General Relativity at the turn of the third millennium

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Abstract. We shall review the development of the notion of SpaceTime and new challenges of theories of gravitation in the third millennium. We shall stress the fundations of GR in the geometry and physics of XIX Century and motivations leading to the Einstein paradigm. Then the mathematical structure of standard GR is briefly reviewed and some possible generalizations are considered. In particular the so-called *alternative theories of gravitation* are considered. Their connection to dark matter/dark energy problem is presented. Finally we shall briefly review some open problems of gravitational theories which lead to motivations for quantum SpaceTimes and gravity.

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1 Introduction and motivation

General Relativity is almost hundred years old. In spite of time passing by and in spite of the many further and important developments that Theoretical and Mathematical Physics had in XX Century (e.g., Quantum Mechanics, Quantum Field Theory, Gauge Theories, Strings and Superstrings) this beautiful theory developed by Einstein in 1915 to describe the gravitational field in terms of the curvature of SpaceTime is still fresh and alive and its paradigms still represent the basis for most attempts to "geometrize Physics".

The keynote Lecture at DGDS 2009 upon which this review paper is grounded aimed to reviewing the mathematical and physical ideas that led Einstein to formulate General Relativity (GR) - starting from Newton's and Galilei Physics and from the Theory of Special Relativity (SR), that he did already develop in 1905 — together with the reasons why GR has been and it is still so important for the development of whole branches of mathematical research, from *Differential and Riemannian Geometry* to the *theory of Ordinary and Partial Differential Equations* (*ODE's* and *PDE's*), from the *Calculus of Variations on Jet Bundles* to the theory of *topological invariants* (characteristic classes, knots, . . .) and many others; just to quote few examples see [19] and [5].

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We shall shortly and schematically present some of the most emblematic and present challenges to General Relativity (*quantization, loop quantization, dark matter, generalizations to higher dimensions and to nonlinear Lagrangians*) — as well as its hopes and its failures, alongwith the mathematical methods necessary to work in General Relativity (that form what can be called the "*mathematical legacy*" of Relativity) and some hints for future developments of GR and its extensions.

2 Euclidean Geometry, Classical Physics and Astronomy until XIX century

The key historical figures of this Section are *Euclid, J. Kepler, G. Galilei* and *I. Newton.*

Classical Physics, until XIX Century, has been based on the hypothesis that *Space* and *Time*, both "*Absolute*", exist separately and are universally valid for whichever external observer.

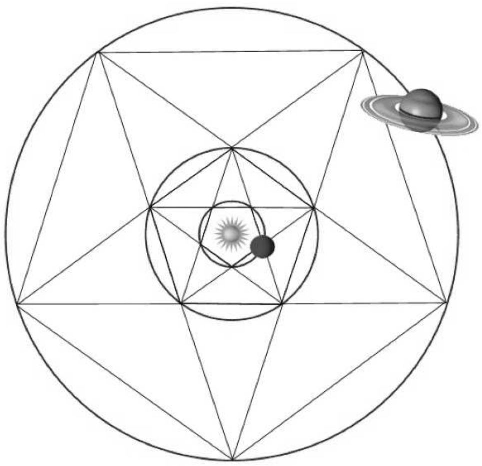

Fig.1. Johannes Kepler - Harmonices Mundi

In Classical Physics it is assumed and believed that *Time* — that continuously flows from *Past* to *Future*, through *Present* — is ticked by a "*Universal Clock*" identical for all observers. Time is *absolute* and *universal.* Also Space is absolute and universal ... Space is ruled by *Euclidean Geometry*, i.e. the Geometry of ordinary 3-dimensional Space, where *Straight Lines* and *Circles* do exist. The first ones define what in Mathematics are called the "*linear (and affine) structure*" of Space (lines, planes, parallelism, triangles, etc...) while the other ones, through the "*Theorem of Pythagoras*", define all the "metric" notions: distances, measures of angles, areas, volumes.

Fig.2. Euclid

The idea that Space and Time — together with Physics — are regulated by the rigid paradigm of Euclidean Geometry and by "absoluteness" pervades the reading of Universe's structure from the time of antiquity (Egyptian, Babylonian, Greek, Indian, Precolombian views) until the eve of XIX Century, when new but somehow pre-existing ideas of "deformability of Geometry" and of relativeness of Space begin to spread in the mind of scientists. The role that Euclidean Geometry plays in reading the structure of Cosmos is emblematically resumed by *Galileo Galilei* - who, in XVI Century, writes the following in "*Il Saggiatore*":

> Questo grandissimo libro che continuamente ci sta aperto innanzi a gli occhi (io dico l'universo), ma non si può intendere se prima non si impara a intender la lingua, e conoscer i caratteri, né quali è scritto. Egli è scritto in lingua matematica e i caratteri son triangoli, cerchi, ed altre figure geometriche senza i quali mezzi è impossibile a intendere umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto

i.e.: "This huge book is continuously open in front of our eyes (say the Universe) but it cannot be understood if one does not learn before to understand the language and the characters through which he is written. He is written in mathematical language and the characters are triangles, circles and other geometrical figures, without which it is impossibile for humans to understand a single word; without them one continuously struggles in an obscure labyrinth".

And also *R. Descartes* will write in his "*Discours sur la Méthode*" (XVII Century):

> All problems in Geometry can be easily referred to terms such that, to be constructed, one has just to know the length of some straight lines.

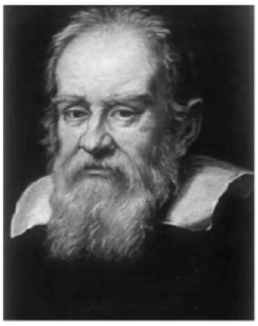

Fig.3. Galileo Galilei — a famous portrait due to Giusto de Sustermans (Firenze, Uffizi Galleries)

Much before that Cosmology begins to be a "Science" in a true sense, mankind investigates the structure of Cosmos and of its elements. It becomes clear that the Earth, the Sun, the Moon and the Planets form a "local system" while "stars" are more faraway. Slowly the center of this system is moved from being the Earth (so-called "geocentric view" of Ptolomeus) to being the Sun (*Kepler*'s view of an "helio-centric system"). Through *Classical Physics*, *Galileo* will finally investigate the Solar System in detail and make a number of fundamental discoveries for the understanding of its structure. Isaac Newton will eventually formulate his *Theory of Gravitation* [26] that exactly rules the motion of all known Planets and allows to predict theoretically all three Kepler's laws rather than establishing them phenomenologically.

Fig. 4. Isaac Newton

Newton's theory will be eventually confirmed when it will allow, on the bases of new astronomical observations, to predict the existence of yet unknown Planets or to calculate with an astonishing precision the periodical passage of celestial objects like *Halley's Comet* (already known to Chinese, in Bayeux' tapestries and painted by Giotto in his famous frescoes of the "*Cappella Scrovegni*" in Padova, Italy).

Newton's theory becomes so the *Universal Theory of Gravitation* and, as such, it will rule Physics and Astronomy until the middle of XIX Century.

3 Is Euclidean Geometry enough to understand Physics. . . .?

Across XVIII and XIX Centuries, however, new developments in Physics begin to require drastic changes in the current views about the structure of the Universe, because of the discovery of new physical phenomena, while the parallel developments of Geometry allow to relax the rigidity of the Euclidean structure to replace it with more flexible paradigms.

To fully understand these changes one should not forget the fundamental work of the Italian mathematician *Joseph Louis Lagrange* (born in Torino in 1736) who wrote his famous treatise on "Analytical Mechanics" and de-facto put the basis of both new Physics (still based to-day, at least in part, on a "variational paradigm") and of new Mathematics. His famous treatise "*Mècanique Analitique*" — at the time this was the correct spelling — was published in 1788, in Paris. Lagrange's intention was to liberate Mechanics from the need of using geometrical constructions ("*On ne trouverà point de figures dans cet ouvrage*" — [21], scholium) to let it develop under the new flag of "Mathematical Analysis", as developed by Leibnitz and Newton themselves. With this work Lagrange was in fact inventing a new way to understand also Geometry, a way that had to culminate in Gauss' and Riemann's investigations on the geometry of curved spaces.

3.1 From Non-Euclidean Geometry to new Astronomy

In his famous treatise "*Disquisitiones supra Superficies Curvas*" (1827) the German scientist *K. F. Gauss* introduces the systematic study of *surfaces*, so establishing the bases of modern *Differential Geometry.* Precursor of Riemann, he founds and develops the theory of curvature, that will later form the base of the subsequent work of Riemann on multidimensional metric spaces and indirectly also of the work of Einstein on gravity.

About twenty-five years later *B. Riemann* will deliver a famous prolusion at the University of Göttingen (given in 1854 but published posthumous only in 1866): "*On the Hypotheses that lie at the Foundations of Geometry*" ([28]). With this celebrated work Riemann lays in fact the foundations of the study of *metric spaces of arbitrary dimension* and understands that *curvature* — that generalizes Gauss' curvature of surfaces - *is a property of space that has to be deduced from astronomical observations.* In doing so he also puts the bases of Einstein's subsequent work, that will see *curvature as the counterpart of gravitational forces*; thus putting - in a renovated Cartesian view (see also [14], [15], [9]) - Space, Time, Matter and Gravity on the same footing in his celebrated gravitational equations.

Writes Riemann: *If the independence of bodies from position does not exist, we cannot draw conclusions from metric relations of the great to those of the infinitely small; in that case the curvature at each point may have an arbitrary value in three directions, provided that the total curvature of every measurable portion of space does not differ sensibly from zero [. . .] The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of space (see [28]).*

Fig.5. Bernhard Riemann

In a later passage Riemann will also write: "*The curvature of Space has to be determined on the bases of astronomical observations*".

The new Geometry of Gauss and Riemann goes well beyond the classical scheme of Euclid. *This Geometry is dominated by curvature*, while Euclidean Geometry is dominated by linearity. It encompasses a much more dynamical way of measuring lengths and areas, something that will be later understood to be also nearer to the reality that surrounds us, that only at a first approximation is linear. It will be later realized that this has to do as well with "observed reality", since also the human eye reflects a "hyperbolic world" (with a constant negative curvature) rather that a rigorously "flat" world (Lunemberg; see [27] — in a sense the eye interprets as curved what often is not, or it tends to see as flat what is instead curved . . . The Geometry of curved spaces is therefore fascinating and interesting; as a matter of fact it also permeates modern forms of Abstract Art (see, e.g., [23]).

3.2 Classical Physics of Galilei and Newton revisited

Before proceeding any further, let us then revisit the Physics of Galilei and Newton, at the light of new ideas of our modern age.

In Galilei and Newton Physics, as we already said, *Space* and *Time* are absolute and given a priori. Physics is a process that takes place in Space and in Time. *Motion* assigns the position as a function of Time. Upon Newton's first fundamental law for the motion of a single particle

$$\mathbf{F} = m\mathbf{a}$$

or, better

$$\mathbf{a} = \mathbf{F}/m$$

(that relates the geometry of motion, i.e. the *acceleration* $\mathbf{a}$, to the sources that generate it, encompassed by the *force* $\mathbf{F}$, by a strict proportionality whose factor m is the "inertial mass" of the particle) one can write the theory of all "more complicated moving systems" (rigid bodies, continua, etc...). This law is written in an absolute Space with an absolute reference system, with respect to absolute Time. Overall there is the *Principle of Inertia*: *in an absolute frame of reference a point that is not subjected to forces moves of a rectilinear and uniform motion.*

A classical way of writing this law passes through the choice of the so-called Frenet's frame (a frame that is naturally associated to any curve in Euclidean space):

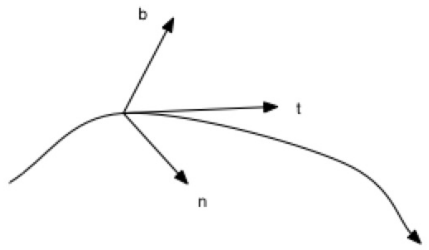

Fig.6. The Frenet reference frame in Euclidean space

In the Frenet frame the force is split as

$$\mathbf{F} = \mathbf{F}_t + \mathbf{F}_n$$

while the acceleration splits as follows

$$\mathbf{a} = (d^2s/dt^2)\mathbf{t} + 1/\rho(ds/dt)^2\mathbf{n}.$$

The force is thence split into its normal and tangential parts, as well as the acceleration, also split into its normal and tangential parts. Force represents — as we said - the source of dynamics, i.e. of motion, while the acceleration represents the curvature of the curve along which dynamics takes place. The curvature is hidden in the normal part, where ρ is in fact the inverse of the curvature (i.e., ρ is the radius of the circle having at the point a contact of order at least two with the curve — this circle is unique and turns into a straight line if the motion is rectilinear). Notice that the first term of $\mathbf{a}$ expresses here the non-uniformity of motion (that is uniform if the arc-length s is a linear function of absolute Time) while the normal part has to do with "spatial non-linearity" and is quadratic in the velocity.

We can now read again the Principle of Inertia: *in an absolute frame of reference a point that is not subjected to forces moves of a rectilinear and uniform motion. Acceleration measures the deviation from linearity in Space and Time.*

3.3 Is Classical Physics enough. . . ? From Electromagnetism to The Theory of Special Relativity

In the year 1879 the physicist J. Clerk Maxwell writes the equations governing all electromagnetic phenomena; in the same year the man destined to change the Physics of XX Century is born in Ulm (Prussia): his name is Albert Einstein.

According to Maxwell equations the speed of a light ray (in vacuum) does not depend either on the motion of the source nor on the motion of the observer. According to this prediction, at least when the light is concerned, velocities cannot sum up linearly, as one expects instead in Galilei Physics.

In a sense Electromagnetism refuses to adapt directly into the Galilean scheme. This crucial fact generates a number of speculations about the nature and immateriality of aether, as well as a flourishing of new theoretical hints for an explanation of this fact, obviously considered as impossible in the context of Galilean and Newtonian absoluteness, taken for granted.

Starting from the theoretical prediction that the speed of light (in vacuum) is a universal constant that does not depend on the motion of the source nor of the observer — something that, being a velocity the ratio of a space and a time measurements, entails the existence of an absolute relation between Space and Time, rather that a separate absoluteness of their own — Albert Einstein is forced to introduce in 1905 a new geometrical scheme to explain Electromagnetism in a combined four-dimensional continuum (as previously introduced by Minkowski) and to give Lorentz' transformations (derived already in 1895) the status of invariance laws of a new geometry rather than a "curious set of equations" to be (possibly) interpreted in Galilei-Newton separate Space and Time.

In this crucial year Einstein publishes — among other things - his celebrated theory of *Special Relativity (SR)*, in which Mechanics and Electromagnetism are finally reconciled ([7]) ... but from which Gravitation and matter are still secluded (Minkowski SpaceTime is in fact rigorously empty and flat).

In this new Physics, no Time and no Space exist a priori as separated and absolute entities. Only *SpaceTime* is given a priori and each observer separates independently Space and Time according to his choice. Space and Time are therefore bound together in an absolute entity, i.e. *SpaceTime*, since — as we already said - there is an absolute constant to tie them together, i.e. the velocity of light in vacuum (that shall eventually be denoted by c). In the new view introduced by SR, Physics is what happens *in* SpaceTime *before* an observer decides how to separate Time from Space. One has therefore to forget about Galilean and Newtonian prejudices; and through its subtle analysis of the problem of "Synchronization" Special Relativity removes Newtonian prejudices (see [22]).

A fascinating prediction of Special Relativity is the fact that *velocities do no longer sum up linearly as in Galileo's Physics* — as it cannot any longer be - but should instead follow *a more complicated non-linear composition rule that keeps into account the need of maintaining the velocity of light in vacuum as a constant* (the conceptual base of Einstein's theory, as it follows from Maxwell's equations).

Relativity entails also that *a small amount of matter contains a huge amount of energy*, as emblematically described by the probably most famous equation of Einstein's theory of SR:

$$E = mc^2.$$

This equivalence between mass and energy is in fact at the bases of Nuclear Physics.

3.4 Is Special Relativity enough...? From Minkowski SpaceTime to General Relativity

As we already recalled above, Special Relativity allows to reconcile Electromagnetism with Mechanics, but Gravitation and matter are still secluded from it. Minkowski SpaceTime is in fact found to be compatible only with an empty Universe — in which only electrically charged particles can travel — but no reasonable way to include matter and Gravitation is found to work on it. The crucial point is understood by Einstein, who — on the bases of Riemann's (and also Clifford's) views on the relation between Space, Matter and Geometry - realizes that a way out of the *empasse* will be found only when general non-linear frame transformations will be allowed and curvature somehow included in the picture from the very beginning.

Minkowski SpaceTime cannot serve for this purpose since it is rigorously flat — as Euclidean Space is — being its curvature zero. In between 1905 and 1915-1916 Einstein will work hard on the problem; after ten years of further investigations Einstein will eventually find the way to accommodate into a single theory both the notion of SpaceTime, born with Special Relativity, together with gravitational forces, that (according to Riemann) will be interpreted as an expression of the curvature of SpaceTime itself.

A first version of the theory is published already in 1912, but it is not satisfactory. Marcell Grossman enters then the scene; he teaches Einstein the "new instrument" of Differential Geometry, due to the German and Italian schools. Einstein already knows that in Newtonian Physics forces determine the curvature of trajectories in a SpaceTime given rigidly from the very beginning. He understands also that forces (and Gravity in particular) have to do the same, but *in a SpaceTime that is not given a priori.* As already pointed out by Riemann (1856) the curvature has to be given by astronomical observations. On these premises, only in 1915 the *Theory of General Relativity* will take its final form, to be finally completed in the year 1916 ([6]).

Fig.7. Albert Einstein

Einstein is led to assume that SpaceTime is no longer the flat Minkowski SpaceTime, but rather a 4-dimensional manifold M endowed with a metric g having Lorentzian signature (i.e., of "hyperbolic character"). In his celebrated equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 4\pi k T_{\mu\nu}$$

the left hand side expresses the curvature of SpaceTime (through a linear combination of the Ricci Tensor $R_{\mu\nu}$ and the Scalar Curvature R, derived linearly from the

Riemann Curvature Tensor of the Metric $g_{\mu\nu}$), that in the right hand side is set equal to the forces that generate it; what in Continuum Mechanics is usually called "the stress".

As in Newton's law, curvature is determined by the sources, but now is no longer a motion trajectory to be curved by forces into a rigid Space and a rigid Time — recall the "Principle of Inertia" - but rather SpaceTime itself to be curved by Gravity and Matter.

3.5 Which are the real differences and analogies between SR and GR. . . ?

SpaceTime is given a priori in Special Relativity; it is flat. Motion is a trajectory in SpaceTime (a "*Universe line*"). Each observer defines his own way to separate Space and Time, to obtain position as a function of Time. The true Physics — in a sense - has no Time. . . !

SpaceTime is no longer given a priori in General Relativity; it is curved. Its Geometry is not assigned a priori, but rather determined by the gravitational field, that in turn is determined by the distribution of matter. Motion is still a trajectory in SpaceTime (a "Universe line"). Again each observer defines his own way to separate Space and Time, to obtain position as a function of Time.

But Gravity does not fit into Special Relativity and action-at-a-distance is not acceptable; this is why SpaceTime Geometry cannot be given a priori. According to a nice view first promoted by Arthur Eddington, SpaceTime in GR is like an elastic canvas, onto which masses determine the curvature, so that the more matter is present the more the canvas is curved.

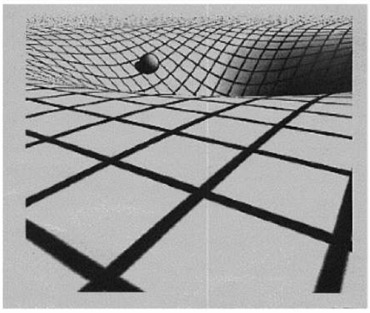

Fig.8. The "Eddington Canvas"

We may say that "*General Relativity is like to play croquet on a field that deforms according to the weight of players*". Violent gravitational perturbations produce *gravitational waves*, i.e. disturbances in the geometric structure of SpaceTime that propagate at the velocity of light and put it in oscillation. The search of gravitational waves (still elusive to "local" laboratory experiments but discovered in binary systems) is one of the present challenges of experimental GR.

Fig.9. An aerial view of the VIRGO experiment

4 General Relativity and Mathematics

4.1 The Mathematical structure of the Theory of General Relativity

As we said above, Einstein released the GR theory in 1916, finally written in a fully coherent form, both on a physical ground (conservation of matter) and on a mathematical ground (Lagrangian formulation). This was in fact the result of a fruitful interaction with David Hilbert, who did in fact obtain the correct Lagrangian, while Einstein was more devoted to understand the problem from the viewpoint of matter conservation.

As we also said, Gravity is identified with the (dynamical) *metric structure* g of a curved SpaceTime M. The field equations are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 4\pi k T_{\mu\nu}$$

afterwards called universally the "*Einstein Equations*". Here the "stress tensor" is defined as follows:

$$T_{\mu\nu} = -\frac{1}{\sqrt{g}} \frac{\partial L_{mat}}{\partial g^{\mu\nu}}$$

as it derives from a "matter Lagrangian".

According to Hilbert the *simplest* variational principle is assumed, based on the following (gravitational) Lagrangian:

$$L_H = \sqrt{g} R ds,$$

where ds denotes the volume element of SpaceTime M, afterwards called universally the "*Hilbert-Einstein Lagrangian*".

The Theory of General Relativity is mathematically and physically based on a few foundational requirements, that have in fact to be satisfied by any reasonable and possible generalization of Einstein Theory. Physically speaking a coherent gravitational theory - according to Einstein's view - has to satisfy the following requirements:

1. the *principle of equivalence* (Gravity and Inertia are indistinguishable; there exist observers in free fall, so called *inertial motion*);
2. the *principle of relativity* (SR holds pointwise, in the sense that the structure of SpaceTime has to be "pointwise Minkowskian" in the limit);
3. the *principle of general covariance* (democracy" in Physics: all laws have to be written in such a way to transform "coherently" in SpaceTime whenever an arbitrary frame of reference is chosen);
4. the *principle of causality* (all physical phenomena propagate respecting the causality relations that still hold in SR, i.e. nothing can exceed the velocity of light and what is past or future for an observer is past or future for all observers);
5. Riemann's teachings about the *link between matter and curvature.*

The Mathematical consequences of these principles in GR are usually assumed to be the following:

1. the principle of equivalence entails that the inertial motion has to be a geodesic motion;
2. the principle of relativity entails that SpaceTime M is endowed with a Lorentzian metric g;
3. the principle of general covariance entails tensoriality of all physical laws;
4. the principle of causality entails that all physical phenomena propagate respecting the light cones structure generated by the metric g;
5. Riemann's teachings about the link between matter and curvature entail that gravitational forces" have to be described by a curvature tensor. In particular, since in GR the gravitational field is described by the metric g there should be exactly 10 independent equations. Since the Riemann curvature tensor Riem(g) has 20 (independent) components (too many...!) the appropriate linear concomitant is assumed to be the Ricci Tensor Ric(g), that is OK since it has exactly 10 (independent) components.

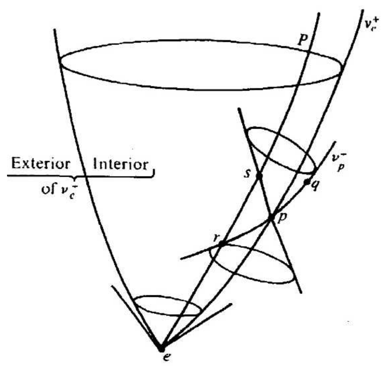

Fig.10. The light cone structure defined in GR

A few remarks are now in order. They will turn to be useful later...

In an absolute SpaceTime, as Newton's one, the equation $\mathbf{F} = m\mathbf{a}$ also tells that the sources of curvature of the motion are the forces. As well as the "Principle of Inertia" ensures that there exist reference frames where in absence of any force motions are straight and uniform (as we said, linearity in four dimensions, but with Space and Time separated).

Accordingly, in Classical Physics the principle of inertia selects at the same time: a) the reference frame (or better an "inertial family of frames ") where the forces are "true"; b) the straight lines of four dimensional space thought as a product of Time (a real line) by a 3-dimensional Euclidean space.

In the "Newtonian limit" of GR (something that is in fact a little bit misleading, but we cannot enter here in details) the principle of equivalence replaces tout-court the principle of inertia and geodesics replace straight lines.

4.2 The "Mathematical Legacy" of General Relativity

General Relativity has been and still is an extremely important and fascinating framework for the development of whole branches of mathematical research, some even stimulated by research in Relativity, ranging from *Differential and Riemannian Geometry* to the *Theory of PDE's*, from the *Calculus of Variations on Jet Bundles* to the theory of *topological invariants* (characteristic classes, knots,...) and many others. As reference books we may quote [18] and [30].

Differential Geometry has to do with describing the global and intrinsic properties of SpaceTimes, including *asymptotic behaviour.*

More in particular, *Riemannian Geometry* has to do with describing the metric properties of SpaceTimes, including *geodesic structure and curvature singularities.*

The *Theory of PDE's* has been influenced by applications in GR; Einstein Equations — as we said - form in fact a system of 10 equations, 6 of which of hyperbolic character and 4 as *(elliptic) constraints* on initial data given on a spacelike 3-surface. Important is then the Cauchy problem for GR, as well as the *conformal techniques* used to solve it. Moreover, when initial data are given on a lightlike 3-surface, the *Cauchy problem becomes ill-posed* and several ad-hoc techniques can be used to approach it.

But also the *Theory of ODE's* has gained insights because of applications to GR; ODE's techniques are in fact necessary to investigate *geodesic motion in SpaceTimes*, to find specific *exact solutions* and to understand the motion of *test particles.*

Of extreme importance is the *Variational Structure of Einstein Equations*, having to do with *families of solutions, conservation laws, entropy, singularities and quantization issues.* This explains the impetus that also the theory of *Jet Bundles* has gained because of GR (and, more generally, because of the "geometrization of physics" developed insofar) — [13], [10].

Differential Topology has also to do with GR; in particular it turns out to be related with understanding the structure of *singular SpaceTimes*, as well as notions such as *horizons, cosmic strings and other topological defects* in the geometrical structure of the Universe thought as a whole. Differential Topology enter also the variational formulation of General Relativity (and its extensions) through the *curvature invariants of manifolds* (see [25]).

But also *Infinite Dimensional Differential Geometry* has gained a lot from GR. It has to do, for instance, with the Hamiltonian description of Einstein Equations as a flow in the space of all 3-geometries, i.e. *metrics modulo diff-invariance. Knots and Graphs* enter the quantization procedures that investigate SpaceTime as a suitably defined *spin network* (see later).

Not to speak of *Numerical Analysis*, that is needed to study numerically particular solutions of Einstein Equations, especially in the *regime of collapsing matter.*

5 Is General Relativity enough. . . ? beyond the Theory of General Relativity

We shall here try to explain why — in spite of its beautifulness and of its immense impact on the Physics and the Mathematics of XX Century — General Relativity "tout-court" is far from being the "ultimate theory of gravity" and, as such, it needs further developments and investigations, that are in principle aimed not to destroying its structure but rather to include it into larger and more flexible frameworks. This because of the need of more general mathematical structures, of more appropriate coincidences with observational data and because of needs related with the emerging "Theory of Strings" and of issued related with the still unsolved problem of appropriately quantizing the gravitational field.

5.1 The theory of linear connections (1916-1919)

Working on the theory of "parallelism" in manifolds, the Italian mathematician Tullio Levi-Civita understands around 1916-1919 that "parallelism" is not a metric property of space, but rather a property of "affine" type, having to do with "congruences of privileged lines".

Fig.11. Tullio Levi-Civita

Generalizing the case of the Christoffel's symbols, Levi-Civita introduces the notion of *linear connection* as the more general object ? such that the *equation of autoparallel curves* (sometimes improperly called also "geodesics")

$$\frac{d^2x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} = 0$$

is generally covariant.

A connection in a 4-dimensional space has 64 components (it has in fact n^3 components in dimension n); only 40 of them survive if (in 4 dimensions) the (linear) connection is assumed to be symmetric (we say also "*torsionless*").

Any linear connection Γ defines a *covariant derivative* operator $\nabla^{(\Gamma)}$ (one for each connection); this, in turn, defines parallelism through the "geodesic equation" of autoparallel lines. A rather interesting and particular case is the case of the *Levi-Civita connection* (LC connection) generated by a metric g; its components are the *Christoffel symbols* of the metric and they totally depend on the metric g together with its first derivatives. The LC connection of a metric g is in fact the only torsionless connection such that the metric g is "parallel along g" (i.e. the covariant derivative of g vanishes along it). Hereafter the symbol $\nabla^{(g)}$ will be used, whenever necessary, to denote the covariant derivative along the Levi-Civita connection of a metric g. Linear connections are "geometric objects" in the sense of [32], [20], [8]. They are not tensors, but rather "natural objects" of order two (i.e., related to bundles of second order jets). Differently from tensors they can be therefore set to be equal to zero at any SpaceTime point, by means of appropriate coordinate transformations that, by "second order" effects, take accelerations into account. They are thence the most natural candidates to represent "inertial motions"; the case of the LC connection of a metric g is just a particular case, a case in which parallelism and free fall are pretended to be regulated by a variational prescription on the arc length (or, better, the "energy" to keep also non-positive signatures into account) defined by the metric g itself. In this case the terminology "geodesics" is appropriate.

5.2 The Unified Field Theories (1919-1955)

Already in 1919 the physicist Hermann Weyl makes a celebrated attempt to unify Gravity with Electromagnetism. He understands that Electromagnetism is what in modern language is called a *gauge field* (see, e.g., [16], [2]). He introduces a scalar factor φ (a "Gauge") that point by point calibrates the interaction. The metric g (Gravitation) and the scalar factor φ (the "Phase") determine in fact together a linear connection in SpaceTime. Weyl chooses a Lagrangian that is quadratic in the curvature of this linear connection (depending on g and φ together) — see [33].

Fig.12. Hermann Weyl

Weyl's idea, unfortunately, fails. The Lagrangian is not appropriate and field equations describe instead a "massive photon" (in modern language it is in fact a Proca-Yukawa interaction). Weyl's idea generates however a keypoint: *connections may have an interesting dynamics.* Fields may be *gauge fields* - i.e. fields with group properties coming from further principles and "internal symmetries". Nevertheless, gauge theories will have to wait for about 30 more years. . . ..

After the pioneering work by Weyl, the attention of Einstein will be immediately switched to the search of more general field theories, based on the use of a linear connection able to encompass at the same time both the gravitational effects and the electromagnetic effects. Einstein, followed also by Eddington, Schrödinger and other scientists, will investigate this kind of theories, unfortunately without any reasonable success, in various attempts from 1923 until the fifties. These efforts will be somehow "wiped off" after 1955 as a consequence of the strong developments of "gauge theories" as theories of a somewhat different character.

These gravitational theories, in which the Lagrangian is of the type $L(\Gamma, \partial\Gamma)$, can be collectively called "*purely affine theories (of gravitation)*". One of the most important outcomes of Einstein's and Eddington's ideas in this respect is that whenever one has a Lagrangian theory depending on just a connection — and because of covariance requirements this has to be a Lagrangian depending on the curvature tensor of the connection Γ — a metric $g(\Gamma)$ can be always obtained as "momentum" canonically conjugated to the connection, by requiring that the metric is proportional to the derivative of the Lagrangian with respect to the (symmetric part) of the Ricci Tensor of Γ.

5.3 The Metric-Affine Field Theories and the Einstein-Palatini Method (1925)

Because of the preliminary failure of his 1923 attempts to unify Gravity and Electromagnetism through "purely affine theories", the attention of Einstein is immediately switched to "*metric-affine theories*", i.e. theories in which a metric g and a linear connection Γ are given from the beginning as independent variables. Einstein himself, in fact, is not so happy with the fact that the gravitational field is not the fundamental object, but just a by-product of the metric. Generalizing a method invented few years before by Attilio Palatini in a purely metric framework (see [11]), he realizes that one can obtain Einstein equations by working on a theory that depends on two variables, varied independently: a metric g and an arbitrary linear connection, assumed to be symmetric. He defines in 1925 a Lagrangian theory and method, nowadays known as *Palatini-Einstein method*, in which the Lagrangian is assumed to be the density associated with the scalar curvature of the pair (g, Γ):

$$R(g, \Gamma) = g^{\mu\nu} R_{\mu\nu}(\Gamma, \partial\Gamma).$$

Varying this simple linear Lagrangian there are (in dimension 4) exactly 10 + 40 independent variables and field equations; if a matter Lagrangian is added these are

$$R_{(\mu\nu)} - \frac{1}{2} R g_{\mu\nu} = 4\pi k T_{\mu\nu}, \qquad \nabla^{(\Gamma)}_{\alpha}(\sqrt{g} g^{\mu\nu}) = 0,$$

provided matter interacts only with g, i.e. $L_{mat}(g, \nabla^{(g)}\Phi)$; here $\nabla^{(\Gamma)}$ is covariant derivative with respect to Γ. A result due to Levi-Civita ensures that if a torsionless linear connection Γ satisfies the second equation for a given metric g, then this connection is (proportional to) the LC connection of the metric itself. This, replaced back into the first field equation, reproduces then Einstein equations for the metric g. The Einstein-Palatini principle applied to the "Hilbert-like Lagrangian" above is therefore nothing but a curiosity, even if it has the advantage of obtaining that Γ is a LC connection as a dynamical effect rather than as an ansatz chosen a priori. The power of this remark remains undisclosed until our days.... (see later).

5.4 Is the Universe accelerating?

One of the remarkable solutions of Einstein equations is the so-called Friedmann-Robertson-Walker model for Cosmology (FRW solution), together with its deSitter extensions (see [18]). These solutions for a perfect fluid tensor account for many properties of cosmological origin; in particular they served, until the new developments of so-called "precision cosmology" (see later), to theoretically predict the so-called "Hubble's law" of Galaxies redshift (expansion of the Universe).

After the seventies new cosmological models took also into account — after the so-called "big bang", i.e. a primordial fireball that should have originated the Universe as we know it — a short "period of inflation". In all these models the rate of the expansion of the Universe depends on the matter content; standard observations until "precision cosmology" seemed to be compatible with a "decelerating Universe", but more recent data seem to strongly entail that the Universe is instead accelerating in its expansion (see [4]). Phenomena are related, e.g., with the behaviour of Ia-type Supernovae; the X-ray CMWB (Cosmic Microwave Background); X-ray Large-scale structure spectrum; etc... (see [4]; and see also all papers in the special issue of GERG Journal that contains this paper).

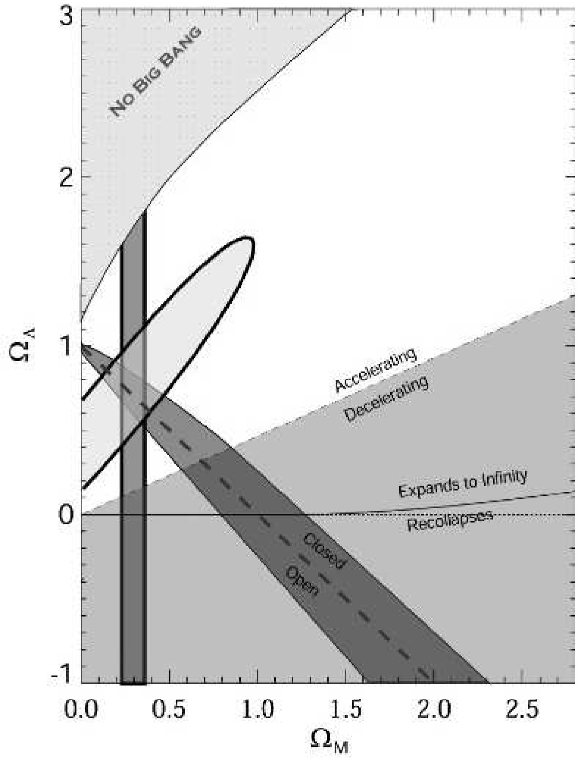

Fig.13. The observational diagram for an accelerating Universe

Possible ways out have been investigated. If one pretends to maintain rigidly the structure of Einstein equations, as above, then the simplest way out seemed to be the introduction of so-called "dark matter" and "dark energy", i.e. components in the right hand side of the equations that would in principle induce the observed acceleration in FRW-based solutions, fitting observational data at the expense of introduce rather exotic kinds of "unobserved" matter; see [4] for a discussion.

According to the "dark paradigm" the content of the Universe is, up today, absolutely unknown for its largest part (up to 95% of matter is considered to be invisible). The situation is in a sense very "dark" from the emotional and theoretical side, too; while the observations are instead extremely good....!

Of course one can accept exotic matter in order to keep the gravitational part of the equations unchanged. Or, viceversa, one could envisage modifications of the gravitational part of the equations, to keep coherence with "observed matter". In a sense, the idea is that dark matter and dark energy could as well be exotic "gravitational effects", i.e. "curvature effects" to be theorized in a larger scheme than Einstein GR.

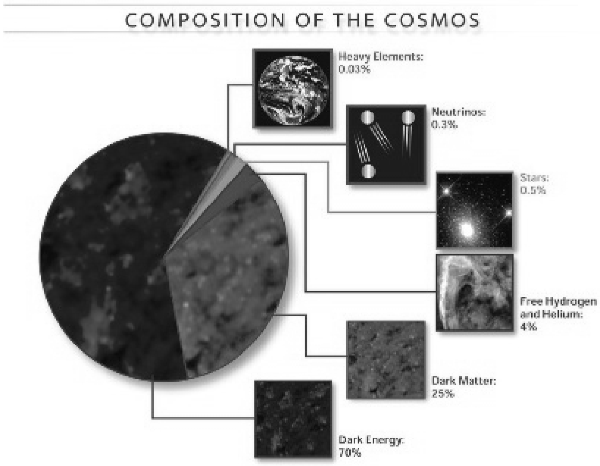

Fig.14. Hypothetical distribution of observed and dark components of matter/energy in the Universe

5.5 Dark matter and alternative theories of gravitation

To summarize, GR is simple, beautiful... and (seems to be) insufficient on the bases of astronomical observations: a) there is the need of a nonzero cosmological constant Λ; b) there is "inflation"; c) an anomalous acceleration is observed in the recession of Galaxies (modifications to Hubble's law).

Observations say that there is way for too few matter in the Universe! Thence the need, in order to save GR, for dark energy and dark matter, by changing as follows the equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 4\pi k (T_{\mu\nu} + T_{\mu\nu}^{dark}),$$

where $T_{\mu\nu}^{dark}$ dark accounts for all observed extra effects and should contain also a cosmological term of the form $\Lambda g_{\mu\nu}$ (with Λ possibly negative). To account for Dark Energy and to fit observations many unconventional hypotheses have been proposed, such as the already mentioned Cosmological constant Λ, that might be even time-varying, extra scalar fields, so-called "Phantom fields", etc...

We need therefore "*Modified Cosmological Models*". However, also because of other new developments in Gravity, it seems that "*Alternative Gravitational Theories*" should also be investigated, in which the Lagrangian for Gravity is assumed to be more general than the simple Lagrangian of Hilbert. Issues come from at least three different new perspectives: 1) *QFT (Quantum Field Theory) on curved SpaceTimes*; 2) *String/M-theory corrections* [17]; 3) *so-called Brane-world models*.

According to these issues (non-linear) curvature invariants more complicated than R should be taken into account, e.g.:

$$R_{\mu\nu} R^{\mu\nu}, R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}, \ldots$$

(see, e.g., [31]). In a sense we do not suitably understand gravity at large scales, so that we are forced to go well beyond the Theory of General Relativity.

5.6 Non-linear metric theories of Gravitation — NLTG - (1950-1965-now)

These are field theories in which the (gravitational part of the) Lagrangian is an arbitrary function of a metric together with its first and second order derivatives. Covariance then imposes that the Lagrangian should be a scalar function of g and of the Riemann tensor $R_{\mu\nu\alpha\beta}$ of g:

$$L_{PM} = f(g_{\mu\nu}, R_{\mu\nu\alpha\beta}, \Phi^A, \nabla_\alpha \Phi^A)\sqrt{g}$$

for any (analytic) function f. Here ∇ is a short-hand for $\nabla^{(g)}$. Of course particular cases are those in which the Lagrangian depends only on the Ricci tensor of the metric

$$L_{PM} = f(g_{\mu\nu}, R_{\mu\nu}, \Phi^A, \nabla_\alpha \Phi^A)\sqrt{g}.$$

NLTG complicate the mathematics of the problem. These genuine second order Lagrangians do in fact produce field equations that are "generically" of the fourth order in the metric, something that cannot be accepted if one believes that physical laws should be governed by second order equations. (A NLTG gives in fact second order field equations if and only if the Lagrangian is degenerated; e.g., if it is linear in curvature or if one chooses a "topological Lagrangian"...)

A particular family of NLTG is that of $f(R)$ *theories in metric formalism*, in which the Hilbert Lagrangian is replaced by any non -linear density depending on R. GR is retrieved in (and only in) the particular case $f(R) = R$. In these theories there is a second order part that resembles Einstein tensor (and reduces to it if and only if $f(R) = R$) and a fourth order "curvature part" (that reduces to zero if and only if $f(R) = R$ and accordingly $f'(R) = 1$):

$$f'(R^{(g)})R^{(g)}_{\mu\nu} - \frac{1}{2}f(R^{(g)})g_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R^{(g)}) + g_{\mu\nu}\Delta f'(R^{(g)}) = 4\pi k T_{\mu\nu}.$$

This (as all theories involving more derivatives than the second order ones) is usually called as "Higher Order Gravity". Pushing the fourth order part to the right hand side and making some easy manipulations allows one to write these equations in the form

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 4\pi k(T_{\mu\nu} + T^{grav}_{\mu\nu}(g)),$$

so that the fourth order terms can be there suitably interpreted as an "extra gravitational stress" $T^{grav}_{\mu\nu}(g)$, much in the spirit of Riemann.

In any case, however, the fourth order character of these equations still hidden in $T^{grav}_{\mu\nu}(g)$ makes them very unsuitable under several aspects, so that they were eventually abandoned and only recently reconsidered as valid.

These metric theories of gravitation can be shown to be dynamically equivalent to Einstein Theory for a conformally related metric, where a scalar factor ϕ arises as $f'(R)$ and obeys non-linear Klein-Gordon equations (see [24],[1]) with a potential $V(\phi)$ depending on the form of the function f.

5.7 Non-linear Palatini formalism - (1994-now)

Andrzej Borowiec, Marco Ferraris, Mauro Francaviglia, and Igor Volovich [12] remarked in 1994 that for non-linear theories of Gravitation written under the Palatini form a *universality property* holds (see [1], [12]).

In this case one assumes for SpaceTime a metric-affine structure and considers the following generalization of the above Lagrangians:

$$L_{NLP} = f(R(g,\Gamma))\sqrt{g},$$

where f is again an arbitrary (analytic) function.

The field equations for this Lagrangian are:

$$f'(R)R_{(\mu\nu)} - \frac{1}{2}f(R)g_{\mu\nu} = 4\pi k T_{\mu\nu}$$
$$\nabla^{(\Gamma)}_{\alpha}(f'(R)\sqrt{g}g^{\mu\nu}) = 0$$

where we have taken into account the existence of a matter Lagrangian L_{mat} depending only on g but not on Γ - i.e. $L_{mat}(g, \nabla^{(g)}\Phi)$ - and where $\nabla^{(\Gamma)}$ is again the covariant derivative with respect to Γ; they reduce to the Einstein-Palatini equations iff $f(R) = R$.

Taking the trace of the first equation one gets (in 4D) the so-called *master equation*:

$$f'(R) - 2f(R) = 4\pi k\tau,$$

where τ is the trace of the stress tensor $T_{\mu\nu}$. If this trace τ is a constant (e.g., if there is no matter at all....) then the master equation forces the scalar curvature R to take specific constant values that depend on the analyticity properties of f. The only degenerate cases are the linear case $f(R) = R$ and the quadratic case $f(R) = R^2$. Then the equations transform again into Einstein equations for g with a (quantized) cosmological constant. If τ is not a constant then the master equations allows to calculate (implicitly) R as a funcion of τ, so that the equations are turned into Einstein equations for a conformally related metric $h_{\mu\nu} = f'(R(\tau))g_{\mu\nu}$. We refer the reader to [17] for more details.

5.8 Is dark matter really "matter" ? Or is it rather a (non-linear) effect of curvature...... ?

Standard GR tell us that we do not understand Gravity at a full scale if we insist in pretending that: 1) the metric is the fundamental field; 2) the Lagrangian is linear in the curvature. Assuming non-linear functions of the curvature allows one to interpret "exotic effects" in Einstein Equations at different scales (Solar System, Galaxy, Extra Galactic, Cosmological) as curvature effects due to non-linearity of the Lagrangian rather that effects due to unseen matter or energy ("the dark side of the Universe" is suggested to depend on the conformal factor $f'(R(\tau))$ as above).

Experiments have to tune-up the form of the Lagrangian. Classes of allowable Lagrangians have to be selected by experimental data, if possible....

The Palatini approach - moreover - tell us why the metric we see generating Gravity is different, point by point, by the metric we use to make experiments. Gravity drives the "gauge" of our instruments. Clocks tick slower in a Gravitational field. Curvature tunes up Gravity and forces us to redefine rods and clocks....!

A review and comments on this may be found in [3].

6 Quantum Gravity

A further big problem with Gravity is the fact that even now it is difficult to reach an agreement on how to "quantize" the theory of Einstein.

To reach an agreement between General Relativity and Quantum Theory it is probably necessary to renounce to describe SpaceTime as a continuum. In this we like to recall one of the most astonishing predictions of Riemann, that in his already quoted prolusion also wrote: "*If the independence of bodies from position does not exist, we cannot draw conclusions from metric relations of the great to those of the infinitely small*", somehow anticipating that the "geometry of the infinitely large" and the "geometry of the infinitely small" might be different and even unrelated.

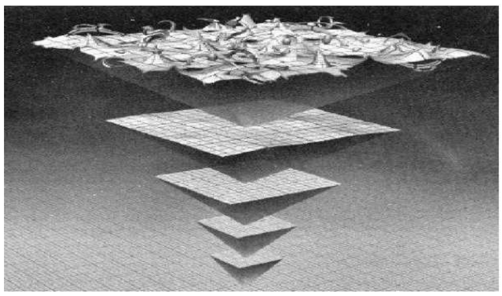

Fig.15. A depiction of "Quantum Fluctuating Universe"

6.1 Loop Quantum Gravity (LQG)

One of the possible extensions of GR to include quantum effects is the so-called "Loop Quantum Gravity" program (see [29]). In LQG physical states of the gravitational field are represented by means of *linear combinations of (trace of) holonomies* (*spin network*" states). These states must be gauge invariant and (spatially) diffeomorphic invariant (as a consequence of the *general covariance principle*).

States are labelled by knots with links coloured by the unitary group $SU(2)$; representations and nodes are coloured by "intertwiners".

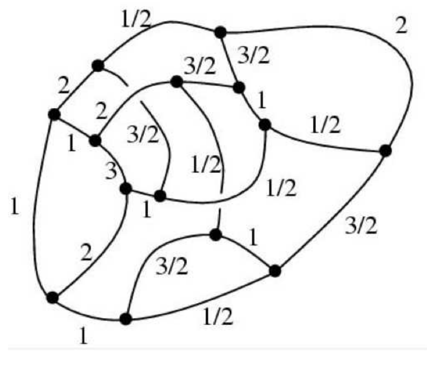

Fig.16. A "Spin Network"

In "spin foam models" the evolution of a spin knot in time is represented by 2-complexes in homology. In this framework these objects are used to represent quantum SpaceTime.

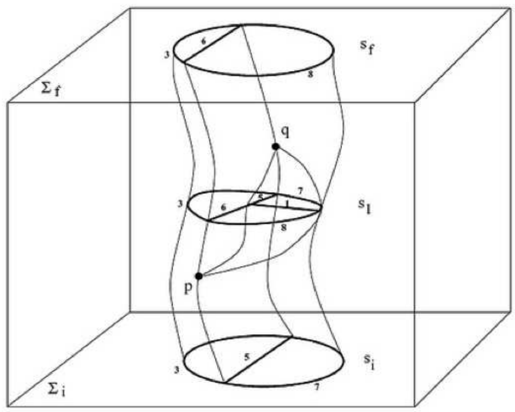

Fig.17. The time evolution of a "Spin Network"

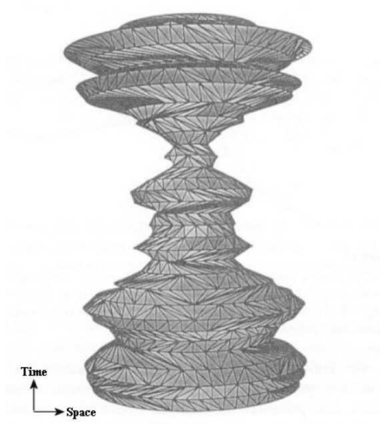

Fig.18. The final evolution of a "Spin Network" into a Quantized SpaceTime

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