proposition 3.1. *On a balanced path, $\dot{k}/k = M$, and all points lie on the curve in $\{k, h\}$-plane defined by equation $(M\sigma^{-1} + \rho + n)/(\alpha u_*^\beta) = k^{\alpha-1}h^\beta$. Along this curve the marginal product of capital is constant, and the marginal product of human capital is falling.*

Proof. From (2.2), when $\dot{u} = 0$, we derive that $\dot{h}/h = \delta(1-u_*) - (n-m)$, where u_* is the balanced growth path value of u. Thus, on the balanced growth path the growth rate of human capital is constant. Combining (2.4) and (2.6) gives

$$\frac{1}{\sigma}\frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda} = -\rho + \alpha k^{\alpha-1}(uh)^\beta - n,$$

so that along the balanced path it must be

$$\frac{M}{\sigma} + \rho + n = \alpha k^{\alpha-1}(u_* h)^\beta. \tag{3.1}$$

Thus, the marginal product of capital must also be constant. Equation (3.1) describes a curve in $\{k, h\}$-plane that intersects the origin and is concave with respect to the h-axis. All the points on this curve, that also satisfy $u = u_*$, are points on the balanced path. Log differentiation of (3.1) with respect to time yields the required balance between the factor inputs h and k, i.e.

$$\frac{\dot{k}}{k} = \frac{\beta}{1-\alpha}\frac{\dot{h}}{h}. \tag{3.2}$$

Equation (3.2) shows that the growth rate of physical capital must also be constant in the balanced path. In addition, it shows that the marginal product of human capital must always be falling faster (or increasing slower) than the marginal product of physical capital on a balanced path. However, from (3.1), we derive that the marginal product of physical capital is constant along the balanced path, so the marginal product of human capital must be falling. Finally, the relationship between the growth rates of consumption and capital can be derived. Let us rewrite equation (2.1) as $\dot{k}/k = k^{\alpha-1}(uh)^{\beta} - c/k - n$. It is now immediate from the above that c/k is also constant on the balanced path. In particular, $\dot{c}/c = \dot{k}/k$. $\square$

We are now going to discuss the existence of the balanced path. First, we rewrite the accumulation equations in terms of variables that are constant along a balanced path. In other words, we express the model in terms of the following variables: the consumption capital ratio, $x = c/k$, the average product of capital, $z = k^{\alpha-1}(uh)^{\beta}$, the average product of capital, and u. Using the necessary conditions, and by log differentiating $x = c/k, z = k^{\alpha-1}(uh)^{\beta}$, and (2.8), we get

$$\left\{\begin{array}{rcl} \dfrac{\dot{x}}{x} &=& x + (1-\sigma)n - \sigma\rho - (1-\sigma\alpha)z, \\ \dfrac{\dot{z}}{z} &=& \dfrac{(1-\alpha-\beta)(x+n)}{1-\beta} + \dfrac{\beta(\delta+m)}{1-\beta} - (1-\alpha)z, \\ \dfrac{\dot{u}}{u} &=& -\dfrac{\alpha}{1-\beta}x + \delta u + \dfrac{\beta(\delta+m)}{1-\beta} + \dfrac{(1-\alpha-\beta)n}{1-\beta}. \end{array}\right. \tag{3.3}$$

Setting these equations equal to zero, and solving the corresponding linear system gives the following balanced path values

$$\begin{aligned} x_* &= \frac{(1-\alpha)(1-\beta)\sigma(\rho+n) + (1-\sigma\alpha)\beta(\delta+m)}{\alpha\left[(1-\alpha-\beta)\sigma+\beta\right]} - n, \\ z_* &= \frac{(1-\alpha-\beta)\sigma(\rho+n) + \beta(\delta+m)}{\alpha\left[(1-\alpha-\beta)\sigma+\beta\right]}, \\ u_* &= \frac{(1-\alpha)\sigma(\rho+n) + (1-\sigma)\beta(\delta+m)}{\delta\left[(1-\alpha-\beta)\sigma+\beta\right]} - \frac{n}{\delta}. \end{aligned}$$

Since $\sigma\alpha < 1$, for fixed δ, ρ, there exist n and β small enough to have $0 < x_*, 0 < z_*$, and $0 < u_* < 1$.

Proposition 3.2. *The unique steady state (x_*, z_*, u_*) is a saddle point.*

Proof. The local dynamic of the dynamical system (3.3) around a steady state equilibrium $(x_*, z_*, u_*$ is determined by the signs of the eigenvalues of the Jacobian matrix corresponding to its linearized system. This means to write

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{u} \end{bmatrix} = J^* \begin{bmatrix} x - x_* \\ z - z_* \\ u - u_* \end{bmatrix},$$

where

$$J^* = \begin{bmatrix} x_* & -(1-\sigma\alpha)x_* & 0 \\ \dfrac{(1-\alpha-\beta)z_*}{1-\beta} & -(1-\alpha)z_* & 0 \\ -\dfrac{\alpha u_*}{1-\beta} & 0 & \delta u_* \end{bmatrix}$$

denotes the Jacobian matrix evaluated at the steady state. We observe that this matrix has three eigenvalues ξ_1, ξ_2, ξ_3, one of which is immediately seen to be $\xi_1 = \delta u_*$, a real and positive number. The signs of the remaining eigenvalues of J^* can be determined by looking at its trace and determinant. We have

$$trace = x_* - (1-\alpha)z_* + \delta u_* > 0 \Rightarrow \xi_1 + \xi_2 + \xi_3 > 0,$$

$$det = -\frac{(1-\sigma)(\beta+\sigma)\delta\alpha x_* z_* u_*}{1-\beta} < 0 \Rightarrow \xi_1\xi_2\xi_3 < 0.$$

Since the trace of a matrix is also equal to the sum of its eigenvalues, there must be at least one eigenvalue with positive real part. On the other hand, the determinant of a matrix is also equal to the product of its eigenvalues. Therefore, we can have either three eigenvalues with negative real parts, or one negative and two with positive real parts (recall that complex eigenvalues must occur as conjugate pairs). Since we already know that there cannot be three eigenvalues with negative real part, we can exclude the first case. Consequently, the system dynamics exhibits saddle point stability with the stable manifold, which is the hyperplane generated by the associated eigenvectors, being two dimensional (see Blume and Simon [2]). □

Theorem 3.1. *There exists a unique optimal solution.*

Proof. From the previous results, we derive that the dynamic optimal model has uniquely feasible path. So, it has uniquely optimal solution. □

4 Conclusions

In the foregoing discussion, we have presented an extension of the Uzawa-Lucas model, to incorporate unskilled labor as a separate factor. In economies with relatively stationary populations, the issue is of little matter. However, in a developing economy, the relationship between population levels and income growth is of considerable importance. In contrast to Robertson [8], we have assumed that total utility does no longer depend on the size of the population, but only on the per capita felicity discounted by the rate of time preference. Within this framework, we have demonstrated that the model has a unique non-trivial steady state equilibrium, which is a saddle point with a two dimensional stable manifold. This proves the statement done by Robertson [8] that ignoring the size of the population in the utility function does not substantially change his results. For future research, it would be interesting to analyze how a logistic population growth hypothesis might affect the dynamics of the model as well as the possibility of writing closed-form solutions in a similar way as done by Germana and Guerrini [3] for the Solow model, and by Guerrini [4] for the Mankiw-Romer-Weil model.

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